New Matriculation Geometry

DELHI

M. Gulab Singh & Sons
Educational Publishers.
1951

Price. **3/12/-**

Geometry

Published by :—
L. Moti Ram, Manager,
for M. Gulab Singh & Sons,
Educational Publishers,
DELHI.

Printed by :—
I. C. Vaish, B.A.,
at Swatantra Bharat Press,
423, Kucha Bulaqi Begum,
Esplanade Road, DELHI.

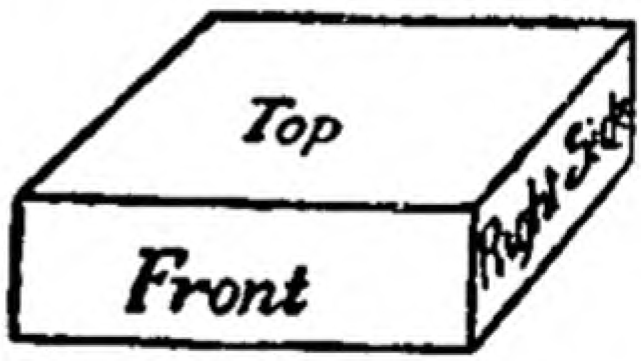
CONTENTS

PRELIMINARY CONCEPTS.

1. **Solid**.--Look at a brick, a table, an iron ball and a piece of stone. All these things are of different shapes and different sizes and are made of different materials ; but they have one property in common, that is, they all *occupy space*. Anything, therefore, which occupies space is, in geometry, called *a solid*. A piece of paper, a tub full of water and an empty box are all solids.

2. **Surface**.—Look at a brick again. You find that it is bounded by six sides, which may be called the top, the bottom, the front, the back, the right side and the left side. These six sides also called faces, separate the brick from the space surrounding it.

The **Surface** of a solid is, therefore, the boundry which separates it from the rest of the space around it.



In the case of a brick, all its six faces are flat, but if you look at a ruler, you will see that it has one surface curved and two flat, *viz.*, the top and the bottom. So it is not necessary for a solid always to have flat surfaces. It may have some flat and some curved surfaces.

A flat **surface** is called **a plane surface** or **a plane**.

If the straight line passing through any two points taken on a surface lies wholly in that surface, the surface is a plane surface.

3. **Dimensions**—A solid has always length, breadth and thickness (or height or depth) and these are called its **dimensions.** A solid, therefore, has *three* **dimensions**.

4. Look at the brick and examine carefully one of its faces, say, the front. You will find that it is joined to the top and the bottom by means of two edges; also it is connected with the right side and the left side by two more edges. So the front face which is a **surface** and which separates the brick from the surrounding space has only length and breadth. but no thickness. A surface, therefore, has *only two dimensions*.

5. **Line.**—An edge which joins any two faces of the brick has only length and is called **straight line.** Similarly the curved surface of a ruler meets one of its flat surfaces at an edge called a **curved line**.

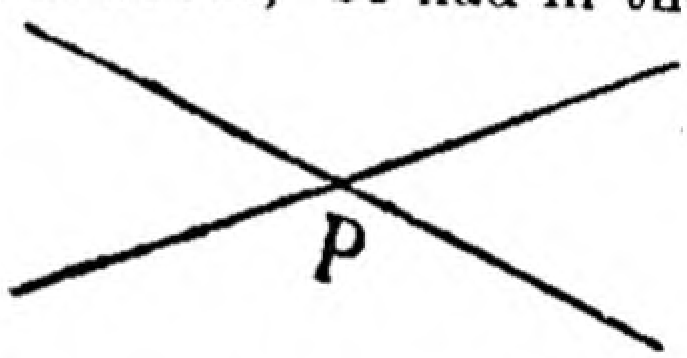
A fair conception of a straight line can be had by stretching tightly between two hands a fine thread. The smaller in thickness the thread, the more nearly it approaches an ideal straight line which has only length but no breadth or thickness.

A straight line has, therefore, only *one dimension.*

6. **Point.**—Stand inside a room and look into one of its corners. You will find that the two sides of the floor meet *at a place* in that corner. This place where the two straight lines representing these two sides meet is called a **point**. Therefore a **Point** is *only the meeting place of two straight lines.* As such it has neither length nor breadth nor thickness and therefore has *no dimensions*, but has only *position*.

A little further observation will show that another straight line representing the height of the room also passes through the same point. From this it should be clear that any number of straight lines may pass through one and the same point but two lines are generally sufficient to fix the position of a point.

In geometry the position of a point is usually denoted by a dot, which again is not the correct way of representing a point because, however small a dot may be, it will still have some size represented by some length and some breadth. A somewhat fair representation of a point can, however, be had in the intersection P of two straight lines which as explained above, are only lengths without breadth.



7. A point is named by one letter from the alphabet and a straight line by two, one placed at each end. For instance, the line AB is the join of two points A and B, and is read as the *line* AB.

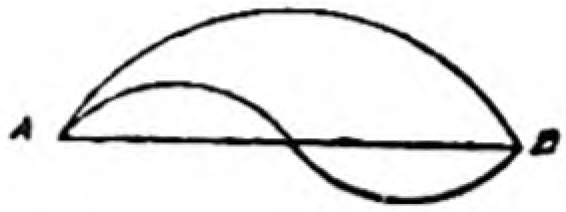
When any two points are joined, it results in a straight line connecting those two points.

"*Join* AB" therefore means "draw a straight line from A to B."

"Produce AB" means "lengthen the line AB beyond B." "Produce BA" means "lengthen the line AB beyond A." "Produce AB both ways" similarly means "lengthen the line AB beyond B as well as beyond A."

8. (*i*) *There cannot be drawn more than one straight line between any two points.*

Suppose you are standing at a point A and you want to reach another point B. Surely there are more ways than one, as shown in the figure, in which it can be done; but the shortest one would be along the straight line AB. Try any other path and every time you will find that it is longer than the straight line AB and is not a straight line. Therefore *one and only one* straight line can be drawn between any two points and that will always be the *shortest distance* between them.

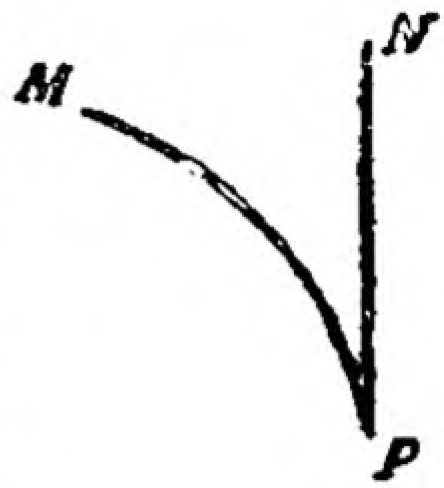


(*ii*) *Two straight lines cannot be drawn so as to enclose a space.*

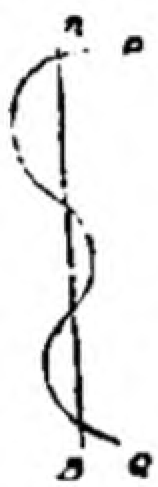
Examine the figure. In this two *lines* are drawn to connect A with B and obviously between them they enclose some white space. One of these lines is a straight line. Can the other also be a straight line? No, it cannot, because in (*i*) above we have shown that one and only one straight line can be drawn between two points. Therefore it follows that two straight lines can *never* be drawn so as to enclose space between them.

(*iii*) *A straight line has the same direction throughout its length.*

Suppose you stand at a point P and straight line PN is drawn due North and you proceed to walk along this line. You will find that all the time you keep walking along it, you will always be facing North. But if you walk along a curved line such as PM, your direction will continuously change. This shows that a straight line keeps the same direction throughout its length.

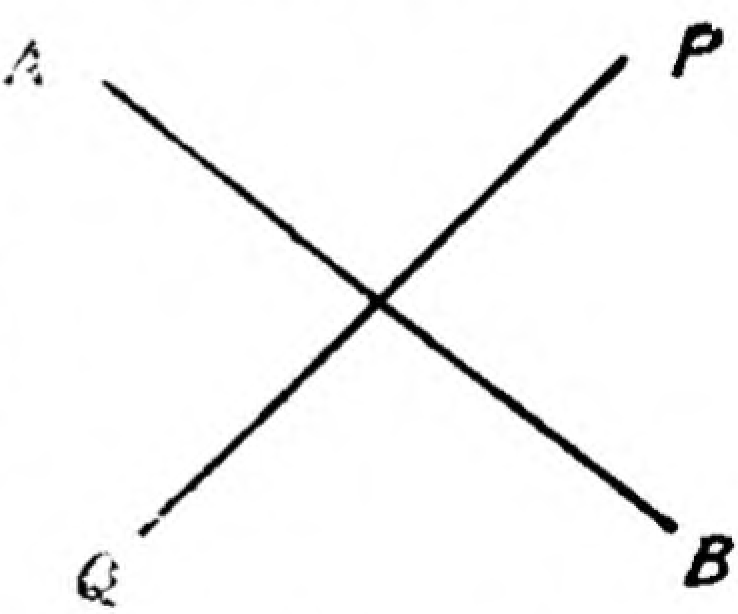


(iv) *If two straight lines cross one another, they can cut only at one point.*

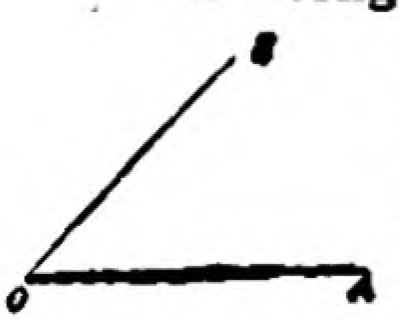


Draw any curved line PQ. Now with a ruler try to rule a straight line AB across it. You see that you always draw a line AB so as to cut PQ at two or more than two points.

Now make PQ a straight line and as before with the help of your ruler try to draw another straight line AB across it so as to cut it at two points. You will soon find that it is impossible, and the st. line AB, in whatever manner you draw it, cuts PQ in one and only one point.



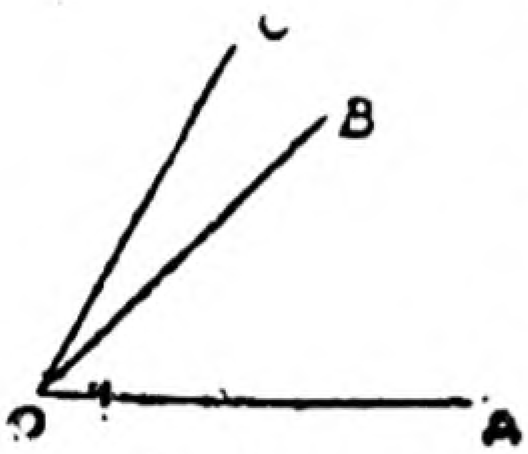
9. **Angle.**—Any two straight lines meeting at a point form an **angle**. For instance, OA and OB meet at O and therefore form an **angle** at O.



The lines OA and OB are called

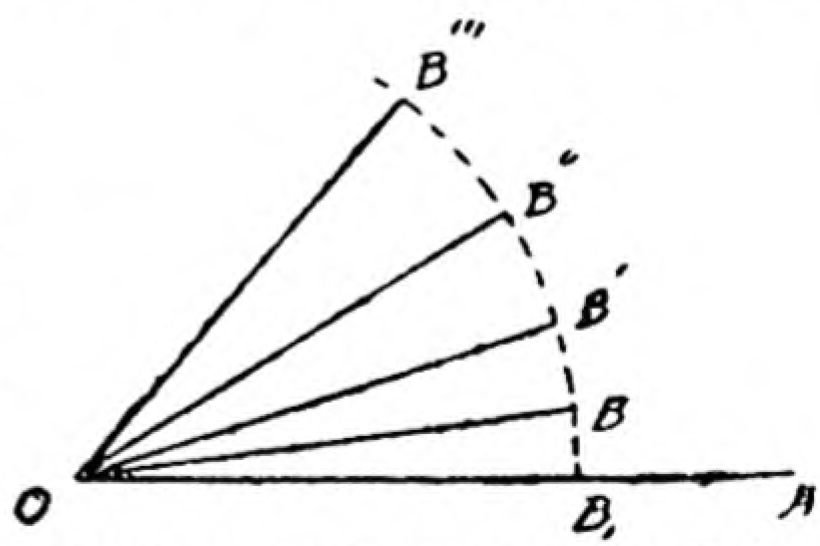
the **arms** of the angle and the point O is called its **vertex.**

10. **How to name an angle.**—An angle is generally named by three letters one placed at the vertex and the other two at the ends of the two arms. While reading an angle care must be taken to keep the *letter at the vertex* in the middle. Thus the angle in art. 9 will be read as **AOB** or **BOA**.



If a single angle is formed at a point it is often named by the letter placed at its vertex. As in the figure given above, the angle could also have been named "**Angle O.**" But if more than one angle is formed at the same point, to avoid confusion, each is named by three letters. Thus in this figure the angles formed ot **O** are **AOB, BOC** and **AOC.**

11. **How angles are produced and how they are measured**—An angle is produced by the rotation of a line about its point of intersection with another fixed straight line. In the fig. we have a line OB intersecting a fixed line OA at the point O. The line OB begins to rotate about O and occupies different positions such as **OB′, OB″, OB‴** at different intervals of time and forms with **OA** angles **AOB, AOB′, AOB″** and **AOB‴.** Students should note that **B′, B″** and **B‴** denote the different positions occupied by the extremity **B** of the rotating line **OB.**



The rotating line is called the **revolving line** and the fixed line the **initial line**.

Evidently the size of an angle formed at any time depends upon the amount of rotation that the revolving line undergoes, supposing it in the beginning to be in coincidence with the initial line. Thus *the amount of rotation of the revolving line gives the measure of the angle formed and obviously the size of the angle does not depend on the lengths of its arms.*

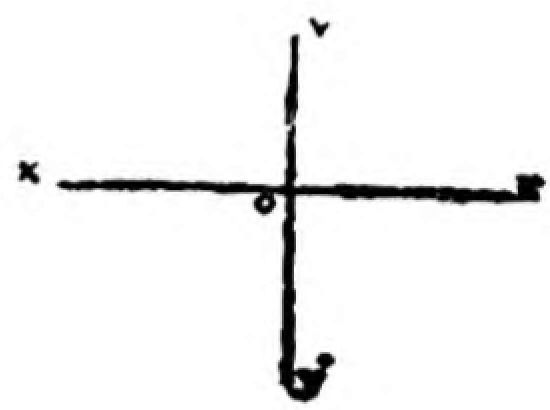
It is also clear from the above consideration that one angle may be greater or smaller than another. For instance, in the above figure angle **AOB″** is greater than the angle **AOB′** but is smaller than the angle **AOB‴**.

The revolving line OB can begin to rotate in two ways either *away* from the initial line or towards the initial line OA. In the latter case the direction of the motion of OB is evidently the same as that of *the hands of a clock* and is therefore called **Clock-wise** motion. In the former case it being opposite direction to that of the hands of a clock, it is known as **Counter-clockwise** motion.

12. **Right angle.**—Look at a page of your book. Where two edges meet you find an angle. This angle is called a **right angle**. To understand it better, suppose you stand out in your play-ground facing due east. Your drill instructor says "Left turn". You slightly turn round and face north. In so doing you have turned round through one *Right angle.* If he repeats the same order you turn round again and now face west. You have turned through another right angle. In this position you are facing in a direction

directly opposite to the first one. If he were to repeat the same order twice over, you would turn due south making three right angles and ultimately come to your first position facing due east and make a complete round turning through one more right angle. Thus a complete round consists of four right angles, and one-fourth of the complete round is called a **right angle** as angle XOY.

When two lines make a right angle, each is said to be perpendicular or at right angles to the other. In the figure, OY is perpendicular or at right angles to LXY.



Generally all angles are measured in terms of right angles and we say, a certain given angle contains so many right angles. But as we have seen above, all angles are not of the same size ; certain angles are less than a right angle and therefore to measure them we should have a unit of measurement smaller than a right angle. With this object in view a right angle is divided into 90 equal parts, each called a **degree**. The symbol ° is used to denote degrees.

Thus **One right angle** $=90^\circ$
Two right angles $=180^\circ$
Three right angles $=270^\circ$
Four right angles $=360^\circ$

Further $1^\circ = 60'$ (read sixty minutes)
$1' = 60''$ (read sixty seconds)

Note :—Measurement into minutes and seconds is beyond the matriculation stage.

13. **Kinds of angles**—A **right angle** is an angle of 90° (see Fig 1).

Fig. 1

An angle less than a right angle is called an **acute angle** (see Fig 2).

Fig. 2.

An angle greater than a right angle is called an **obtuse angle** (see Fig. 3).

Fig. 3.

An angle greater than two right angles but less than four right angles is called **a reflex or reentrant angle**. (see Fig. 4).

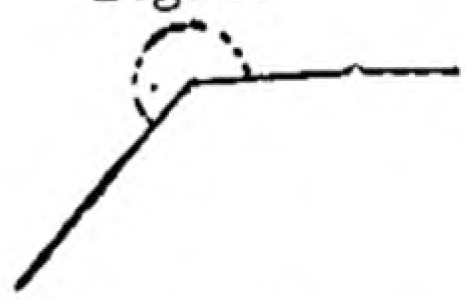


Fig. 4.

An angle equal to 2 right angles is called a **straight angle** (See Fig 5).

Fig. 5.

14. **The Circle**.—Open the points of a compasses to some distance. Place its needle end at a point O, marked on the paper and let the pencil point P rotate about O, taking care that the needle end sticks firmly at O and the distance between the two points of the compasses remains *unaltered.* The end of the pencil will trace out a figure enclosed by a curved line.

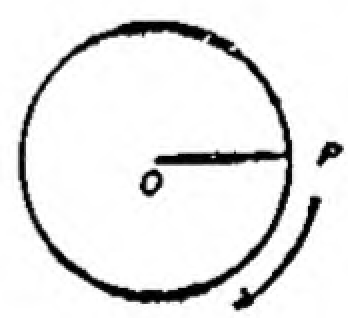


Fig. 6.

The figure so drawn is called a **circle** and **the** curved line is called the **circumference** of this circle.

The point O is called the **centre** of the circle and any straight line drawn from the centre to a point on the circumference is called a **radius**. Thus OP in Fig. 6 is a radius.

A straight line joining any two points on the circumference is called a **chord** of the circle. In Fig. 7, PQ is a chord.

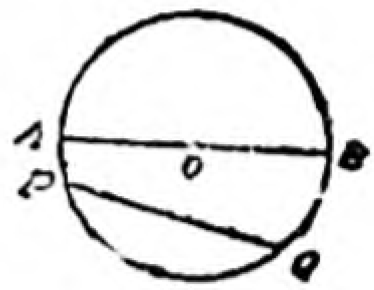


Fig. 7.

Any straight line drawn through the centre and terminated both ways by the circumference of a circle is called a **diameter** such as AOB in Fig. 7. It is thus obvious from the above definitions that a diameter of a circle is only a *chord* of the circle passing through its centre.

Any part of the circumference of a circle is called an **arc** of the circle. For instance, AB in Fig. 8 is an arc. and so is ALB. ALB is called the **major** arc, being greater than half the circumference and AB the **minor**. Both taken together are called the **conjugate** arcs.

Fig. 8.

A **Sector** is a figure enclosed by two radii and the arc lying between them. Thus the portion of the circle enclosed by OA and OB and the arc AB is a sector.

A chord of a circle divides it into two parts. The figure enclosed by that

chord and the arc it cuts from the circumference is called a **Segment** of the circle. Thus the figure PLQ enclosed by the chord PQ and the arc PLQ in Fig 9, is a segment, and so is PMQ, which is enclosed by the same chord and the arc PMQ.

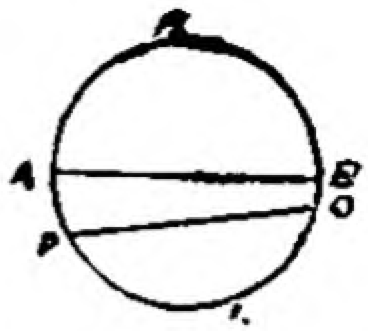
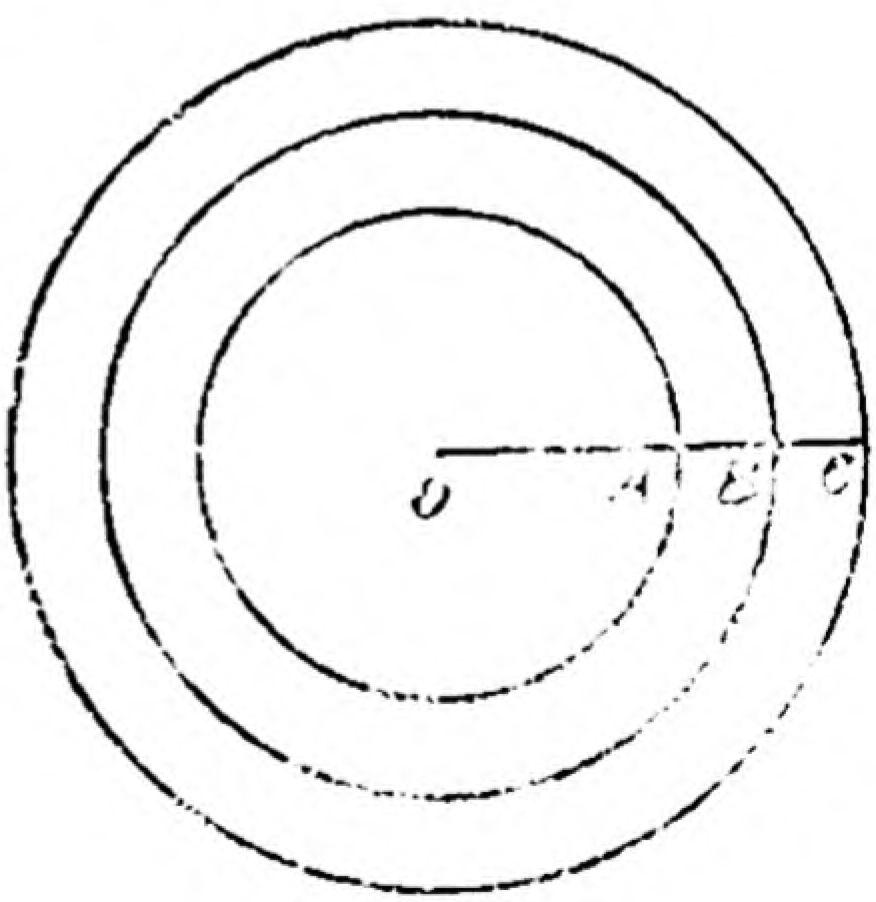


Fig. 9.

A diameter of a circle divides the circle into two *equal parts*. The figure enclosed by a diameter and the portion of the circumference cut off by it is called a **Semi circle** such as AMB and ALB in the previous figure. Any other chord except the diameter divides the circle into two unequal parts. The greater is called **Major Segment** and the smaller the **Minor Segment** ; as in Fig. 9 the chord PQ divides the circle into two unequal parts PMQ and PLQ. PMQ is a major segment and PLQ is a minor segment. Taken together they are called conjugate segments.

Concentric circles are circles having the same centre but different radii.

Here three circles have the same centre O but different radii, OA, OB and OC.



PROPOSITION AND ITS PARTS.

Demonstrative Geometry is the science that treats of the shape, size, and position of figures by pure reasoning based on definitions, certain axioms and other established geometrical facts.

A Geometrical fact proposed for discussion is called a **Proposition**.

There are two kinds of propositions :—

(*a*) **Theorems**—In these a statement is made of geometrical facts which are to be demonstrated or proved.

(*b*) **Problems**—In these geometrical constructions are required to be made.

The **Enunciation** of a proposition is the statement of facts required to be proved or a description of the construction to be made.

A theorem consists of two parts :—

(*i*) The **Hypothesis**, or that which is supposed to be true ; and

(*ii*) The **Conclusion** or that which is asserted to follow from the hypothesis.

A problem also consists of two parts :—

(*i*) The **Data** or that which states what is given ; and

(*ii*) The **Quaesita** or that which states what is required to be constructed.

One theorem is said to be the **converse** of another when the hypothesis of the first is the conclusion of the second and *vice versa*.

A **Corollary** is a proposition the truth of which follows immediately from a proved proposition.

AXIOMS AND POSTULATES

All reasoning in this book will be based on certain principles which are self-evident truth. Such principles are called **Axioms**.

All constructions are based on certain elementary statements called **Postulates**.

The axioms are :—

1. *Things which are equal to the same thing, are equal to one another.*

2. *If equals be added to equals, the sums are equal.*

3. *If equals be subtracted from equals, the remaining parts are equal.*

4. *If equals be multiplied by equals, the products are equal.*

5. *If equals be divided by equals, the quotients are equal.*

6. *If equals are added to unequals, the sums are unequal in the same order.*

7. *If equals are subtracted from unequals, the remainders are unequal in the same order.*

8. *Doubles of the same thing or of equal things are equal.*

9. *The whole is greater than its parts, and is equal to the sum of its parts.*

10. *Of three quantities, if the first is greater than the second and the second greater than the third, then the first is greater than the third.*

11. *A quantity may be substituted for its equal in an equation or in an inequality.*

12. *Magnitudes which coincide are equal to one another.*

The postulates are :—

1. *A straight line may be drawn from any one point to another.*

2. *A finite straight line may be produced to any length in that straight line.*

3. *A circle may be described with any point as centre and with any length as radius.*

4. *A geometrical figure may be moved from position to another without any change in its form and size.*

5. *All right angles are equal.*

6. *A finite straight line can be bisected at a point.*

7. *An angle can be bisected by a straight line.*

Note.—It should be borne in mind very clearly that for making geometrical constructions the use of only the *ruler* and the *compasses* is allowed.

SYMBOLS AND ABBREVIATIONS.

In all books on geometry certain recognised symbols and abbreviations are used. Such a device not only makes the solutions brief, simple and elegant, but also enables the writer to present various parts of his arguments in a concise and clear form.

The following symbols and abbreviations are used in this book :

Symbols

$\angle$	for angle	$\angle$s	for angles
$\odot$	,, circle	$\triangle$	,, triangle.
$\perp$	,, perpendicular or is perpendicular to.	rt. $\angle$	,, right angle.
$\parallel$	,, parallel or is parallel to	$\parallel^{m}$	,, parallelogram.
$\therefore$	,, therefore.	$\because$	,, because or since.
$>$	,, is greater than.	$<$	,, is less than.
$\not>$	,, is not greater than.	$\not<$	,, is not less than.
$=$	,, is equal to, or is equal to in area.	$\neq$	,, is not equal to.
$\odot^{ce}$	,, circumference.	$\equiv$	,, congruent or is congruent to.
$\sim$	,, the difference between.		

Abbreviations.

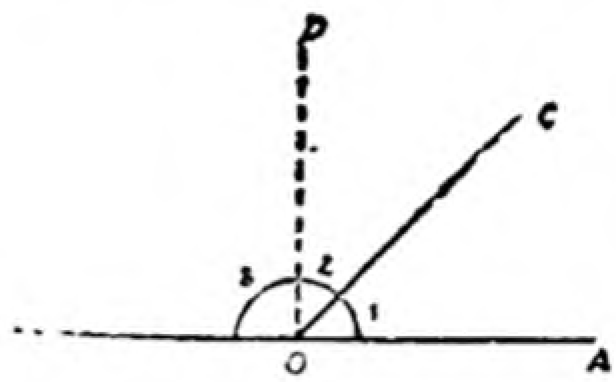
line	for straight line	st.	for straight.
pt.	,, point	sq.	,, square.
alte.r	,, alternate.	corresp.	,, corresponding.
hyp.	,, hypothesis	const.	,, construction.
adj.	,, adjacent.	rect.	,, rectangle.
def.	,, definition.	ax.	,, axiom.
perp.	,, perpendicular.	cor.	,, corollary.
ex.	,, exterior.	in.	,, interior.
fig.	,, figure.	ex.	,, exterior.
alt.	,, altitude.	ver.	,, vertically.
quad.	,, quadrilateral.	opp.	,, opposite.
		sup.	,, supposition.

SECTION I

ANGLES AT A POINT.

Proposition 1. (*Theorem*)

If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles.



Given :—The st. line OC meeting the st. line AB in O.

Required :—To prove that $\angle$AOC+ $\angle$COB=2 rt. angles.

Construction :—Draw OD$\perp$AB. Denote $\angle$AOC by $\angle$1, $\angle$COD by $\angle$2 and $\angle$DOB by $\angle$3.

Proof :—$\angle$AOC+$\angle$COB=$\angle$1+$\angle$2+$\angle$3.
Also $\angle$AOD+$\angle$DOB=$\angle$1+$\angle$2+$\angle$3.
$\therefore$ $\angle$AOC+$\angle$COB=$\angle$AOD+$\angle$DOB.
=2rt. angles.

Q. E. D.

Another Proof :—$\because$ $\angle$AOC=$\angle$AOD—$\angle$COD and $\angle$COB= $\angle$COD+ $\angle$DOB
$\therefore$ $\angle$AOC+$\angle$COB=$\angle$AOD+ $\angle$DOB
= 2 rt. $\angle$s.

Q. E. D.

Cor. 1.—If two straight lines cut each other, the four angles so formed are together equal to four right angles.

Cor. 2.—The sum of all the consecutive angles about a point in a plane on the same side of a straight line passing through the point is equal to two right angles.

Cor. 3. If any number of straight lines meet at a point the sum of all the angles between successive lines is equal to four right angles.

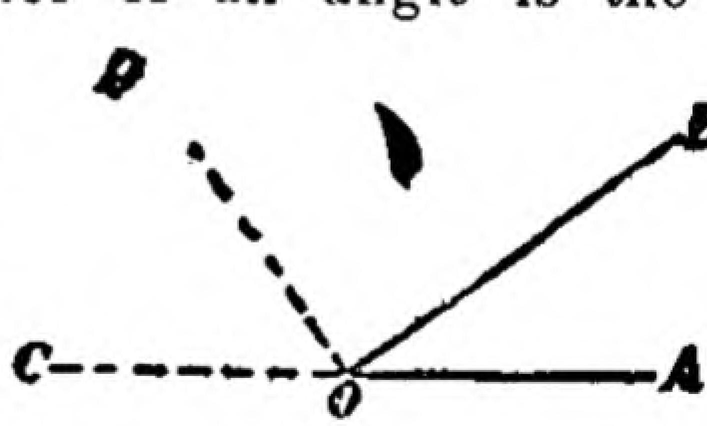
Def.—Two angles (such as ∠COA and ∠COB in the figure) whose sum is equal to two right angles, are called **Supplementary angles** and each angle is said to be the **Supplement** of the other.

Two angles (such as COA and COD in the figure) whose sum is equal to one right angle, are called **complementary angles** and each angle is called the **complement** of the other.

Cor. 4.—Supplements of the same angle or equal angles are equal to each other.

Cor. 5.—Complements of the same angle or equal angles are equal to each other.

Def.—The external bisector of an angle is the bisector of the supplement of that angle. AOB is an angle ∠BOC is its supplement. OD the bisector of angle, BOC is called the *external bisector of the angle* BOA.



Exercises.

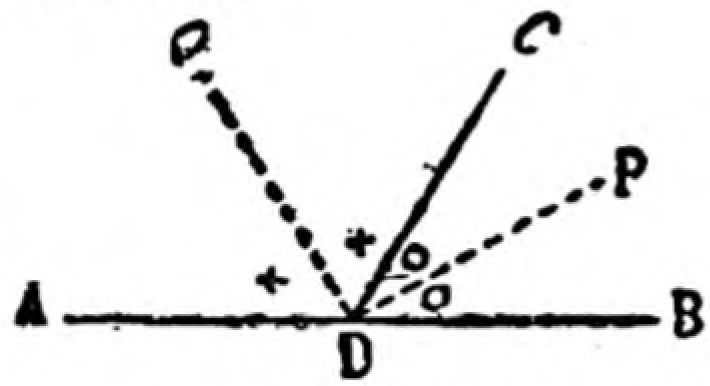
1. Give the complements of :—
   15°, 67½°, 30°, 60°, 45° and 75°.
2. Give the supplements of :—
   135°, 150°, 120°, 45°, 60°, 90° and 30°.

3. Name the angle which is equal to (*i*) its complement, (*ii*) its supplement.

4. What angle is one third of its supplement ?

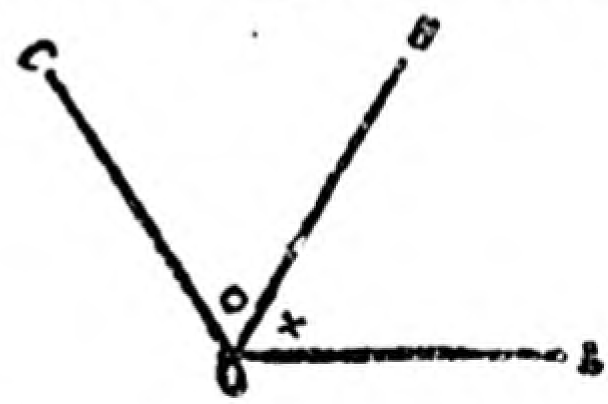
5. If two straight lines intersect each other and if one of the angles so formed is a right angle all the others are also right angles.

6. *The internal and external bisectors of an angle are at right angles to each other.*



7. In the figure of the above proposition prove that $\angle COD = \frac{1}{2}(COB - \angle COA)$.

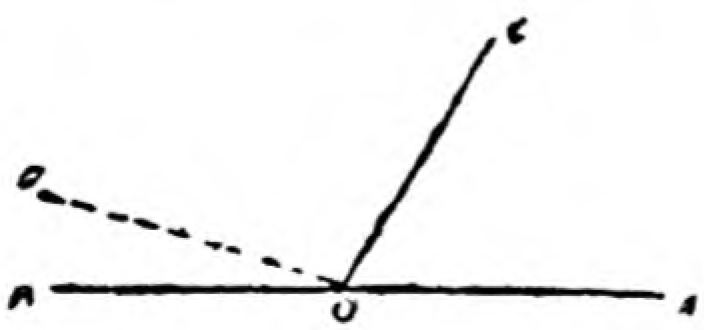
Def. Two angles are said to be **Adjacent** when they lie on opposite sides of a common arm, e.g., angles AOB, BOC.



Proposition 2. (*Theorem*).

(*Converse of Prop. 1*).

If the sum of two adjacent angles is equal to two right angles, the external arms of the angles are in the same straight line.



Given :—$\angle AOC + \angle COB = 2$ rt. angles.

Required.—To prove that AOB is a st line.

Construction.—If AOB is not a st. line produce AO to D.

Proof: ∴ AOD is a st. line and OC meets it in O. (Sup.)

∴ $\angle AOC + \angle COD = 2$ rt. angles.

But $\angle AOC + \angle COB = 2$ rt. angles. (Given)

∴ $\angle AOC + \angle COD = \angle AOC + \angle COB$

∴ $\angle COD = \angle COB$.

But this is possible only when OB and **OD** coincide.

Hence OA and OB are in the same st. line.

Q. E. D.

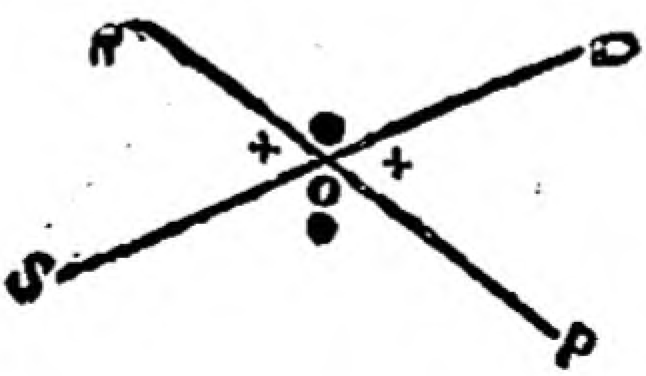
Note.—The method of proof used in the converse proposition above is called the **indirect method** or the **Reductio ad absurdum Method,** which means reducing to absurdity or something impossible.

In this method we suppose, for the purpose of reasoning, that the stated conclusion is untrue, and then prove and then arrive at an impossible conclusion and, consequently, infer the stated conclusion to be true.

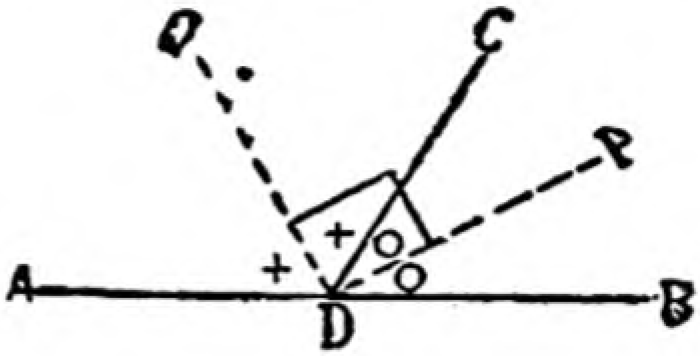
Exercises.

1. Four straight lines OA, OB, OC and OD meet at a point and make angles AOB, BOC, COD, equal to 50°, 60° and 70° respectively. Prove that OA and OD are in the same st. line.

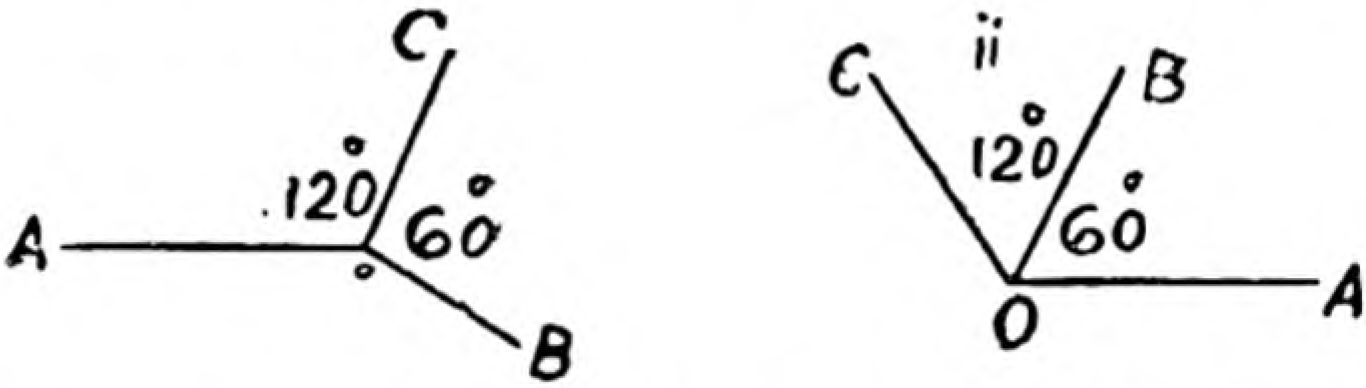
2. Through a point O four straight lines OP, OQ, OR and OS are drawn *in order*, such that $\angle POQ = \angle ROS$ and $\angle QOR = \angle SOP$. Prove that POR and SOQ are straight lines.



3. If the bisectors of two adjacent angles be at right angles, the exterior arms of the angles form a st. line.

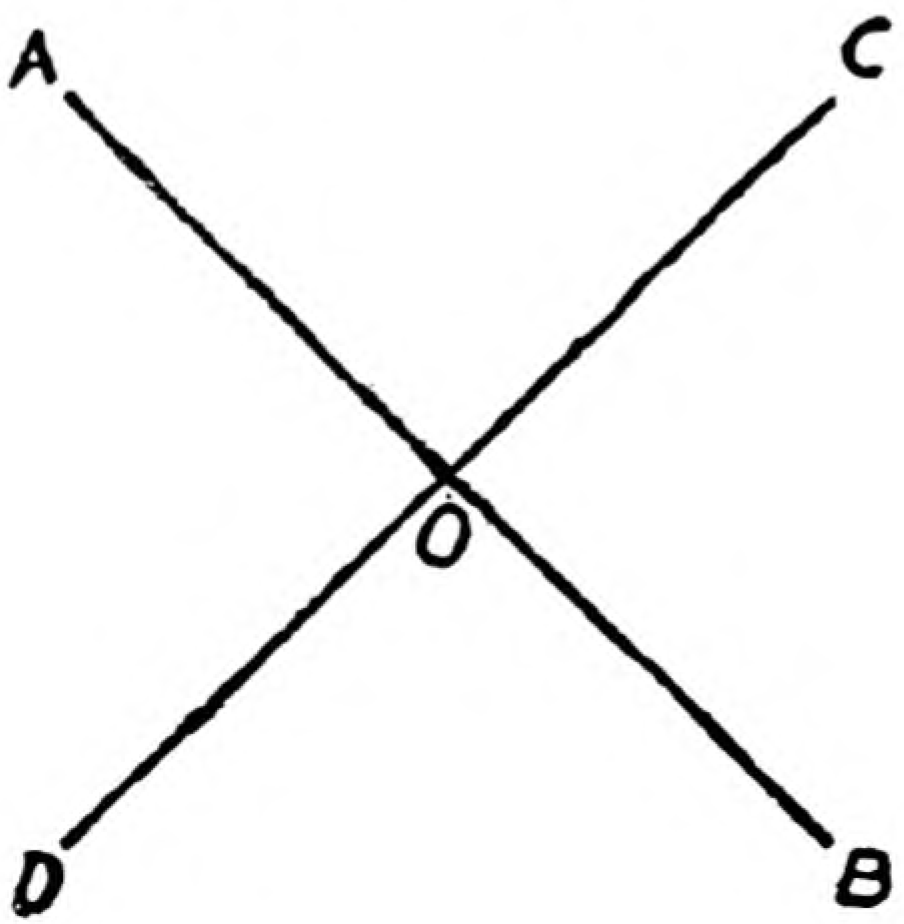


4. What absurdity do you find in the following diagram, and why ?



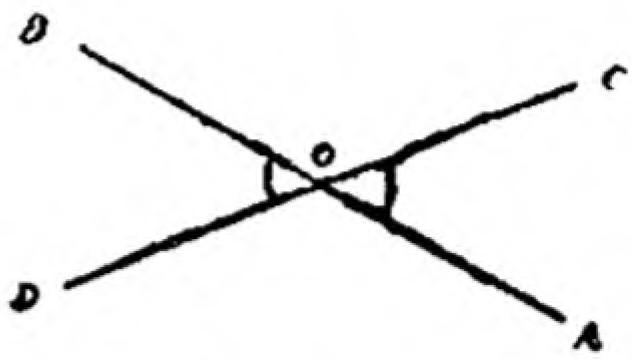
Def.—The opposite angles made by two intersecting st. lines are called **vertically opposite angles**.

In the fig. angles AOC, BOD are vertically opposite, and so are angles COB, AOD.



Proposition 3. (*Theorem*).

If two straight lines intersect, the vertically opposite angles are equal.



Given : -AB and CD two st. lines intersecting at O.

Required :—To prove that $\angle AOC = \angle BOD$ and $\angle COB = \angle DOA$.

Proof :—$\because$ OC stands on the st. line AOB.
$\therefore$ $\angle AOC + \angle COB = 2$ rt. angles.
Also $\because$ OB stands on the st. line COD.
$\therefore$ $\angle COB + \angle BOD = 2$ rt. angles.
$\therefore$ $\angle AOC + \angle COB = \angle COB + \angle BOD$.
Hence $\angle AOC = \angle BOD$.
Similarly $\angle COB = \angle DOA$.

Q. E. D.

Exercises.

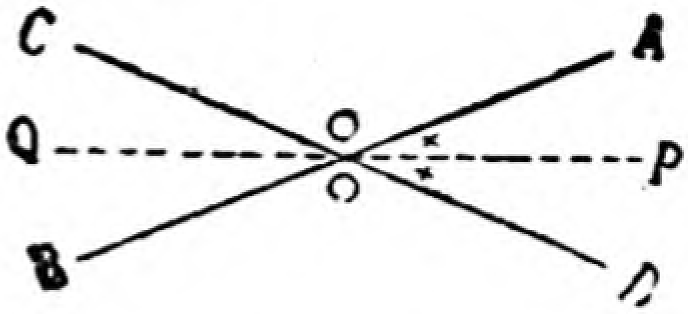
1. If in the figure of the proposition, $\angle AOD$ is 135°, find the other three angles.

2. If the sum of the angles AOC and BOD is equal to 60°, calculate the other angles.

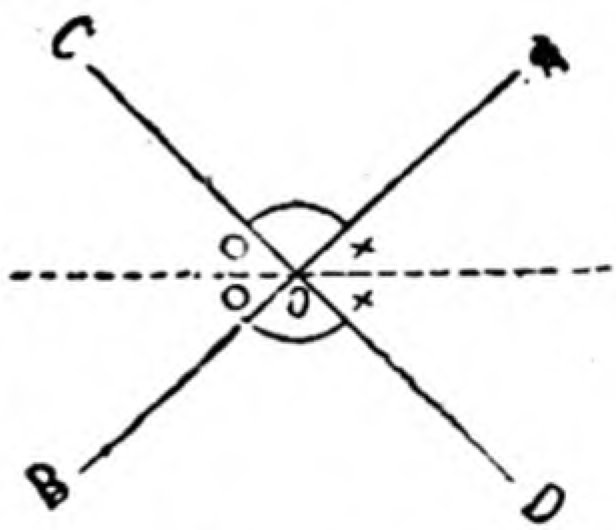
3. If the sum of $\angle AOC$, $\angle AOD$ and $\angle BOD$ is oqual to $202\frac{1}{2}°$, find the size of each of the angles at O.

4. The bisector of an angle also bisects the

vertically opposite angle.

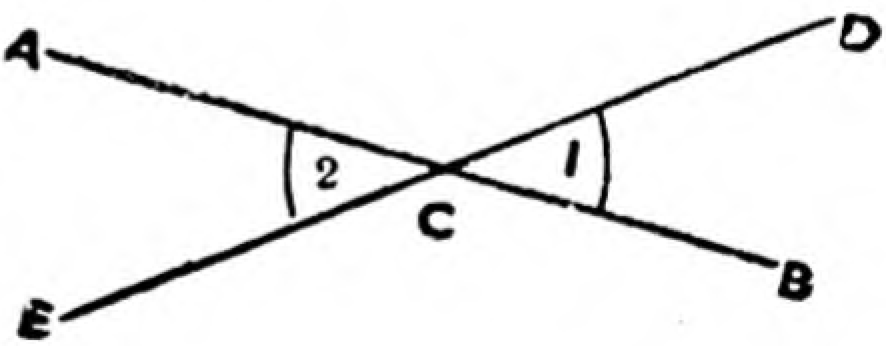


5. *The bisectors of the vertically opposite angles are in the same st. line.* (Punjab, 1914)



Converse of Prop. 3.

If from any point in a given st. line two straight lines be drawn on opposite sides of it making equal angles with the two parts of the given line, then these two st. lines are in one and the same st. line.



Given. C is a point in a given line AB, CD and CE are drawn making $\angle 1 = \angle 2$.

To prove. CD and CE are in the same st. line.

Proof. $\angle 1 = \angle 2$.

Add $\angle$ACD to each

$\therefore \angle 1 + \angle ACD = \angle 2 + \angle ACD$.

But $\angle 1 + \angle ACD = 2$ rt.$\angle$s. (DC stands on AB)

$\therefore \angle 2 + \angle ACD = 2$ rt. angles.

Hence DC and CE are in one and the same st. line.

Q. E. D.

Method of proof. The whole proof amounts to the following statement :—

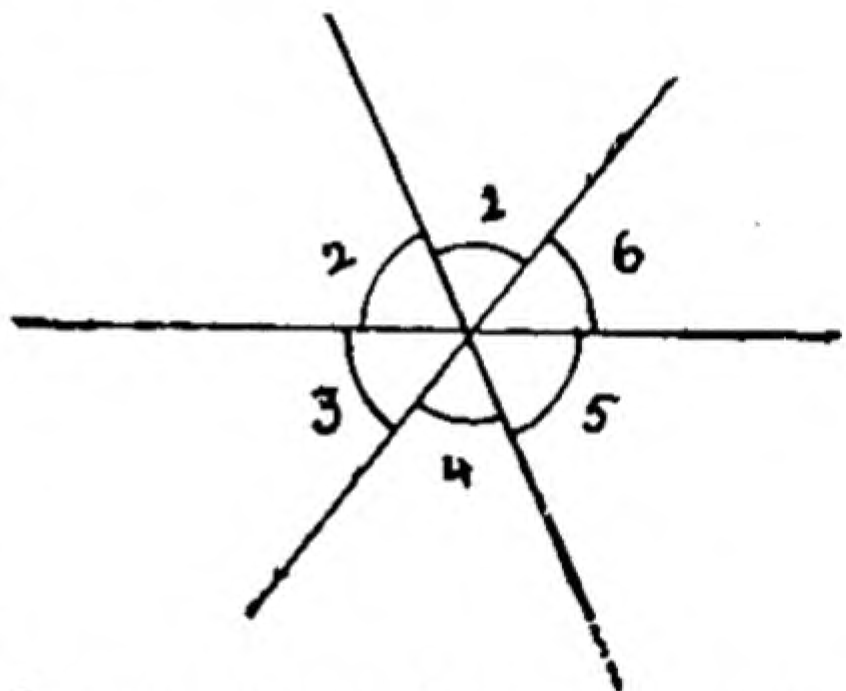
$\angle 1 = \angle 3$, because each is the supplement of the same $\angle 2$.

Application. The Prop. states an important fact applicable in establishing equality of angles while its converse gives a method of proving two straight lines in one and the same st. line.

Exercises.

1. In the accompanying figure, find (*i*) $\angle$s 2, 4, 5 and 6 when $\angle 1 = 30°$ and $\angle 3 = 60°$.

(*ii*) $\angle 1 + \angle 2 = 120°$ and $\angle 6 + \angle 5 = 150°$.



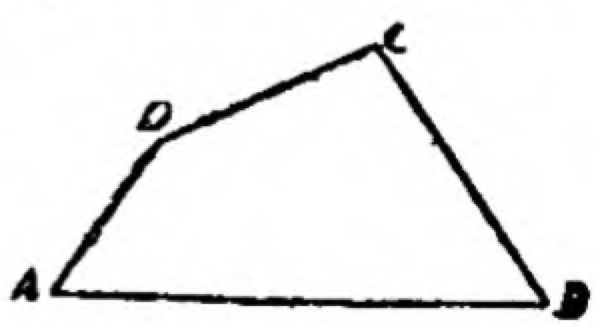
2. If $\angle 6 + \angle 5 = \angle 6 + \angle 1$ in the accompanying figure, prove that $\angle 2 = \angle 4$.

RECTILINEAL FIGURES.

Definitions.

A figure bounded by straight lines is called a *rectilineal figure.*

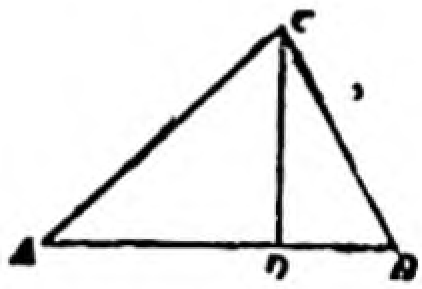
If a rectilineal figure lies wholly in one and the same plane, it is called a **rectilineal plane figure.** For instance ABCD is a figure bounded by four straight lines AB, BC, CD and DA, and lies wholly in the plane of the paper It is, therefore, a rectilineal plane figure.



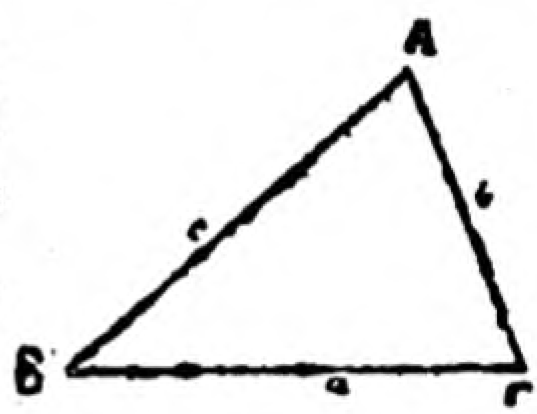
A plane figure bounded by three straight lines is called a **Triangle**.

The three straight lines which go to make a triangle are called the **sides** of the triangle and the points where these sides meet are called the **vertices** of the triangle.

In the figure ABC is a triagle. AB, BC and CA are its sides ; A, B and C, are its vertices.



In any triangle ABC, the capital letters, A, B, C are often used to denote the magnitude of the $\angle$s at A, B, C, and the letters, a, b, c to denote the lengths of the sides opposite to the $\angle$s A, B, C.

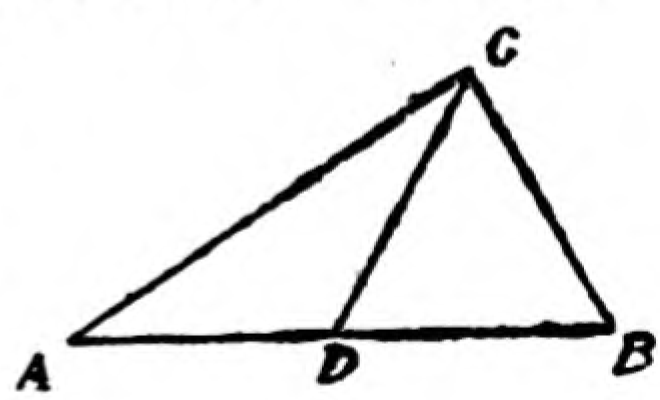


Any side of a triangle being taken as its *base*, the perpendicular upon it from the opposite vertex is called its **altitude** or **height**.

In the figure let AB be called the base, then C is called the vertex, CD the altitude and angle AC the **Vertical angle**.

Angles, at A, B and C are called the three angles of the triangle. The three sides and the three angles taken together are called the six **elements**, of a triangle.

A st. line joining a vertex with the middle point of the opposite side of a triangle is called a **median**. In the figure D is the middle point of AB and CD is the median.



Every triangle has three medians.

Kinds of triangles *with respect to side :—*

An **equilateral triangle** is a triangle having three equal sides.

An **isosceles triangle** is a triangle having two equal sides.

A scalene triangle is a triangle having no two of its sides equal.

Kinds of triangles *with respect to angles :—*

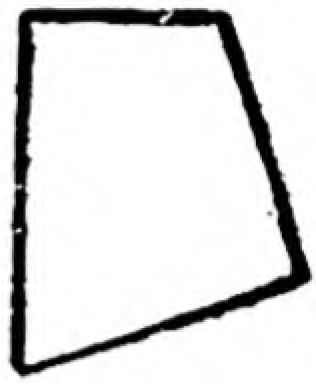
If all the angles of a triangle are equal, it is called an **equiangular triangle**.

An **obtuse-angled triangle** is a triangle having one of its angles obtuse.

An **acute-angled-triangle** is a triangle which has three angles acute.

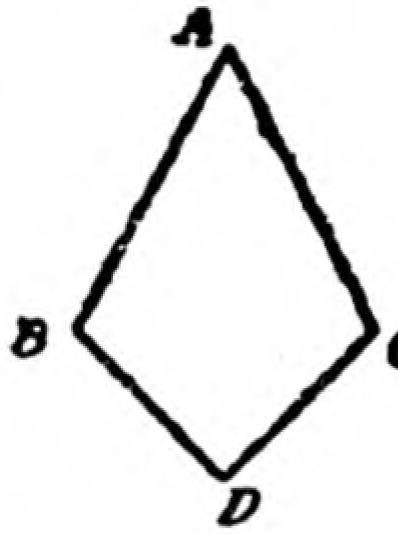
A **right-angled triangle** is a triangle having one of its angles a right angle. The sides opposite to the right angle is called the **hypotenuse**.

A plane figure bounded by four sides is callled a **quadrilateral**.



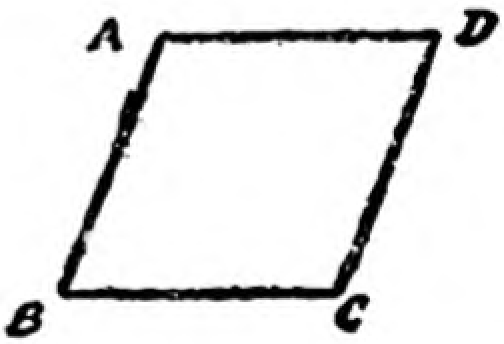
A quadrilateral which has two pairs of equal adjacent sides is called a **kite**.

In the figure, AB=AC and BD=CD. ABCD is a kite.



A quadrilateral whose sides are all equal but whose angles are *not* right angles is called a **Rhombus**.

Thus, in the figure, AB=BC =CD=DA, but the angles are not right angles. ABCD is a rhombus.



Two figures which can be made to coincide (fit exactly) by superposition (placing one figure over the other) are said to be **Congruent** or **identically equal.**

Two figures which are congruent are **equal in all respects**, *i.e.*, they are equal in area and have all sides equal and all angles equal, each to each.

In two congruent triangles the sides opposite to the equal angles are called **Corresponding sides**, and the angles opposite to the equal sides are called the **Corresponding angles**.

Proposition 4. *(Theorem)*

If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, the triangles are congruent.

Given :—Triangles ABC and DEF, having AB=DE, AC=DF, $\angle A = \angle D$.

Required :—To prove that the two triangles are congruent.

Proof :—Place $\triangle$ABC on $\triangle$DEF so that AB coincides with its equal DE, A falling on D and B on E.

Now $\because$ $\angle A = \angle D$ $\therefore$ AC falls along DF

and $\because$ AC=DF $\therefore$ C falls on F.

$\therefore$ the side BC must coincide with the side EF :

$\therefore$ $\triangle$ABC coincides with $\triangle$DEF.

$\therefore$ $\triangle ABC = \triangle DEF$. **Q.E.D.**

Exercises.

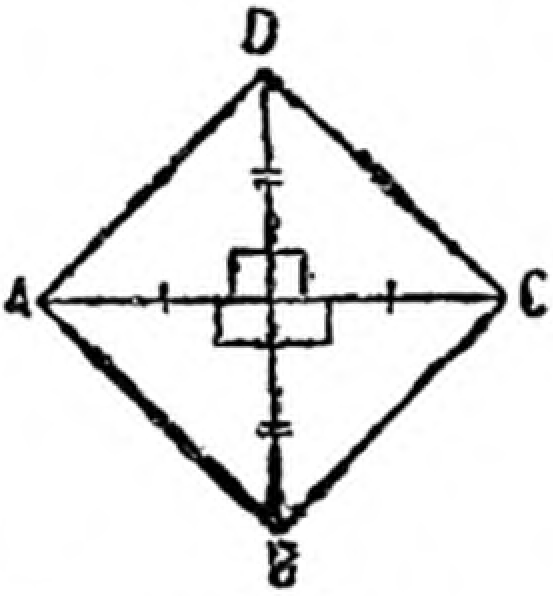
1. The bisector of the vertical angle of an isosceles triangle bisects the base at right angles.

2. Every point on the perpendicular bisector of a straight line is equidistant from the extremities of the line.

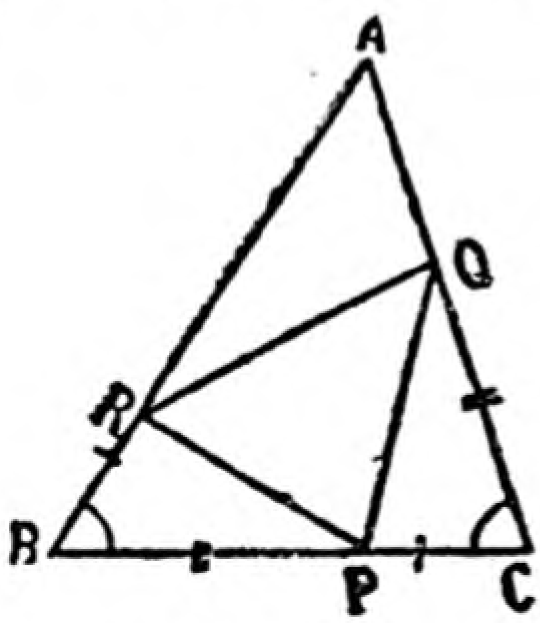
3. From two lines AM and AF inclined at any angle cut off AB=AC and AD=AE ; join BE and CD. Prove that the triangles ABE, ACF are equal.

4. ABC is an isosceles triangle, D and E respectively are the middle points of the equal sides AB, AC ; prove that the triangles ABE and ACD are congruent.

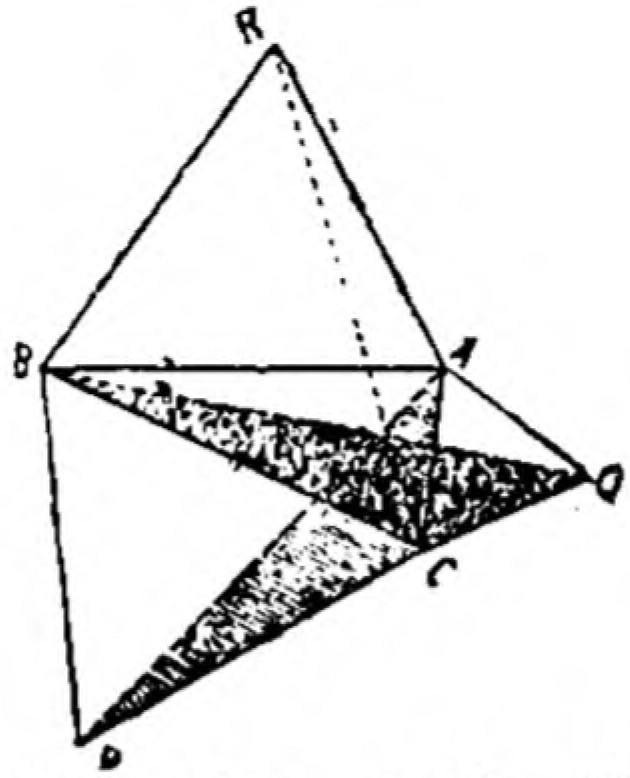
5. If the diagonals of a quadrilateral bisect each other at right angles, show that the quadrilateral is either a rhombus or a square. (Punjab, 1907, 1914).



6. ABC is an equilateral triangle ; P, Q, R, are points in BC, CA, AB respectively, so that BP=CQ=AR. Show that the triangle PQR is equilateral. (Bombay)



7. If BCP, CAQ, ABR be equilateral triangles described externally on the sides of a triangle ABC, show that AP=BQ=CR. (Bombay, 1916).



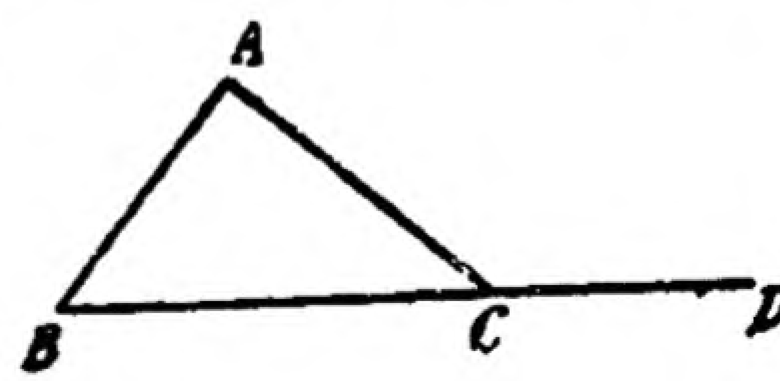
Hint.—$\angle$BCQ=ACP (each=60°+$\angle$ACB); CQ=AC, BC=PC $\therefore$ BQ=AP, similarly AP=CR, etc.

8. If the sides BC, CA, AB of the equilateral triangle ABC are produced to P, Q, R respectively so that CP=CB, AQ=AC and BR=BA; prove that PQR is an equilateral triangle.

9. QA, OB, OC are three radii of a circle equally inclined to one another, show that ABC is an equilateral triangle.

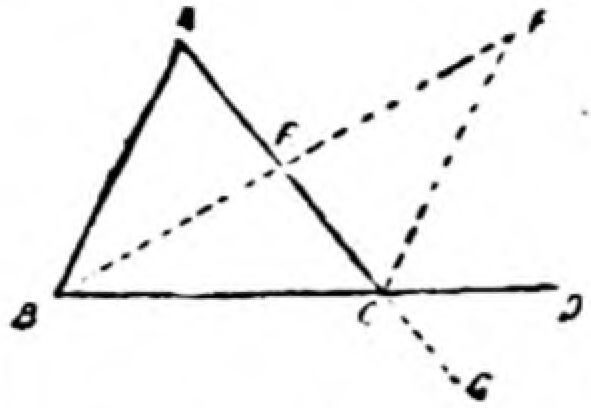
Hint.—$\triangle$s AOB, BOC, and AOC are congruent. Hence AB=BC=AC.

Def. If in the triangle ABC, the side BC is produced to D, the angle ACD so formed is called an **exterior angle** of the triangle.



Proposition 5. (*Theorem*)

If one side of a triangle be produced, the exterior angle so formed is greater than either of the interior opposite angles.



Given :—ABC a triangle having the side BC produced to D.

Required :—To prove that $\angle ACD > \angle BAC$ or $\angle ABC$.

Construction :—Suppose E is the middle point of AC.

Join BE.

Produce BE to F so that EF=BE.

Join FC.

Proof :—In the $\triangle$s, AEB and CEF,

$$\begin{cases} BE = EF & \dots \text{(Const.)} \\ AE = EC & \dots \text{(Const.)} \\ \angle AEB = \angle CEF & \dots \text{(Vert. opp. } \angle\text{s).} \end{cases}$$

$\therefore$ The triangles are congruent.

Hence $\angle A = \angle ECF$.

But $\angle ACD >$ its part $\angle ECF$.

$\therefore$ $\angle ACD > \angle A$.

Again, produce AC to G, and bisecting BC, it can similarly be proved that $\angle BCG > \angle CBA$.

But $\angle BCG = \angle ACD$ (Vert. Opp. $\angle$s).

$\therefore$ $\angle ACD > \angle B$. **Q. E. D.**

Cor. 1.—Any two angles of a triangle are together less than two right angles.

Exercises.

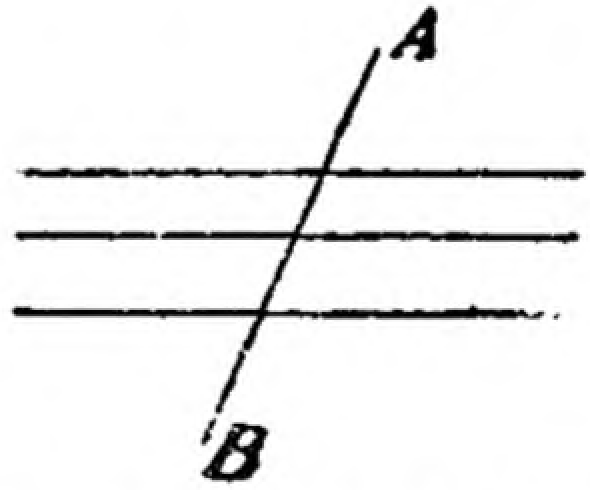
1. P is a point within the triangle LMN and PM and PN are joined ; prove that $\angle MPN > \angle MLN$.

2. In a quadrilateral ABCD, the sides AB and AD are produced to E and F. Prove that the sum of the exterior angles so formed is greater than $\angle BAD$.

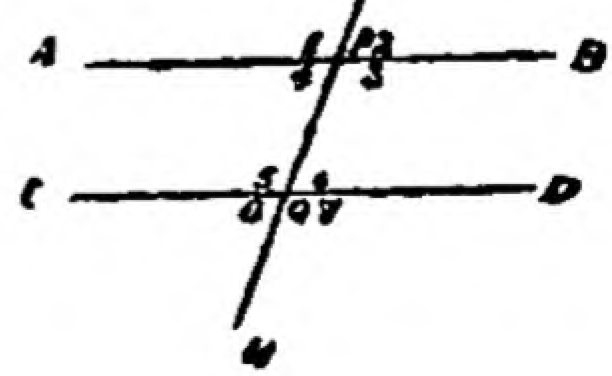
Parallel Straight Lines.

Def.—Two straight lines which are in the same plane, and do not meet however far they may be produced both ways are said to be **parallel**, as AB and CD.

A straight line which cuts a number of other lines is called a **Transversal** as AB in the figures below.



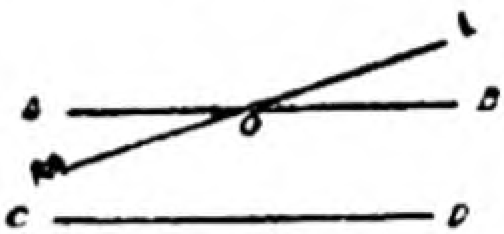
AB and CD are two straight lines and LM a transversal cuts them at P and Q. It makes with the lines *eight angles* marked, 1, 2, 3, 4, 5, 6, 7, and 8 in the figure of which 1, 2, 7, and 8 are called the **exterior angles** and 3, 4, 5, and 6 are called the **interior angles.**



A pair of non adjacent interior angles on opposite sides of LM such as 4 and 6, also 3 and 5, are called **alternate angles**.

A pair of non-adjacent angles on the same side of LM, one interior and the other exterior such as 2 and 6, 1 and 5, 4 and 8, 7 and 3 are called **corresponding angles**.

Playfair's axiom—*Two intersecting straight lines cannot both be parallel to the same straight line*, e.g., AB and LM intersecting at O cannot both be parallel to CD.



Note.—Parallel straight lines have the **same** direction, whereas intersecting lines have **different** directions.

Note 2.—Through a given point only *one straight line* can be drawn parallel to another straight line.

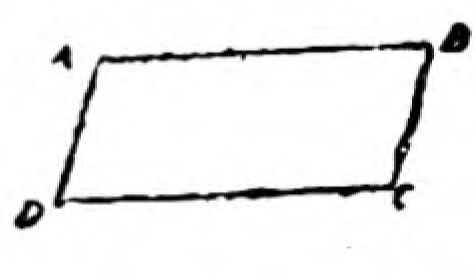
Def. A quadrilateral two of whose sides are parallel is called a **trapezium**.

In the figure, AB is parallel to DC.

If *the non-parallel sides of a* trapezium are equal, it is called an **isosceles trapezium**.

A quadrilateral whose opposite sides are parallel is called **parallelogram**. Opp. fig. ABCD is a ||^{m}.

In the figure AB is parallel to DC, and AD is parallel to BC.



A *parallelogram* which has one of its angles a right angle is called a **rectangle.**

A *rectangle* which has a pair of adjacent sides equal is called a **square.**

A plane figure bounded by more than four st. lines is called a **polygon.**

A polygon of	five	sides	is called	a	**Pentagon**
”	six	”	”	”	**Hexagon**
”	seven	”	”	”	**Heptagon**
”	eight	”	”	”	**Octagon**
”	nine	”	”	”	**Nonagon**
”	ten	”	”	”	**Decagon**
”	twelve	”	”	”	**Do-decagon**
”	n sides	”	”	”	**N-sided polygon** or **n-gon.**

A polygon is said to be **equiangular** when all its *angles* are equal.

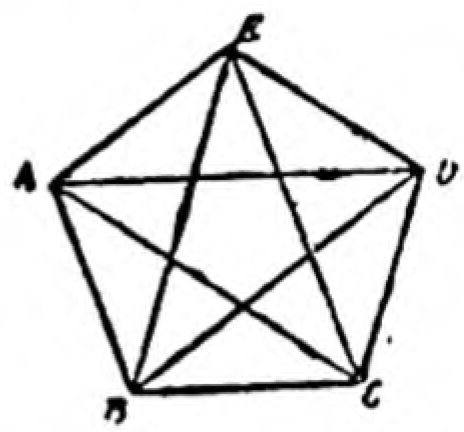
A polygon is said to be **equilateral** when all its *sides* are equal.

A polygon is said to be **regular** when *all* its sides are equal and *all* its angles are equal.

Two rectilineal figures are said to be **equiangular** when the angles of the one *taken in order* are equal to the angles of the other *taken in order*.

A straight line joining any two *non-adjacent* vertices of a rectilineal figure is called a **diagonal**.

AD, AC, BD, BE and EC are all diagonals of the pentagon ABCDE.

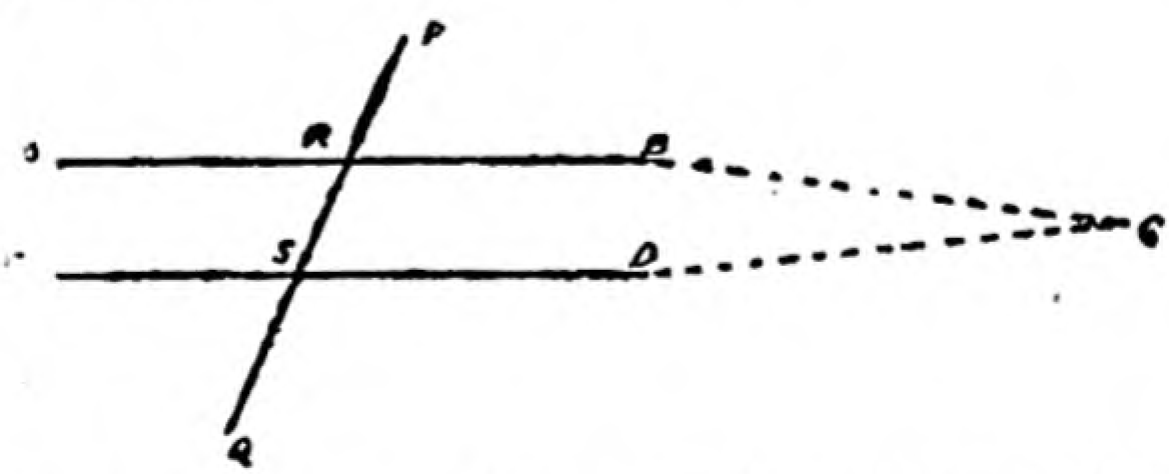


The whole length of the boundary of a plane figuer is called its **perimeter**.

Proposition 6 (*Theorem*)

Part I

If a straight line cuts two other straight lines, so as to make a pair of alternate angles equal, then the two straight lines are parallel



Given :—PQ, a st. line cutting AB, CD in R and S respectively and making ∠ARS=alt. ∠RSD.

Required :—To prove that AB ‖ CD.

Construction :—If AB is not ‖ to CD, let AB and CD when produced beyond B and D meet at G.

Proof :—RBG is a st. line (sup).

Similarly SDG is a st. line.

∴ RSG is a △

In the △RSG, the exterior ∠ARS must be greater than the interior opp. ∠RSD.

But this is contrary to what is given (for ∠ARS = ∠RSD)

∴ AB and CD when produced do not meet towards B and D.

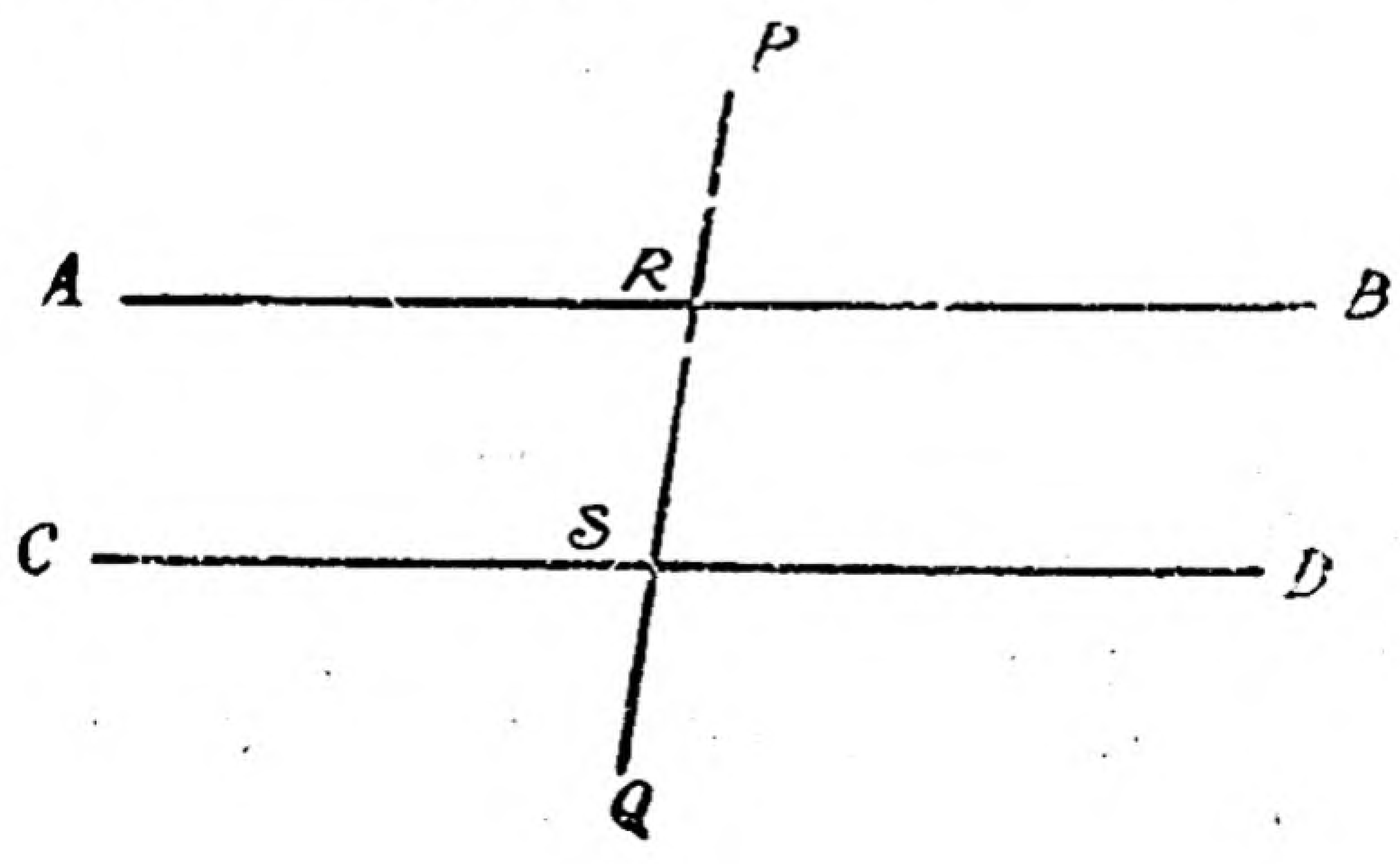
In like manner, it can be proved that they do not meet if produced towards A and C.

∴ AB and CD do not meet when produced both ways. ∴ AB ‖ CD. **Q. E. D.**

Proposition 6. Theorem. (*Contd.*)

Part II.

If a straight line cuts two other straight lines, so as to make a pair of corresponding angles equal, then the two straight lines are parallel.



Given :—PQ, a st. line cutting AB and CD in R and S making ∠PRB = corresp. ∠RSD.

Required :—To prove that AB ∥ CD.

Proof — ∠PRB = ∠RSD. (Given)

Also ∠PRB = vert. opp. ∠ARS.

∴ ∠ARS = RSD.

But these are alternate angles.

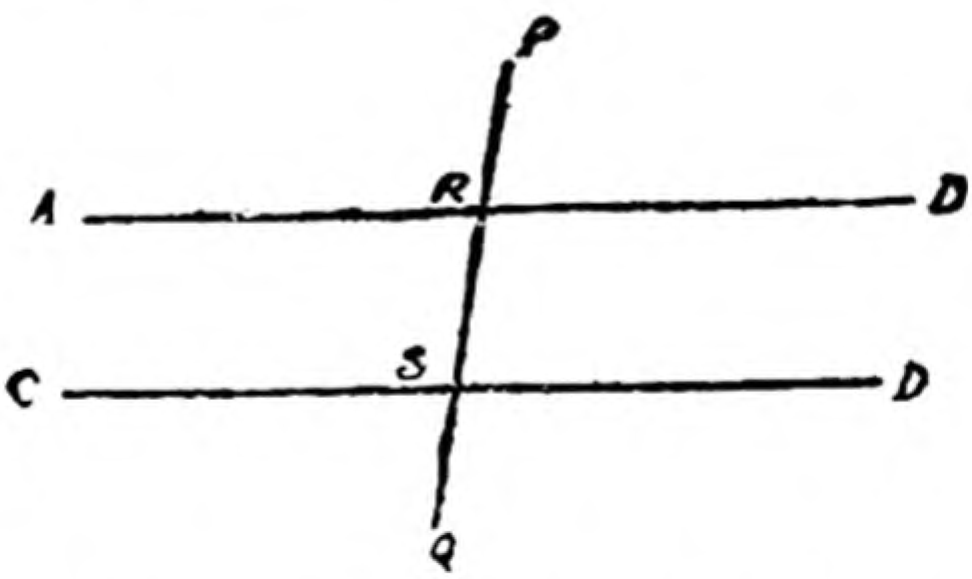
∴ AB ∥ CD.

Q. E. D.

Proposition 6. Theorem (*Contd.*)

Part III.

If a straight line cuts two other straight lines, so as to make a pair of interior angles on the same side of the cutting line together equal to two right angles, then the two straight lines are parallel.



Given :—PQ, a st. line cutting AB and CD in R and S, so that.

∠BRS + ∠RSD = 2 rt. angles.

Required :—To prove that AB ∥ CD.

Proof :—∠BRS + ∠ARS = 2 rt. angles, (SR stands on AB).

Also $\angle BRS+\angle RSD=2$ rt. angles. (given)

$\therefore\ \angle BRS+\angle ARS=\angle BRS+\angle RSD$

$\therefore\ \angle ARS=\angle RSD.$

But these are alternate angles.

$\therefore\ AB \parallel CD.$

Q. E. D.

Exercises.

1. *Two st. lines at right angles to the same straight line are parallel.*

(Calcutta.)

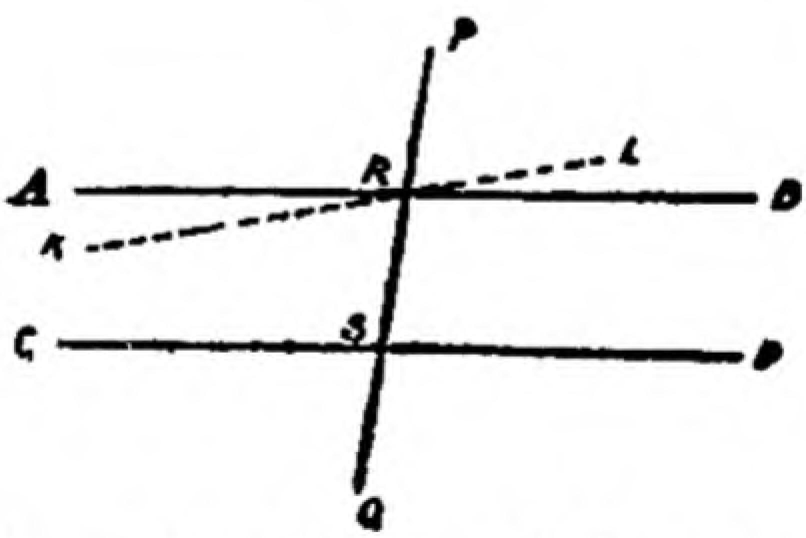
2. If in the figure of the proposition $\angle PRB$ is equal to $\angle CSQ$, prove that AB and CD are parallel st. lines.

3. If XY and PR are two diameters of a circle and XR and PY are joined, prove that these lines are parallel.

Proposition 7. (*Theorem*).

Part I.

If a straight line cuts two parallel lines, the alternate angles are equal.



Given :—PQ cutting the parallel st. lines AB and CD at R and S.

Required :—To prove that $\angle ARS=\angle RSD$.

Construction :—If $\angle ARS$ is not $=\angle RSD$, suppose a st. line KRL is drawn so that $\angle KRS=\angle RSD$.

Proof :—∵ ∠KRS=alt. ∠RSD. (sup.)

∴ KRL ∥ CD.

But ARB ∥ CD (Given).

∴ The two st. lines KL and AB which intersect at R are both ∥ to CD.

But this is impossible. (Playfair's axiom).

∴ ∠ARS is not unequal to ∠RSD.

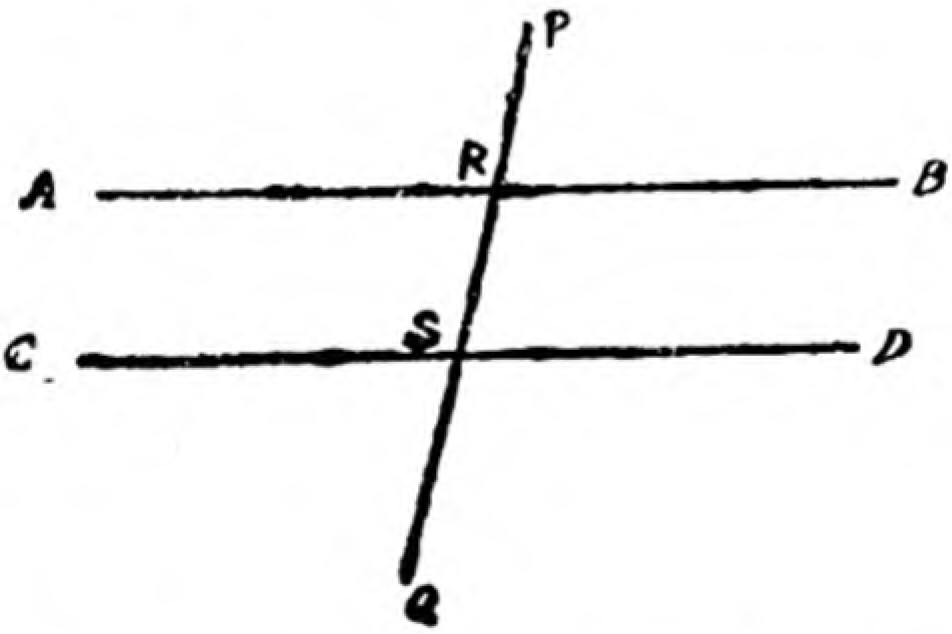
∴ ∠ARS=∠RSD.

Q. E. D.

Proposition 7. Theorem (*Contd.*)

Part II.

If a straight line cuts two parallel lines; the corresponding angles are equal.



Given :—PQ cutting the parallel st. lines AB and CD at R and S.

Required :—To prove that ∠PRB=corresp. ∠RSD

Proof :—∵ AB ∥ CD and PQ cuts them.

∴ ∠ARS=alt. ∠RSD.

Also ∠ARS=vert. opp. ∠PRB.

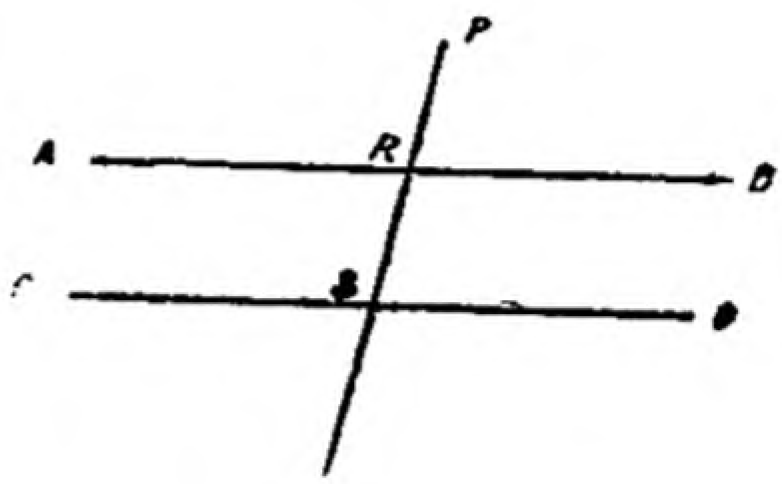
∴ ∠PRB=∠RSD.

Q.E.D.

Proposition 7. Theorem (*Contd.*)

Part III.

If a straight line cuts two parallel lines, the interior angles on the same side of the cutting line are together equal to two rt. angles.



Given :—PQ cutting the parallel st. lines AB and CD at R and S.

Required :—To prove that $\angle BRS + \angle RSD = 2$ rt. angles.

Proof :—$\angle BRS + \angle ARS = 2$ rt. angles ($\because$ SR stands on AB).

But $\angle ARS =$ alt. $\angle RSD$ ($\because$ AB $\parallel$ CD and PQ cuts them)

$\therefore \quad \angle BRS + \angle RSD = 2$ rt. angles.

Q. E. D.

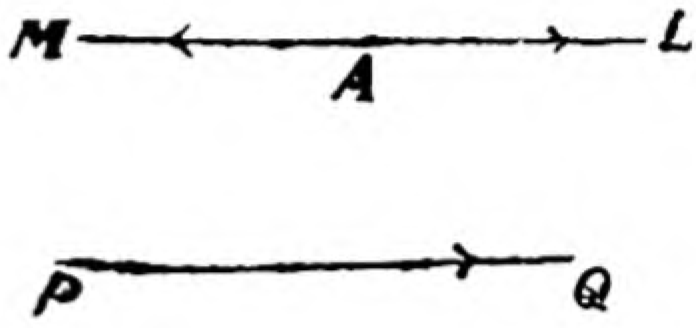
Note 1.—Proposition 7 is the converse of Proposition 6

Note 2.—*The two senses of a straight line.*

If a point O be taken on a straight line XOX′, then lengths measured from O towards X′ are said to be measured in the *opposite* sense to those measured from O towards X.

In the figure, lines OP and OQ are drawn in the same sense but OP and OR are in the opposite senses.

If from a point A a line is to be drawn parallel to a given line PQ, this can be done obviously in two ways; (*i*) AL, (*ii*) AM.



In the first case AL is drawn in the same sense as PQ, but in the second case AM is drawn in the opposite sense.

Generally an arrowhead ⟶ is made on the line to indicate the sense in which it is drawn.

Exercises.

1. Prove that the sum of the interior angles of a trapezium is equal to four right angles.

2. *If one angle of a parallelogram is a right angle, prove that all its angles are right angles.* (Punjab, 1913).

3. *A straight line perpendicular to one of two parallel st. lines is also perpendicular to the other.*

4. If two st. lines AG, GB be parallel to a third st. line CD, then AG, GB are in one and the same st. line.

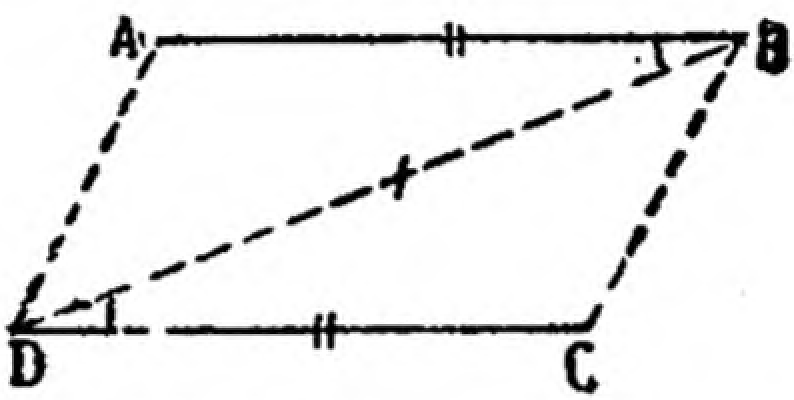
5. If a straight line cuts two parallel st. lines; prove that

(*i*) the bisectors of any pair of alternate angles are parallel;

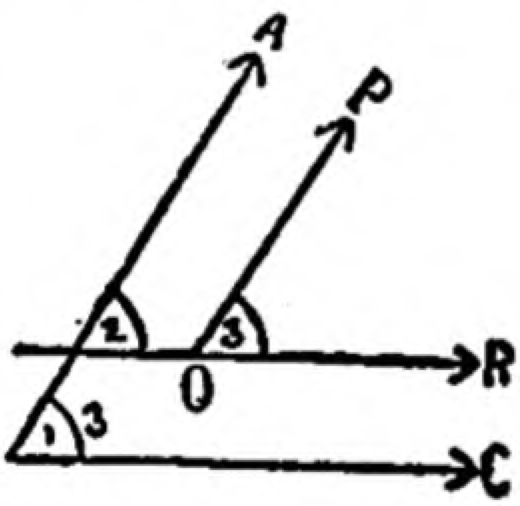
(*ii*) the bisectors of any pair of corresponding angles are parallel.

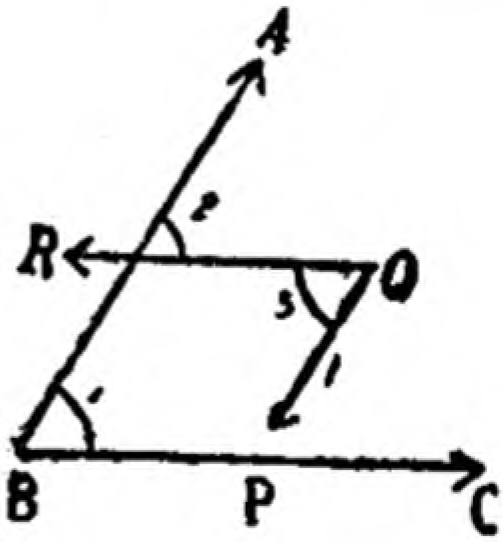
6. *If two st. lines are equal and parallel, the straight lines joining their ends towards the same parts are themselves equal and parallel.*

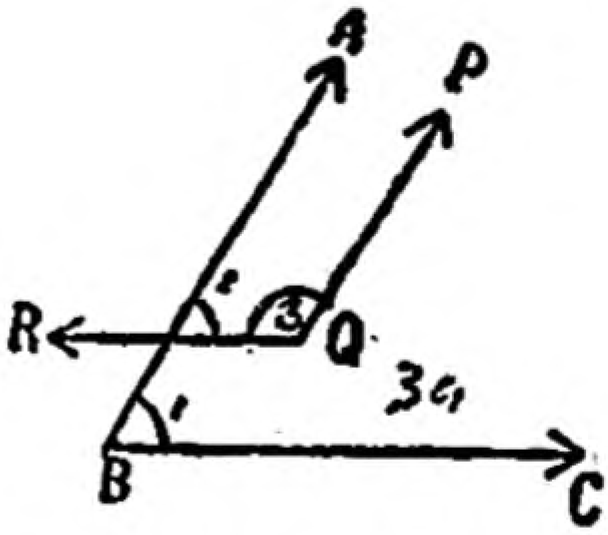
Hint. $\triangle ABD \equiv \triangle BDC$.



7. Two angles which have their arms parallel are either equal or supplementary.



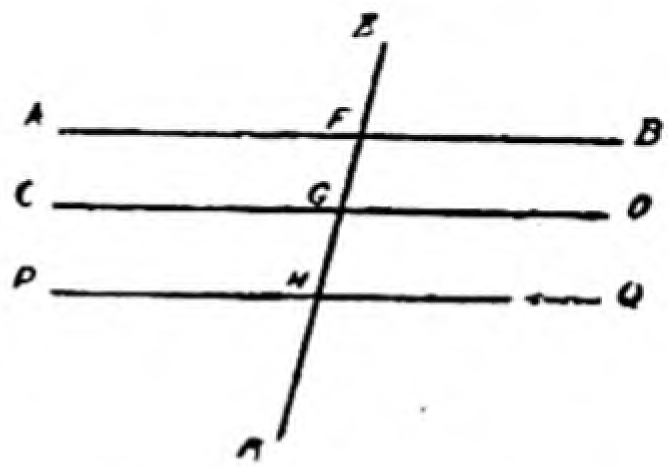




Hint.—$\angle 1 = \angle 2 = \angle 3$ in figs. 1, 2.
$\angle 1 = \angle 2 =$ supp. of $\angle 3$ in fig. 3.

Proposition 8. (*Theorem*).

Straight lines, which are parallel to the same straight line, are parallel to one another.



Given :—AB and CD, each ∥ to PQ.

Required :—To prove that AB ∥ CD.

Construction :—Draw a st. line EK, cutting AB, CD, PQ in F, G and H respectively.

Proof :—∵ AB ∥ PQ.

∴ ∠EFB = corresp. ∠EHQ

also, ∵ CD ∥ PQ.

∴ ∠EGD = corresp. ∠EHQ.

∴ ∠EFB = ∠EGD.

But these are corresponding angles.

∴ AB ∥ CD.

Q. E. D.

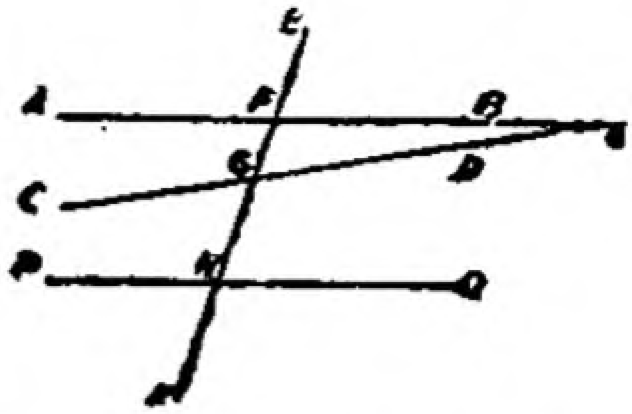
Proof by Playfair's axiom.

Given :—AB and CD, each ∥ to PQ.

Required :—To prove that
AB ∥ CD.

Proof :—If AB is not ∥ to CD.
let them intersect at O.

Then we have two intersecting st. lines ABO, CDO, both parallel to the same st. line PQ, which is impossible. (Playfair's axiom).



∴ AB and CD are parallel.

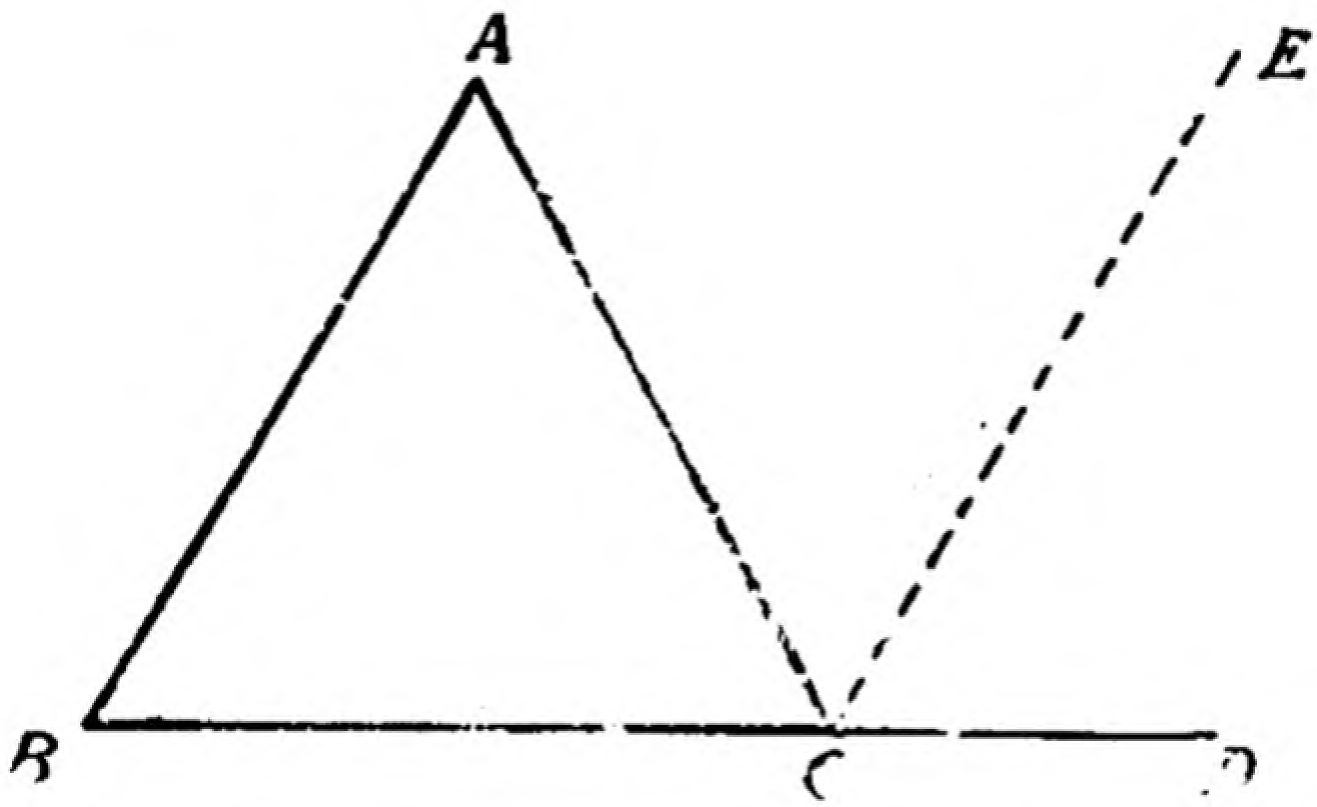
Q. E. D.

Exercises.

1. If a st. line is parallel to one of two parallel st. lines, it is also parallel to the other.

Proposition 9. (*Theorem*).

The sum of the three interior angles of a triangle is equal to two right angles.



Given :—A triangle ABC.

Required :—To prove that $\angle A + \angle B + \angle C = 2$ rt. angles.

Construction :—Produce BC to D. Draw CE ∥ BA.

Proof :—∵ CE ∥ BA and BC cuts them.

∴ $\angle B =$ corresp. $\angle DCE$.

Also ∵ CE ∥ BA and AC cuts them.

$\because \quad \angle A = \text{alt. } \angle ECA.$

Adding, $\angle A + \angle B = \angle DCE + \angle ECA = \angle DCA.$

Add $\angle ACB$ to both sides.

Hence $\angle A + \angle B + \angle C = \angle DCA + \angle ACB$

$= 2$ rt. angles (AC meets BD at C).

Q. E. D.

Cor. 1. If a side of a $\triangle$ is produced, the exterior angle is equal to the sum of the two interior opposite angles.

Cor. 2. If a side of a triangle be produced, the exterior angle so formed is greater than either of the interior opposite angles.

Cor. 3. Any two angles of a triangle are together less than two rt. $\angle$s.

Exercises.

1. If two triangles have two angles of the one respectively equal to two angles of the other, their third angles are equal.

2. If one angle of a triangle is equal to the sum of the other two, that angle is a rt. $\angle$.

3. If any two angles of a triangle are together greater than the third angle, prove that the third angle is acute.

4. The angle formed by the bisectors of any two angles of a triangle is always obtuse.

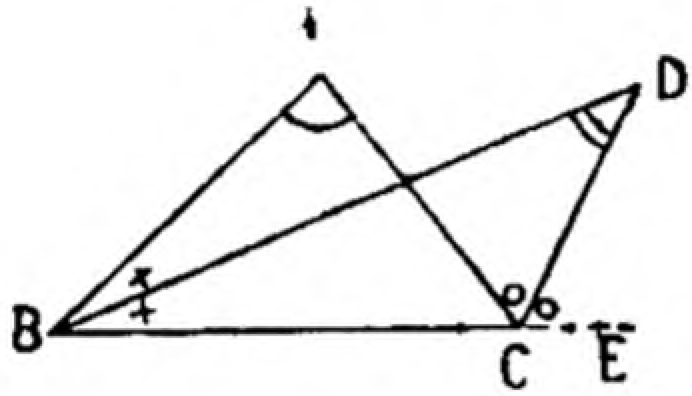
5. *The sum of the angles of a quadrilateral is equal to four rt. angles.* (Bombay, Matric)

Hint.—Draw a diagonal and proceed.

6. *The angle between two st. lines is equal to the angle between any two st. lines at right angles to them.*

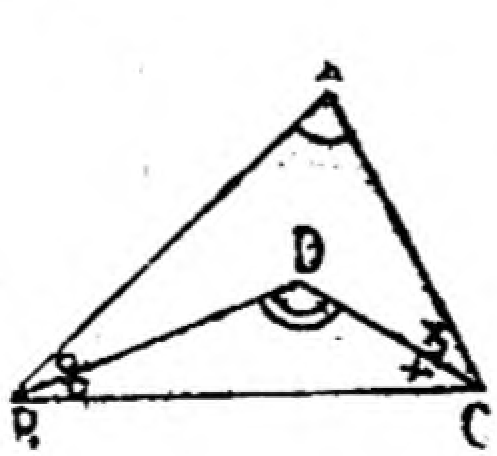
7. The angle between the bisectors of any two consecutive angles of a quadrilateral is equal to half the sum of the remaining two angles of the quadrilateral.

8. The angle formed by the internal bisector of one base angle of a triangle and the external bisector of the other base angle is equal to one half of the vertical angle.



Hint. BD, CD, internal and external bisectors of $\angle$s B and C of $\triangle$ABC meet at D ; BC is produced to E ; then $\angle D = 2\angle A$. $\angle DCE = \angle D + \angle DBC$. $\therefore 2\angle DCE = 2\angle D + 2\angle DBC$ or $\angle B + 2\angle D$ but $\angle ACE = \angle B + \angle A$. $\therefore 2\angle D = \angle A$. $\therefore \angle D = \frac{1}{2}\angle A$.

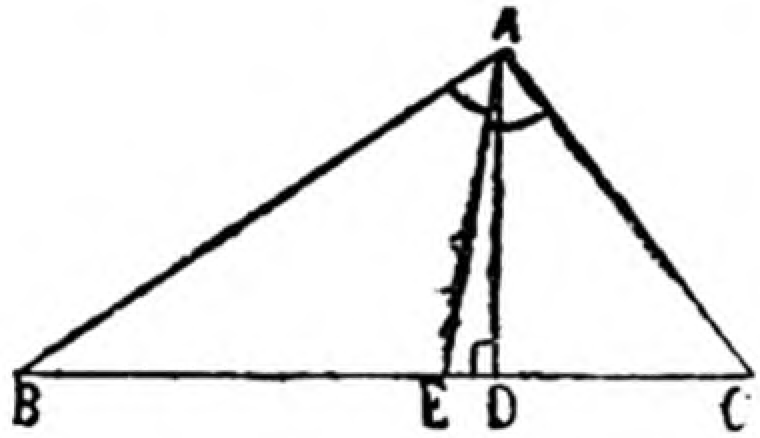
9. **In a triangle ABC the base angles at B and C are bisected by two straight lines meeting at D. Show that** $\angle BDC = 90 + \frac{A}{2}$



10. **In a triangle ABC the sides AB and AC are produced beyond B and C and the exterior angles thus**

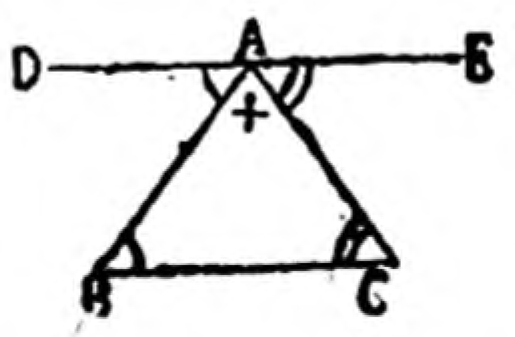
formed are bisected by two straight lines meeting at D. Show that angle $BDC = 90 - \frac{A}{2}$

11. In a triangle ABC, AD is perpendicular to BC, and AE is the bisector of the vertical angle A; prove that angle DAE is half the difference of the base angles of the triangle. (Punjab, 1927)

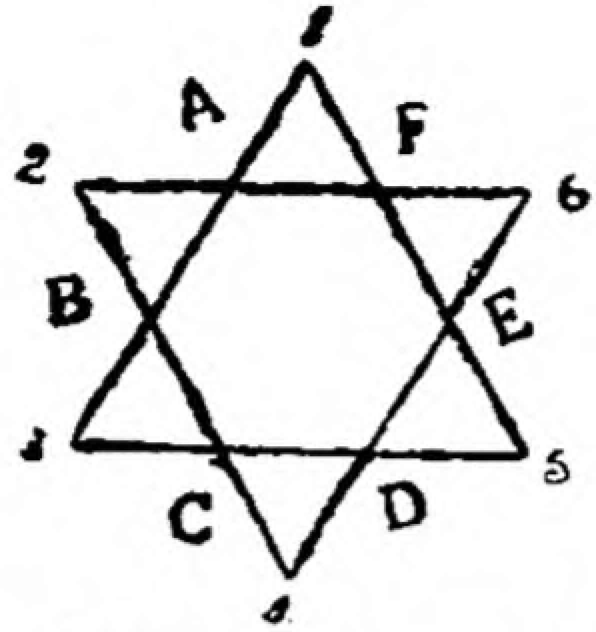


Hint.—$\angle B + \angle BAD = 1$ rt. $\angle = \angle C + \angle CAD$. $\therefore \angle B - \angle C = \angle CAD - \angle BAD = 2\angle DAE$ ($\because \angle BAE = \angle CAE$), etc.

12. By drawing a line through one of the vertices of a triangle parallel to the opposite side without producing any of the sides, prove that the sum of the three interior angles is equal to two right angles.



13. The alternate sides of a hexagon are produced to meet so as to form a star-shaped figure. Show that the sum of the angles at the vertices of the star is equal to four right angles. (Punjab, 1924).

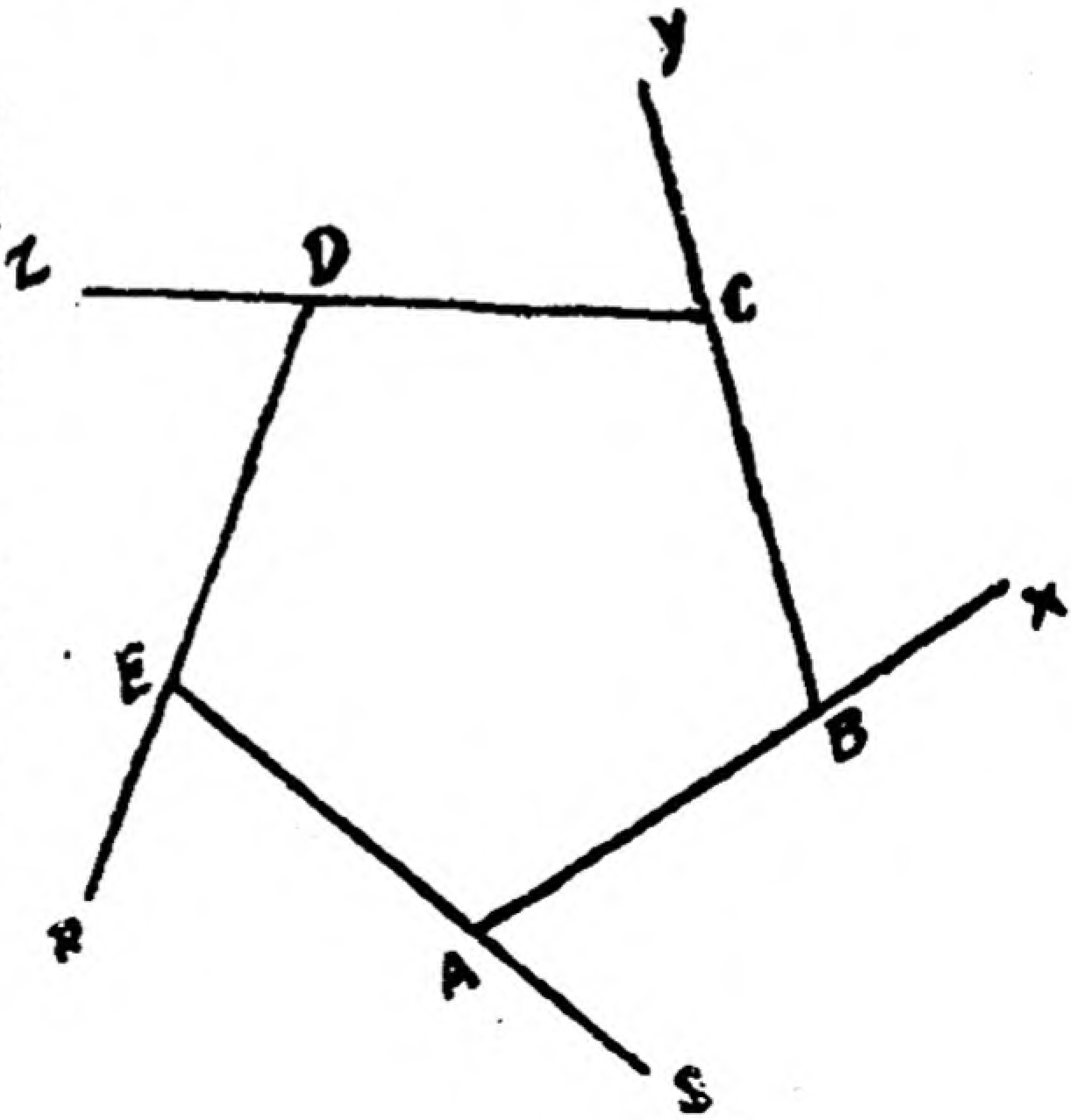


14. The exterior angle formed by producing one of the sides of a right-angled triangle is of 126°. Find all the angles.

15. The angles at the base of a triangle are equal, and the exterior angle at the vertex is 120°. Determine all the angles of the triangle.

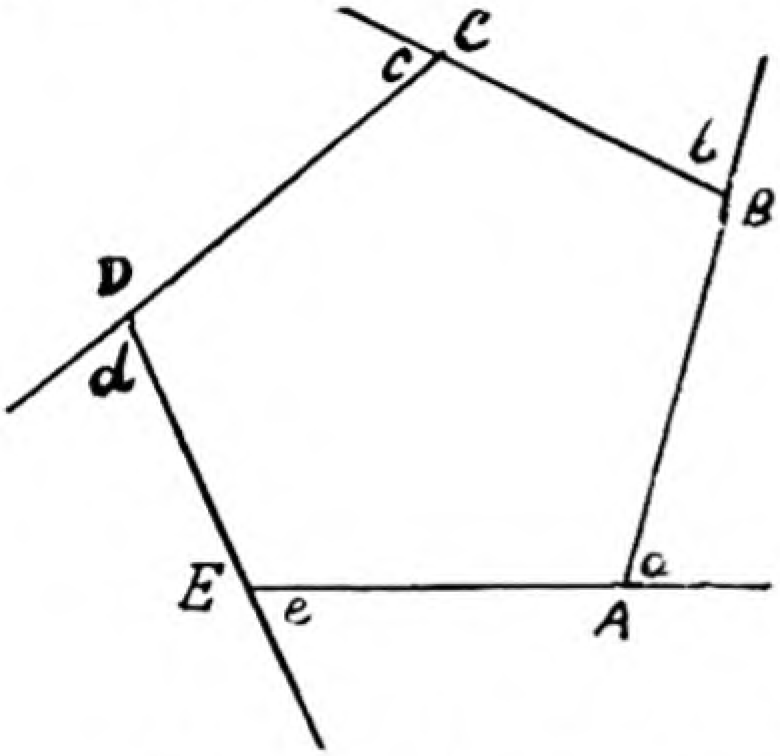
Def.—A polygon is said to be CONVEX when every one of its angles is less than two right angles. But a polygon is not convex, if any interior angle is more than two right angles.

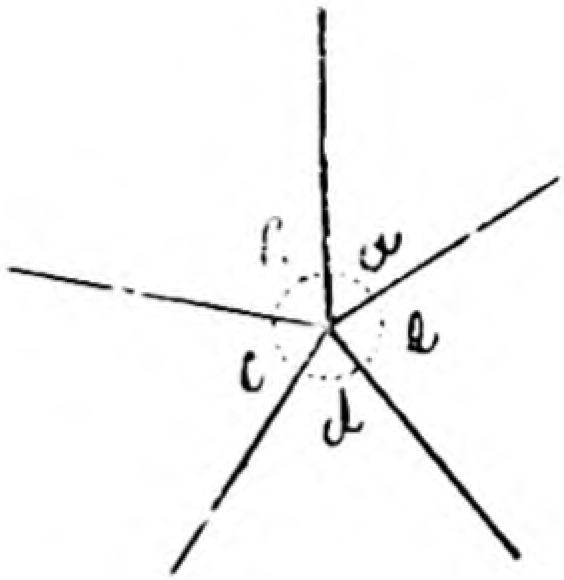
Being given a polygon ABCDE, if AB is produced beyond B to X and BC is produced beyond C to Y and so on, the sides are said to have been produced *in order.*



Proposition 9. (*Theorem*)

If the sides of a convex polygon are produced in order, the sum of the exterior angles so formed is equal to four right angles.





Given :—A convex polygon ABCDE with itss ides produced in order forming the exterior angles a, b, c, d, e.

Required :—To prove that $\angle a+\angle b+\angle c+\angle d+\angle e=$ 4 rt. angles.

Construction :—From any point O suppose lines are drawn parallel to the sides of the polygon and in the same sense in which the sides are produced, forming $\angle$s a', b', c', d', and e'.

Proof :—The arms of $\angle a$ and $\angle a'$ are parallel, and drawn in the same sense, $\therefore \angle a=\angle a'$

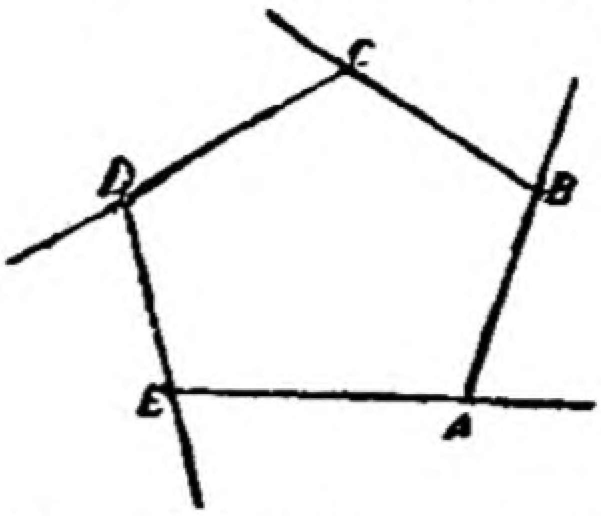
Similarly, $\angle b=\angle b'$, $\angle c=\angle c'$, $\angle d=\angle d'$, $\angle e=\angle e'$.

$\therefore \angle a+\angle b+\angle c+\angle d+\angle e$

$=\angle a'+\angle b'+\angle c'+\angle d'+\angle e'$

$=4$ rt. $\angle$s (angles at the pt. O). **Q. E. D.**

Cor. 1—*In any convex polygon of n sides, the sum of the interior angles is equal to (2n—4) right angle.*



Given :—A convex polygon ABCDE having n sides.

Required :—To prove that the sum of all the interior angles $=(2n-4)$ right angles.

Construction :—Produce the sides in order.

Proof :—At every corner, the sum of the interior and exterior angles is two rt. angles. ∴ there being n corners, the sum of all the interior and exterior angles $=2n$. rt. $\angle$s. But the sum of all the exterior angles$=4$ rt. angles.

(Proved.)

∴ the sum of all the interior angles$=(2n-4)$ rt. angles.

Q. E. D.

Cor. 2. Each angle of a convex regular polygon of n sides$=\dfrac{2n-4}{n}$ rt. angles.

Exercises.

1. How many degrees are there in each exterior angle of (*i*) a regular pentagon, (*ii*), a regular octagon ?

2. How many degrees are there in an angle of (*i*) a regular hexagon, (*ii*) a decagon.

3. Each exterior angle of a regular polygon is equal to $\frac{3}{5}$ of the right angle. Find the number of its sides.

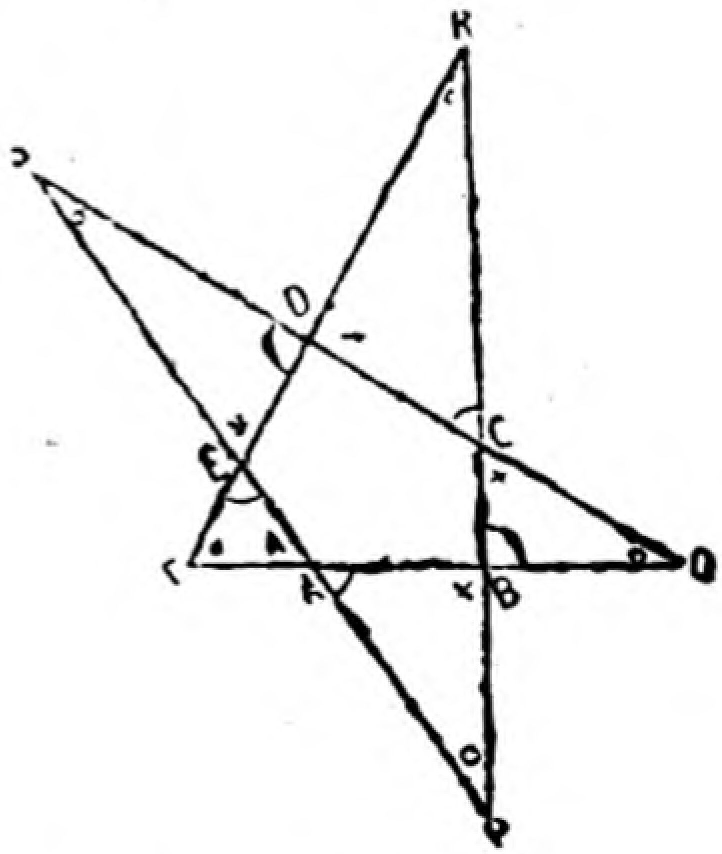
4. The sum of the interior angles of a polygon is equal to 12 rt. angles. How many sides has it ?

5. A polygon has each of its angles 144°, find the number of its sides.

6. Can a regular polygon be drawn, if one of its interior angles is (*i*) 150°, (*ii*) 108°, (*iii*) 125°, (*iv*) 60°.

7. In a convex hexagon the sum of the interior angles is equal to twice the sum of the exterior angles.

8. *The alternate sides of a polygon of n sides are produced to meet so as to form a star-shaped figure. Show that the sum of the angles at the vertices of the star is equal to* $(2n-8)$ *rt. angles.*



Hint.— One set of ext. $\angle$s$=4$rt. $\angle$s. Other set of ext. $\angle$s$=4$rt. $\angle$s, $\therefore$ sum of all ext. $\angle$s$=8$rt. $\angle$s. But sum of all

ext. $\angle$s. + sum of all the $\angle$s. at the star $= 2n$ rt. $\angle$s $= \angle$s. of n outer $\triangle$s. $\therefore$ sum of $\angle$s at the vertices of the star $= 2n - 8$ rt.$\angle$s.

9. If the sides of one triangle are $\perp$ to the sides of another triangle, the triangles are equiangular.

Proposition 11. (*Theorem*).

If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.

Given :—The $\triangle$s ABC, DEF, having $\angle B = \angle E$, $\angle C = \angle F$ and $BC = EF$.

Required :—To prove that $\triangle ABC \equiv \triangle DEF$.

Proof :—Place $\triangle ABC$ on $\triangle DEF$, so that BC coincides with its equal EF, B falling on E and C on F.

Now $\because$ $\angle B = \angle E$, $\therefore$ BA lies along ED.

and $\because$ $\angle C = \angle F$, $\therefore$ CA lies along FD.

$\therefore$ A falls on D, the point of intersection of ED and FD.

$\therefore$ $\triangle$s ABC and DEF coincide.

Hence $\triangle ABC \equiv \triangle DEF$.

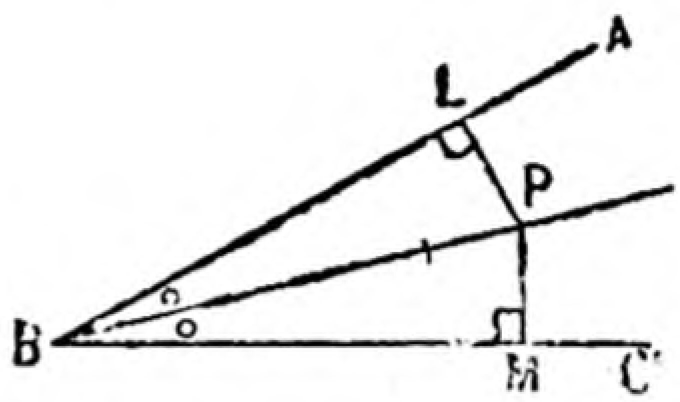
Q. E. D.

Note :— It is not necessary that the given side be adjacent to the given angles. When two angles of a triangle are known—practically all the three angles are known, for their sum is equal to two right angles. Thus in this proposition all the three angles and one of the sides are known, and the demonstration is, therefore, general.

Exercises.

1. *If the bisector of the vertical angle of a traingle be perpendicular to the base, the triangle is isosceles.*

2. *The perpendiculars drawn to the arms of an angle from any point in its bisector are equal.*



3. *In a triangle the perpendiculars drawn from the extremities of a side to its median are equal.*

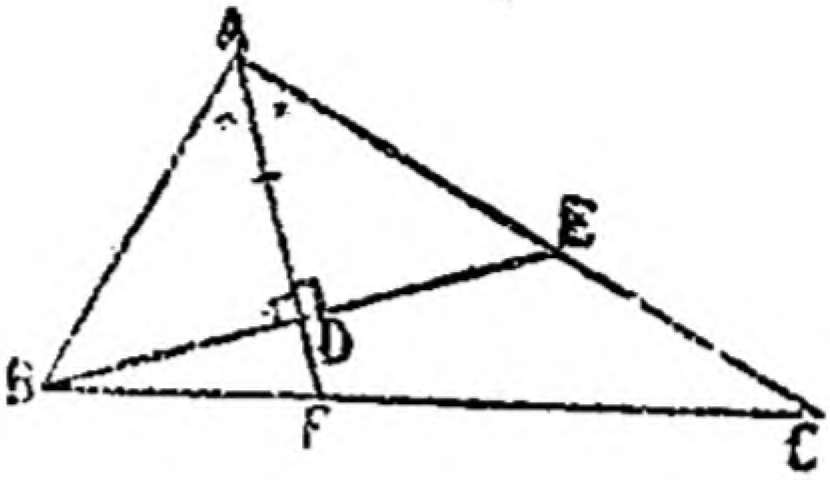
4. If a diagonal of a quadrilateral bisects the angles through which it passes, the quadrilateral is a kite.

5. If the perpendiculars drawn from the base angles of a triangle to the opposite sides are equal, the triangle is isosceles.

6. If the three perpendiculars from the angular points of a triangle on the opposite sides are equal, the triangle is equilateral.

7. ABC is a triangle, BDE is drawn $\perp$ to the bisector of the angle A, meeting it at D and AC at E; show that BD = DE.

8. If in a triangle a perpendicular be drawn from one extremity of the base to the bisector of the vertical angle, prove that :—

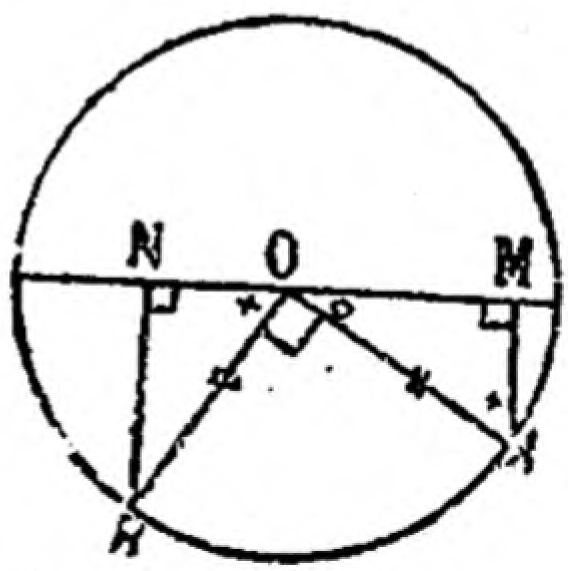


(*i*) it will make with either of the sides containing the vertical angle an angle equal to half the sum of the base angle.

(*ii*) it will make with the base an angle equal to half the difference of the base angles.

9. If one diagonal of a quadrilateral bisect the two angles at its ends, it bisects the other diagonal at right angles.

10. OA, OB are radii of a circle at right angles to each other. AM and BN are drawn $\perp$s to any diameter of the same circle meeting it in M and N. Prove that AM=ON. (Bombay, 1928).

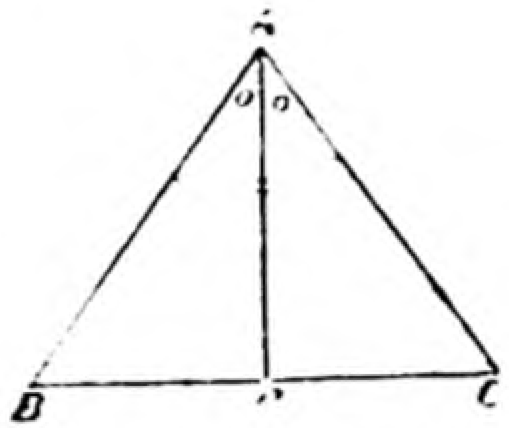


11. If the alternate extremities of two equal and parallel straight lines are joined, the connecting lines bisect each other.

(Calcutta)

Proposition 12. (*Theorem*)

If two sides of a triangle are equal, the angles opposite these sides are equal.



Given :—A triangle ABC, having AB=AC.

Required :—To prove that $\angle$B=$\angle$C.

Construction :—Let AD be the bisector of $\angle$A.

Proof :—In the $\triangle$s ABD, ACD.

$$\because \begin{cases} AB=AC & \text{(Given)} \\ AD=AD. & \\ \text{incl. } \angle BAD=\text{incl. } \angle CAD & \text{(Const.)} \end{cases}$$

$\therefore$ $\angle$ABD$\equiv\triangle$ACD.

Hence $\angle$B=$\angle$C. **Q. E. D.**

Note.—This theorem can also be enunciated otherwise thus :—

The angles at the base of an isosceles triangle are equal.

Cor. 1. If a triangle is equilateral, it is also equiangular.

Cor. 2. If the equal sides AB and AC of a triangle ABC are produced, the angles on the other side of the base BC are also equal.

Exercises.

1. *The perpendicular from the vertex of an isosceles triangle to the base divides the triangle into two equal parts.*

2. The bisector of the vertical angle of an isosceles triangle divides the triangle into two equal parts.

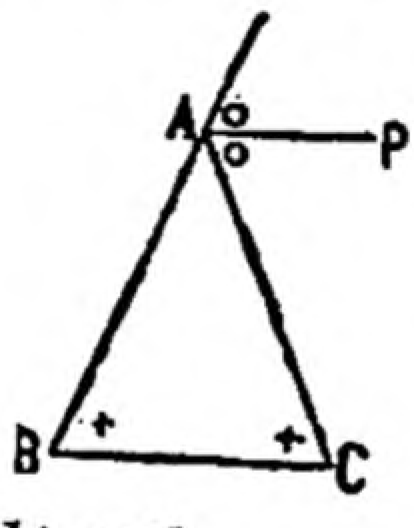
3. ABC is an isosceles triangle. D. E., F. are the middle points of BC, CA, AB. Prove that DEF is an isosceles triangle.

4. The straight line joining the mid-points of the sides of an isosceles triangle to the opposite ends of the base are equal. (U. P., E. S.)

5. The three medians of an equilateral triangle are equal.

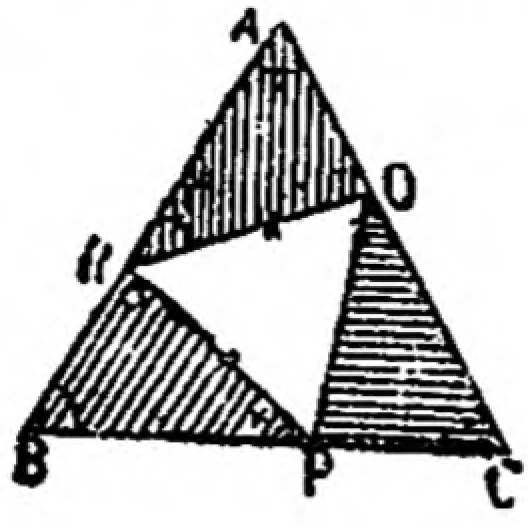
6. The perpendiculars from the ends of the base of an isosceles triangle to the opposite sides are equal.

7. *The straight line bisecting the exterior angle at the vertex of an isosceles triangle is parallel to the base.*

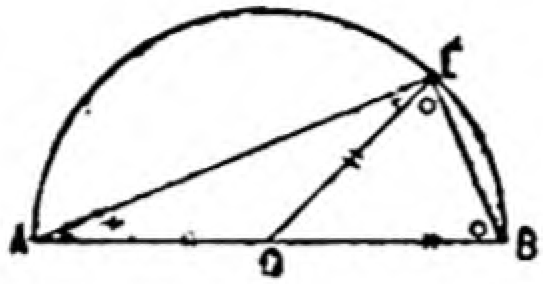


8. *Trisect a right angle.*

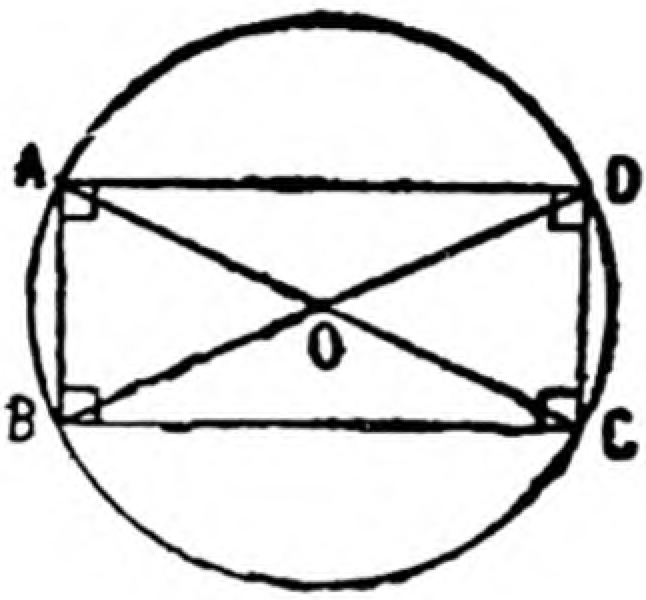
9. An equilateral triangle PQR has its angular points on the sides of another equilateral triangle ABC so that P falls on BC, Q falls on CA, R falls on AB. Prove that the triangles QAR, BPR, PCQ are congruent.



10. Prove that an angle in a semicircle is a right angle.

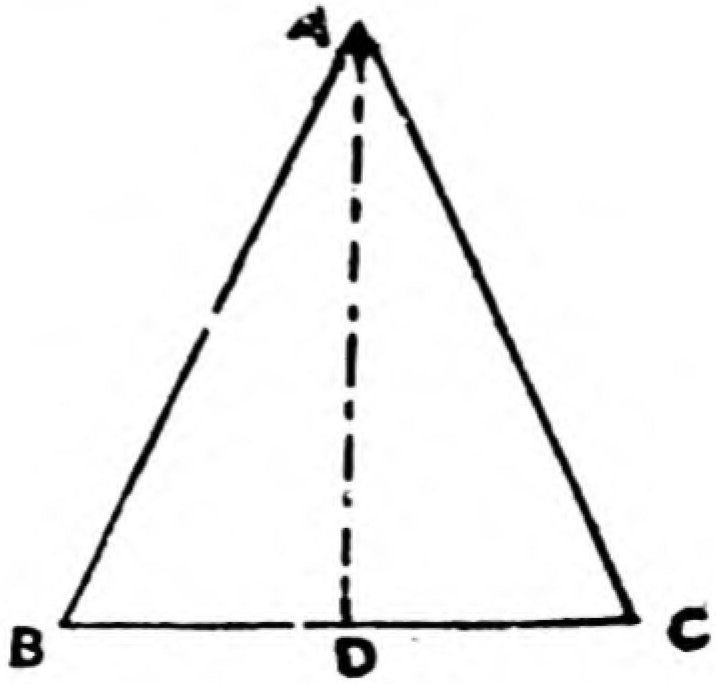


11. Prove that the figure formed by joining the extremities of any two diameters of a circle is a rectangle.



Proposition 13. (*Theorem*).

If two angles of a triangle be equal, the sides opposite them are also equal.



Given :—A △ ABC having ∠B=∠C.

Required :—To prove that $AB=AC$.

Construction :—Let AD be the bisector of $\angle A$.

Proof :—In the $\triangle$s ABD, ACD.

$$\because \begin{cases} \angle B=\angle C, & \text{(Given)} \\ \angle BAD=\angle CAD. & \text{(Const.)} \\ AD=AD. & \end{cases}$$

$\therefore\ \triangle ABD=\triangle ACD$.

Hence $AB \equiv AC$.

Q. E. D.

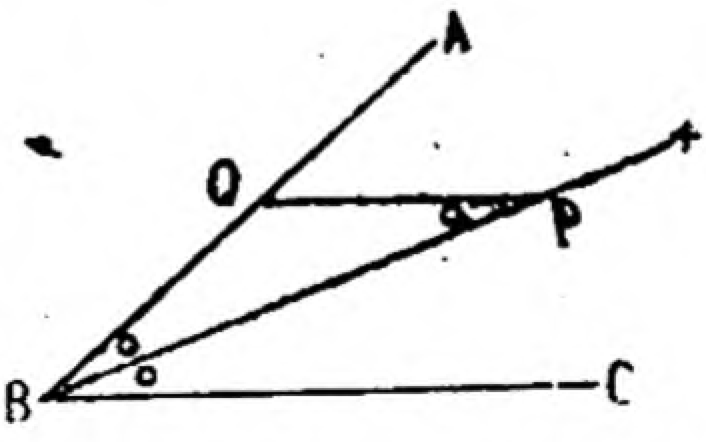
Note.— This proposition is the converse of Proposition 12.

Cor. 1. An equiangular triangle is also equilateral.

Cor. 2.—If any two sides of a triangle are produced and the exterior angles thus formed with the base are equal, the triangle is isosceles.

Exercises.

1. Through any point P on the bisector BX of an angle ABC a straight line PQ is drawn parallel to one of the containing arms BC meeting the other arm AB in Q, prove that the triangle BPQ is isosceles.



2. If OX and OZ the bisectors of the angles X and Z of the triangle XYZ be equal to one another, the triangle is isosceles.

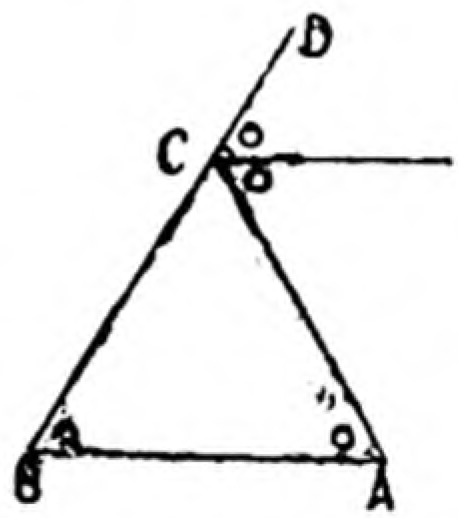
3. The bisectors OB and OC of the base angles of an isosceles triangle form with the base BC another isosceles triangle.

4. The equal sides of an isosceles triangle ABC are produced and the bisectors of the angles on the other side of the base B; C meet at O; prove that OB=OC.

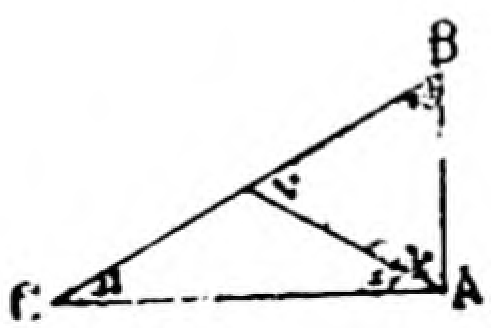
5. In the sides AB, AC of an isosceles triangle two points D and E are taken such that DE ∥ BC. Prove that AD=AE.

6. The side BC of a triangle ABC is produced to D. If the bisector of the angle ACD is ∥ to AB, prove that CA =CB.

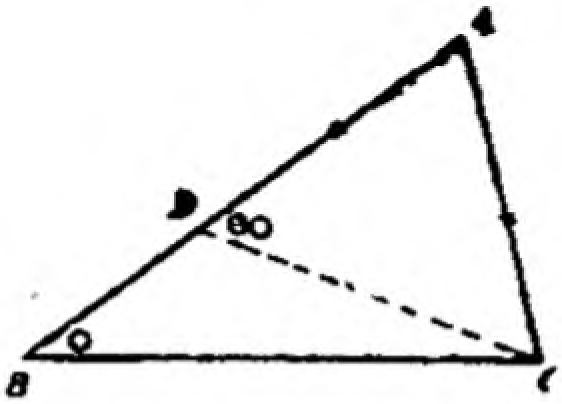
(Bombay Matric).



7. In a right-angled triangle if one angle is double the other, the hypotenuse will be equal to twice the smallest side.

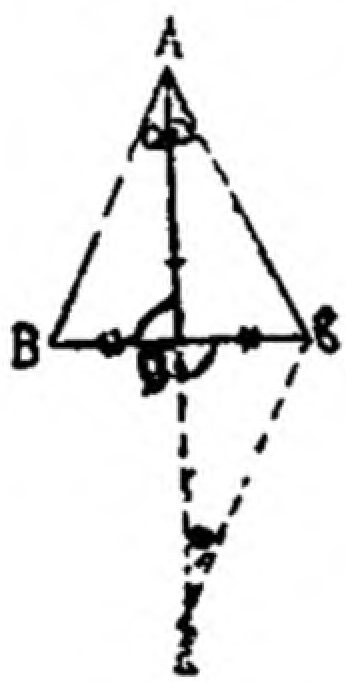


8. *In a right-angled triangle the line joining the middle point of the hypotenuse to the right angle is half of the hypotenuse.*



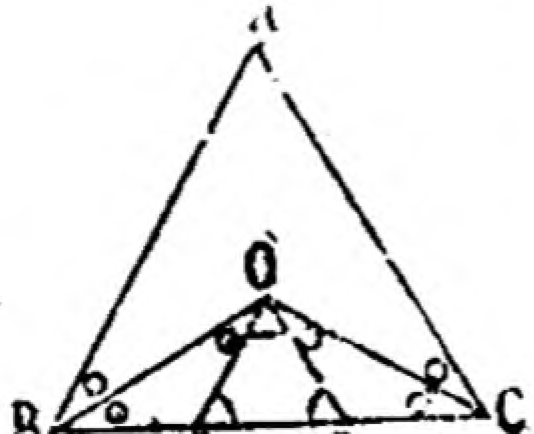
9. If the bisector of the vertical angle of a triangle also bisects the base, the triangle is isosceles. (Punjab, 1930).

Hint.—In ABC produce the bisector AD to E, making AD=DE; join EC, $\triangle$s ADB and CDE are congruent $\angle$E $=\angle$BAD$=\angle$DAC $\therefore$ AC=CE=AB.



10. Use the properties of an equilateral triangle to trisect a straight line. (Punjab, 1876)

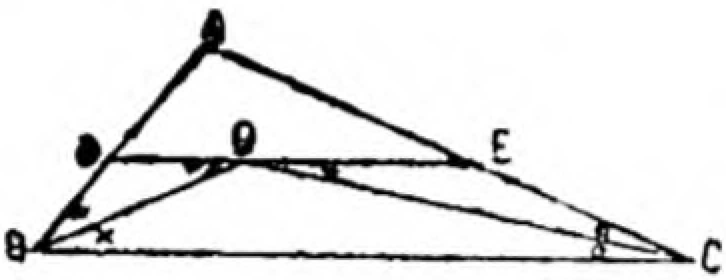
Hint :—Through O the point of intersection of bisectors of base angles of an equilateral $\triangle$ABC on the given line BC, parallels are drawn to the sides, these parallels trisect the base BC.



11. ABC is a triangle in which AB=AC. The ∠s ABC and ACB are bisected by BD and CE meeting AC in D and AB in E. Prove that BE=ED=DC.

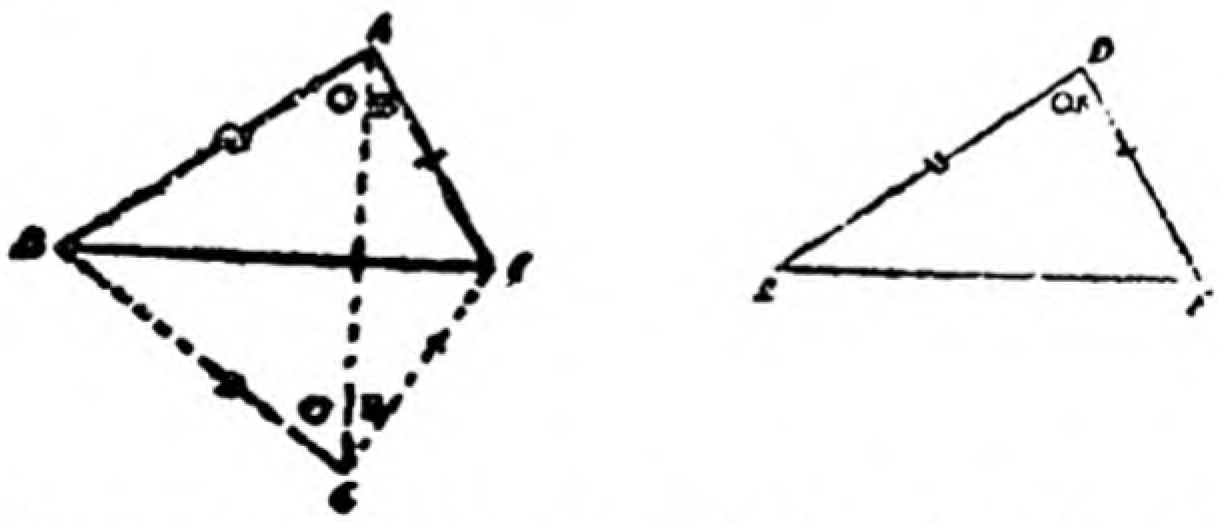
(Bombay, 1929).

12. The bisectors BO and CO of the base angles B and C of a triangle ABC intersect at O. Through O a line is drawn parallel to the base BC cutting the sides AB and AC at D and DE. Prove that DE=DB +EC.



Proposition 14. (*Theorem*).

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.



Given :—△s ABC and DEF, having AB=DE, BC =EF, CA=FD.

Required :—To prove that △ABC≡△DEF.

Construction :—**Suppose BC and EF are not smaller than the other sides.** Place △DEF so that EF coincides with its equal BC, E falling on B and F on C ; and the point D falls on the side of BC opposite to A. Let G be the new position of pt. D. Join AG.

Proof :—$\because$ BA=BG, (each being equal to DE).

$\therefore \angle BAG = \angle BGA$.

Also $\because$ CA=CG, (each being equal to DF).

$\therefore \angle CAG = \angle CGA$.

Adding,

$\angle BAG + \angle CAG = \angle BGA + \angle CGA$

or $\angle BAC = \angle BGC = \angle EDF$. (Const.)

Hence in the $\triangle$s BAC, EDF.

AB=DE, AC=DF, incl. $\angle A$=incl. $\angle D$.

$\therefore$ the triangles are congruent. **Q. E. D.**

Note.—The condition that *BC and EF are not smaller than the other sides* has been introduced into the construction to avoid the possibility of two more cases, which are shown below.

Case I. When $\angle ACB$ is a right angle, in which case AG coincides with AC and CG. (Fig 1)

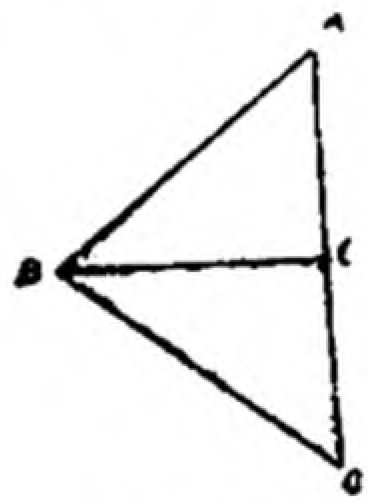


Fig. 1.

Case II. When $\angle ACB$ is obtuse, in which case the line AG falls outside the angles BAC and BCG. (Fig. 2)

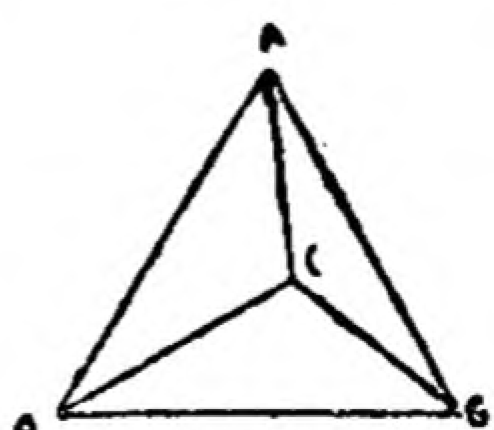
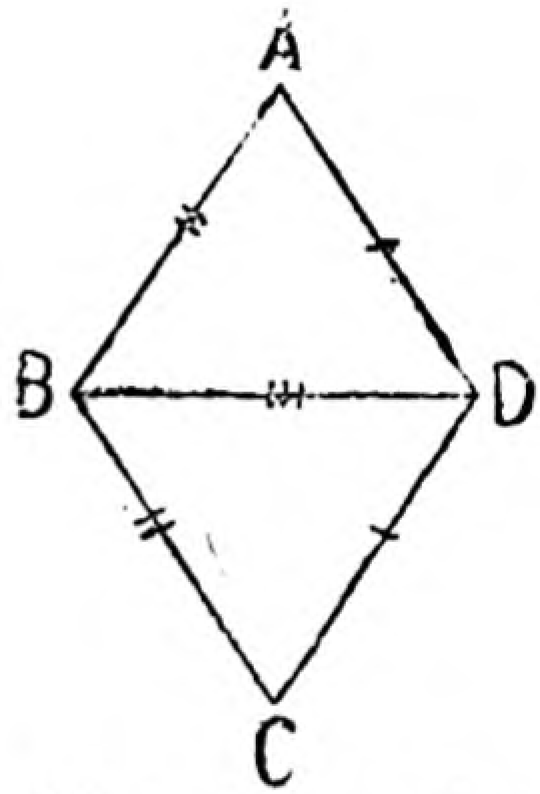


Fig. 2.

Students may have noticed that in the case discussed in the propositions $\angle C$ is acute and the line AG falls within the angles BAC and BGC.

Exercises.



1. The opposite angles of a rhombus are equal.

2. In a circle the straight line joining the middle point of a chord to the centre is $\perp$ to the chord.

3. If the three sides of one triangle are respectively equal to the three sides of another, the three angles of the one are equal to three angles of the other each to each. (Madras, Matric).

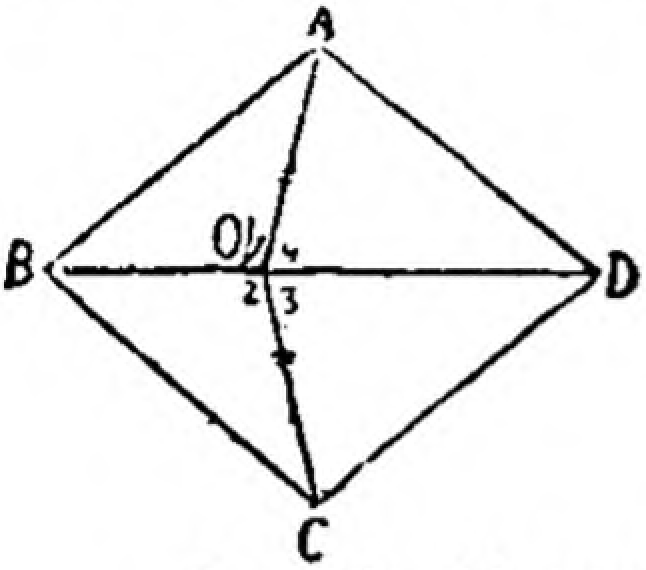
4. *Equilateral triangles on equal bases are congruent.*

5. In a quadrilateral ABCD, if AB=AD and CB =CD prove that the diagonal AC divides it into two equal parts.

6. A point equidistant from the ends of a straight line lies on the perpendicular bisector of the st. line.

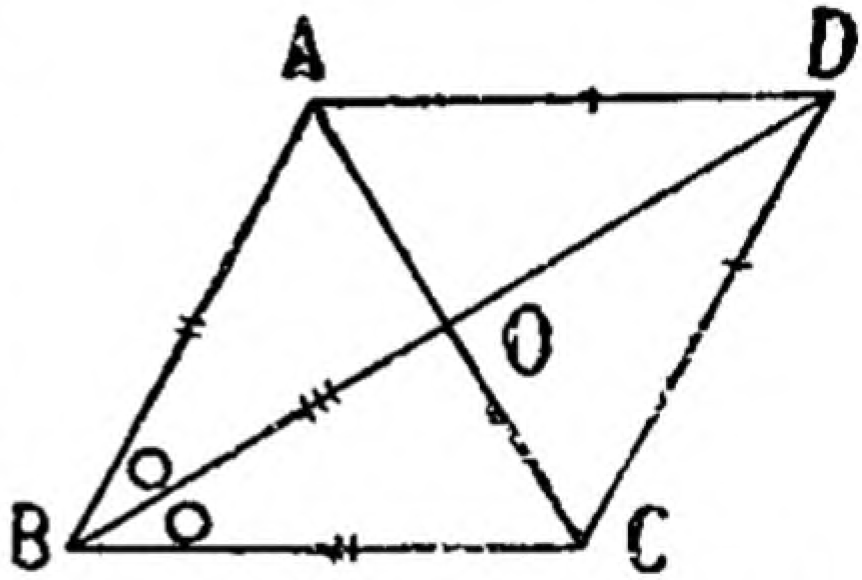
7. The straight line joining the vertex of an isosceles triangle to the Middle point of the base bisects the vertical angle and is $\perp$ to the base. (Calcutta).

8. A point O is taken within an equilateral four sided figure ABCD, such that its distances from the angular points A and C are equal. Show that OB and OD are in one and the same straight line. (Punjab, 1929).



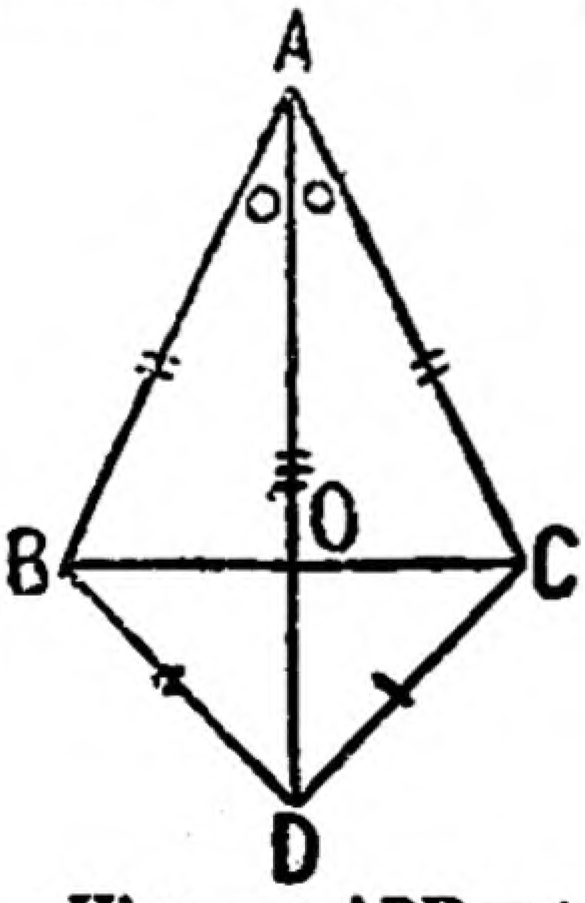
Hint.—$\angle 1=\angle 2$, $\angle 3=\angle 4$, $\therefore \angle 1+\angle 3=\angle 2+\angle 4=2$ rt. $\angle$s.

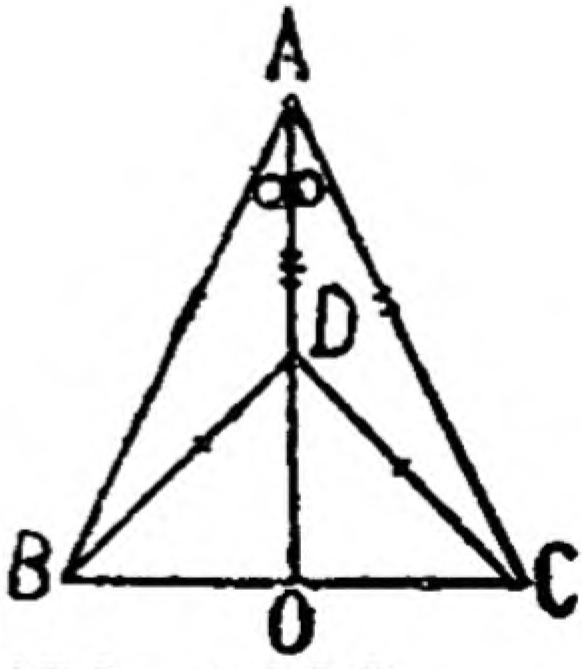
9. *The diagonals of a rhombus bisect each other at right angles.*



Hint.—$\triangle$ ABD $\equiv$ $\triangle$BCD, $\triangle$ABO$\equiv\triangle$ BCO.

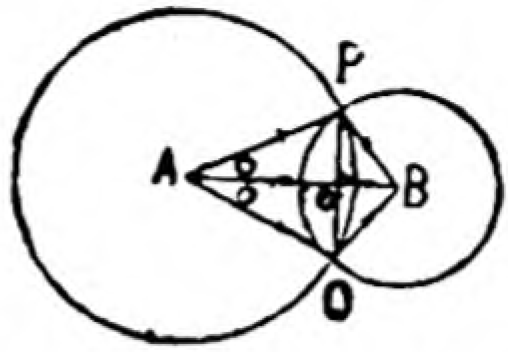
10. Two isosceles triangles stand on the same base; show that the straight line joining their vertices bisects their common base at right angles.





Hint.—$\triangle$ABD$\equiv\triangle$ACD, $\triangle$ABO$\equiv\triangle$AOC.

11. *If two circles intersect, the line joining the centres is the right bisector of the common chord.*
(Cambridge and Oxford 1905 ; Punjab 1916).



12. If one diagonal of a quadrilateral bisects the other at right angles, it divides the quadrilateral into two congruent triangles. (Madras, Matric.)

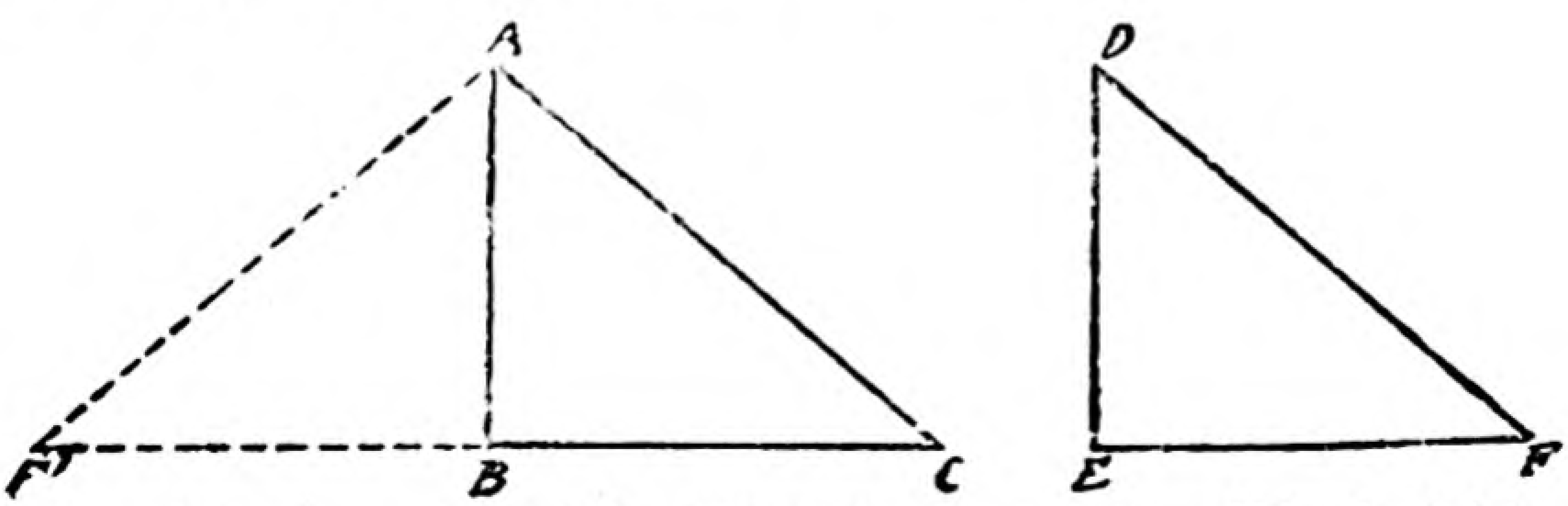
13. If two triangles have two sides of the one equal to two sides of the other and the angles opposite a pair of equal sides equal, then the angles opposite the other pair of equal sides are either equal or supplementary.

14. ABCD is a kite, AB=AD and BC=CD. L is any pt. within the kite equidistant from B and D. Prove that AL and CL form one and the same st. line.

Hint.—$\triangle ABL \equiv \triangle ADL$; $\triangle BLC \equiv \triangle CLD$ and $\angle ALB + \angle CLB = \angle ALD + \angle CLD = 2$ rt $\angle$s.

Proposition 15. (*Theorem*).

If two right angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are congruent.



Given :—$\triangle$s ABC, DEF, having $\angle B = \angle E = 1$ right angle. AC=DF, AB=DE.

Required :—To prove that $\triangle ABC \equiv \triangle DEF$.

Construction :—Apply $\triangle DEF$ to $\triangle ABC$, so that DE coincides with its equal AB, D falling on A and E on B ; and F falls on the side of AB remote from C. Let ABF′ be the new position of $\triangle DEF$.

Proof :—$\because \angle ABC + \angle ABF' = 2$ right angles,
$\therefore$ F′BC is a straight line. $\therefore$ AF′C is a triangle.
Since $AF' = DF = AC$, $\therefore \angle C = \angle F' = \angle F$
Now in the $\triangle$ ABC, DEF

$$\because \begin{cases} \angle C = \angle F. & \text{(Proved)} \\ \angle ABC = \angle DEF & \text{(Rt. angles)} \\ AB = DE & \text{(Given)} \end{cases}$$

$\therefore \triangle ABC \equiv \triangle DEF$.

Q. E. D.

Exercises.

1. A, B are two points on the circumference of a circle with centre O. Prove that the $\perp$ from O on AB divides AB into two equal parts.

2. KE and KF are drawn perpendiculars from K, the midpoint of the base on the sides AC and AB of a triangle ABC. If KE and KF are equal, prove that the triangle ABC is isosceles.

3. *The perpendicular from the vertex to the base of an isosceles triangle divides the triangle into two equal parts.*

4. Every point equidistant from the arms of an angle lies on the bisector of the angle.

5. *Prove that the three bisectors of the angles of a triangle are concurrent.*

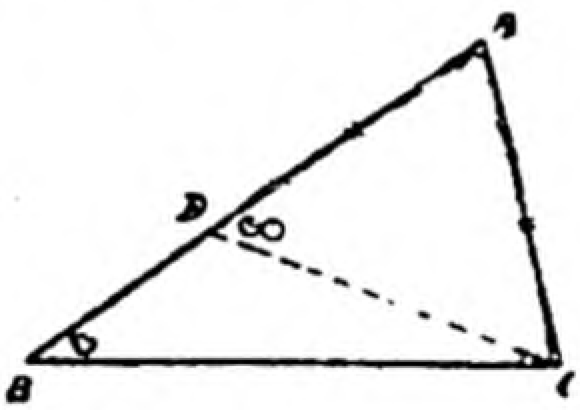
6. Prove that the bisectors of two external angles and one internal angle of a triangle are concurrent.

7. The bisectors of the internal angles of a regular polygon are concurrent.

8. *If the three perpendiculars from the angular points of a triangle on the opposite sides are equal, the triangle is equilateral.*

Proposition 16. (*Theorem*)

If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.



Given :—$\triangle$ABC, having AB>AC.

Required :—To prove that $\angle$ACB > $\angle$ABC.

Construction:—From AB cut off AD=AC. Join CD.

Proof :—$\because$ AD=AC. (Const.)

$\therefore \angle$ADC= $\angle$ACD.

But $\angle$ADC is > the int. opp. $\angle$ABC.

$\therefore$ $\angle$ACD is >$\angle$ABC.

But $\angle$ACB is >$\angle$ACD.

$\therefore$ $\angle$ACB is > $\angle$ABC.

Q. E. D.

Another Proof.

Given :—$\triangle$ABC, having AB> AC.

Required :—To prove that $\angle$ACB>$\angle$ABC.

Construction :—Cut off AD=AC.

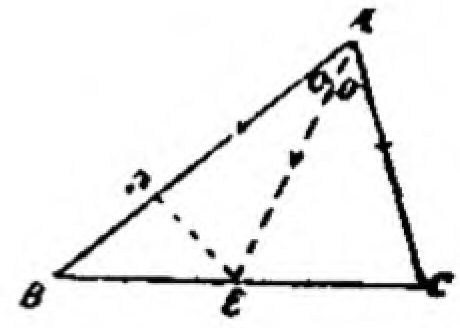
Suppose AE bisects the $\angle$A. Join DE.

Proof :—$\triangle$ ADE, ACE are congruent, for AD=AC, AE=AE, $\angle$DAE=$\angle$CAE

$\therefore$ $\angle$ADE=$\angle$C.

But $\angle$ADE>int. opp. $\angle$B.

$\therefore$ $\angle$C >$\angle$B.



Exercises.

1. *The greatest side of a triangle has the greatest angle opposite to it.*

2. ABCD is a quadrilateral having AB=AD but BC<DC. Prove that $\angle$ABC>$\angle$ADC.

3. ABCD is a quadrilateral whose opposite sides are parallel. If AB>BC, prove that AC lies between DC and the bisector of $\angle$BCD.

Hint :—Prove that $\angle$BCA>$\frac{1}{2}\angle$C.

4. In a quadrilateral ABCD, AD is the longest side and BC the shortest. Prove that $\angle$ABC>$\angle$ADC and $\angle$BCD>$\angle$BAD. (U. P.)

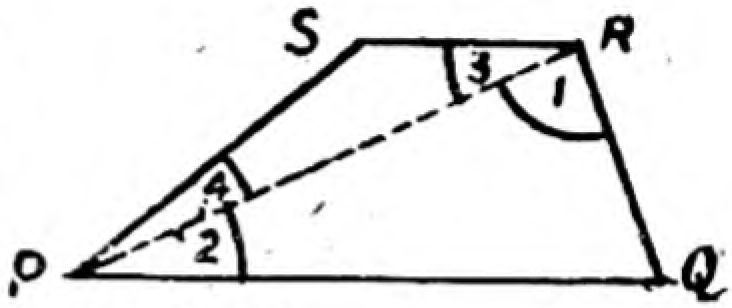
5. The base of a triangle whose sides are unequal is divided into two parts by the bisector of the vertical angle. Prove that the greater part is adjacent to the greater side. (Bombay Matric).

Hint :—In ABC, AB>AC : AD bisects $\angle$A, then BD>CD. From AB, cut AE=AC, join DE. $\triangle$s. ACD and AED are congruent. $\therefore$ DE=DC ; $\angle$ADC=$\angle$ADE. But $\angle$ADC>$\angle$EBD. $\therefore$ $\angle$ADE>$\angle$EBD and $\angle$BED>$\angle$ADE $\therefore$ $\angle$BED>$\angle$EBD $\therefore$ AD>DE. Hence BD>CD.

6. The angles adjacent to the greatest side of any $\triangle$ are both acute.

7. In quad. PQRS, the side PQ is the largest and RS the smallest. Prove that $\angle R > \angle P$.

Join PR.



$\because$ $PQ > RQ$ (Given)

$\therefore$ $\angle 1 > \angle 2$ ......(*i*)

$\because$ $PS > SR$ (Given)

$\therefore$ $\angle 3 > \angle 4$ ......(*ii*)

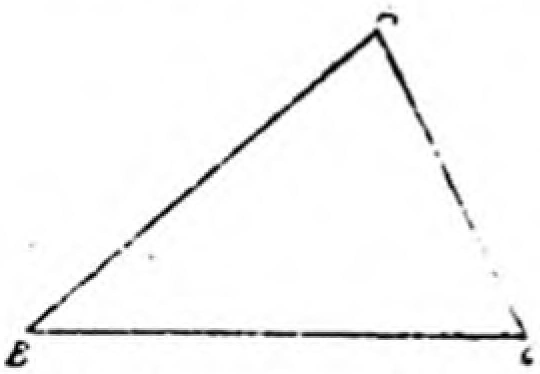
Adding (*i*) and (*ii*) $\angle 1 + \angle 3 > \angle 2 + \angle 4$

or $\angle R > \angle P$.

Q. E. D.

Proposition 17 (*Theorem*)

If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.



Given :—$\triangle ABC$, having $\angle C > \angle B$.

Required :—To prove that $AB > AC$.

Proof :—There can be only three possibilities: either $AB = AC$, or $AB < AC$, or $AB < AC$.

(1) If $AB = AC$.

Then $\angle B = \angle C$.

This is against what is given.

$\therefore$ $AB \neq AC$

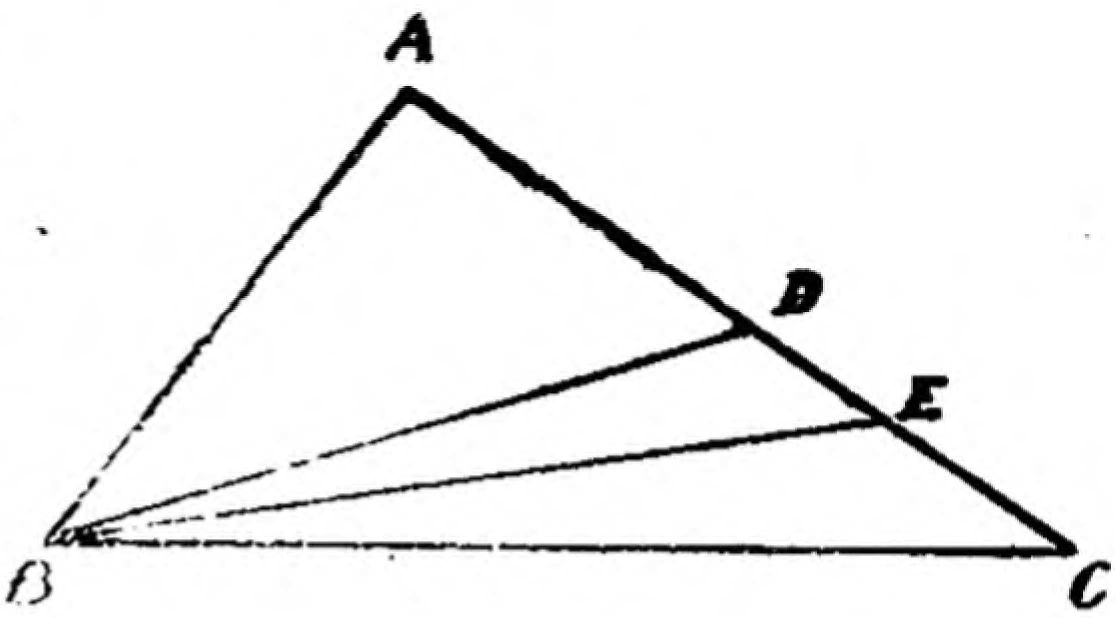
(2) If $AB < AC$,

then $\angle C < \angle B$.

This, too, is against what is given. $\therefore$ AB is not $< AC$. Hence the only possibility left is that $AB > AC$.

Q. E. D.

Another Proof.



Given :— $\triangle$ ABC having $\angle B > \angle C$.

Required :— To prove that $AC > AB$.

Construction :— From $\angle ABC$ cut off $\angle ABD = \angle C$ and let BD meet AC at D.

Bisect $\angle DBC$ by BE meeting AC at E.

Proof :— $\angle ABE = \angle ABD + \angle DBE = \angle C + \angle DBE$ also $\angle AEB = \angle C + \angle EBC$.

$\therefore \quad \angle ABE = \angle AEB \quad \because \angle DBE = \angle EBC,$ (Const.)

$\therefore \quad AE = AB$

But $\quad AC > AE$

$\therefore \quad AC > AB.$

Note 1.—This proposition is the converse of proposition 16.

Note 2.—The mode of proof employed in this proposition (first proof) is known as the **Proof by Exhaustion,** which is applicable to cases in which only *one* of the two or more alternatives must necessarily follow from the given hypothesis. It consists in inferring the truth of one of these alternatives by showing that each of the others is inconsistent with the given hypothesis.

Exercises.

1. In an obtuse-angled triangle the greatest side is opposite the obtuse angle.

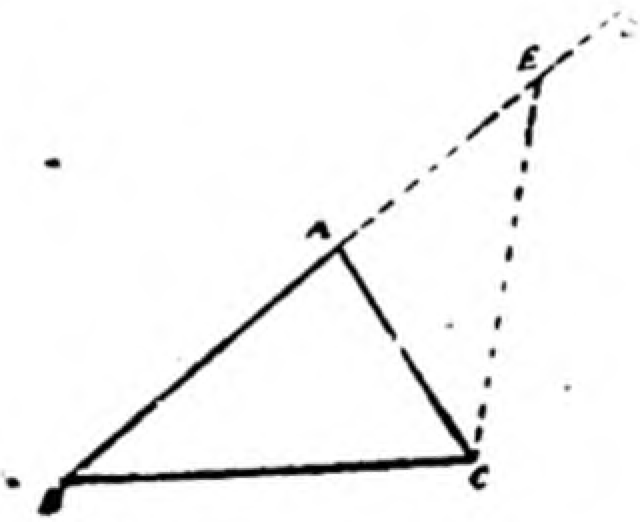
2. In a right-angled triangle the hypotenuse is the greatest side.

3. Show that in an isosceles $\triangle$ the base is greater or less than either of the equal sides according as the vertical angle is greater or less than 60°.

(Punjab, 1919).

4. *The greatest angle of any triangle has the greatest side opposite to it.*

5. *Any two sides of a triangle are together greater than the third side.*



Given :—$\triangle$ABC.

Required :—To prove that $AB+AC>BC$.

Construction :—Produce BA to D.
From AD cut off $AE=AC$.
Join EC.

Proof :—$\because AE=AC$.
$\therefore \angle ACE=\angle AEC$.
$\therefore \angle BCE$ is $>\angle AEC$.
Now in the $\triangle$ BEC.
$\because \angle BCE$ is $>\angle BEC$.
$\therefore$ opp. side BE is $>$ BC.
But $BE=BA+AE$
$=BA+AC$. (Const.)
$\therefore BA+AC>BC$.

Similarly it may be proved that

$AC+CB>AB$

and $CB+BA>CA$.

Q. E. D.

Another Proof : –

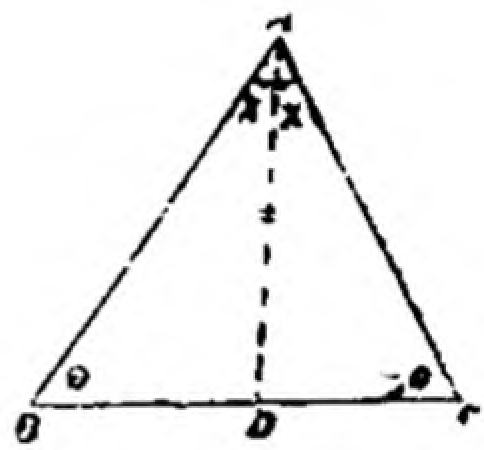
Given :– $\triangle ABC$.

Required :—To prove that

$AB+AC>BC$.

Construction :—

Bisect $\angle A$ by AD.



Proof :—$\angle ADB$ >int. opp. $\angle DAC$

$\because$ $\angle ADB>\angle BAD$ (for $\angle DAC=\angle BAD$).

$\therefore$ $AB > BD$.

Similarly $\angle ADC>\angle DAC$.

$\therefore$ $AC > DC$.

Hence $AB+AC>BD+DC$.

i.e., $AB+AC>BC$.

6. *The difference of any two sides of a triangle is less than the third side.* (Punjab, 1917)

7. Any three sides of a quadrilateral are together greater than the fourth side. (Calcutta, 1913)

8. Show that the perimeter of a quadrilateral is greater than the sum of its diagonals.

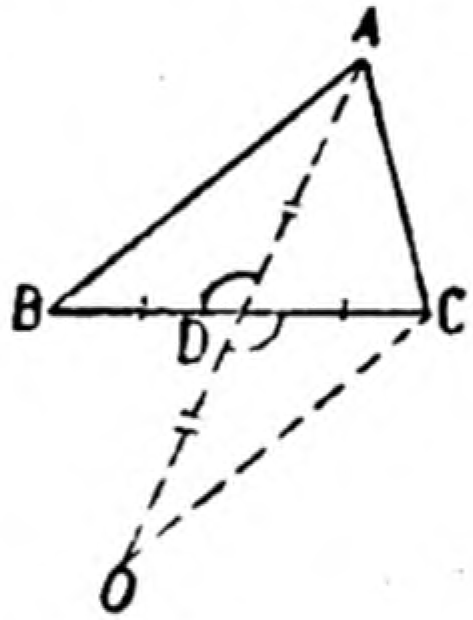
(Calcutta, 1920)

9. Show that in a quadrilateral the sum of the diagonals is greater than its semi-perimeter.

10. K is a point within the equilateral **triangle** PQR and it is joined to the angular points. **Prove** that any two of the st. lines KP, KQ and KR **are** together greater than the third.

Hint.—$KP+KQ>PQ$ but $PQ=RQ$ $\therefore$ $KP+KQ>RQ$ and $RQ>RK$ ($\because$ either side of an isosceles $\triangle>$st. line drawn from the vertex to any point in the base.) $\therefore$ $KP+KQ>RK$.

11. *Any two sides of a triangle are together greater than twice the median drawn to the third side.*



Hint:—In $\triangle ABC$, AD is median of BC: produc AD to O making $DO=AD$; join OC, and prove it equal to AB. Apply Prop 17, Ex. 5.

12. *The perimeter of a triangle is greater and the semi-perimeter less than the sum of the three medians.*

Hint:—First part follows immediately from Exercise 11. Second part. In $\triangle ABC$, medians AD, BE, CF cut in O, $AO+OE>AE$ (or $\frac{1}{2}$ AC). $CO+OD>CD$ (or $\frac{1}{2}$ BC). $OB+OF>\frac{1}{2}AB$. Adding $AD+BE+CF>\frac{1}{2}(AC+BC+AB)$.

13. If O be a point within a triangle XYZ, then

$$OX+OY+OZ>\tfrac{1}{2}(XY+YZ+ZX).$$

14. If P is a point within the triangle ABC, **then**

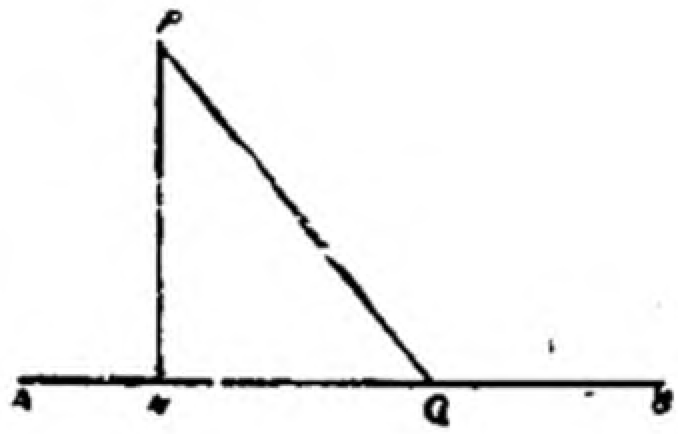
(*i*) $AB+AC>PB+PC$.

(*ii*) $AB+BC+CA>PA+PB+PC$.

15. ABCDEF is a hexagon. Prove that its perimeter is greater than two-thirds of the sum of the three diagonals, AD, BE, CF.

Proposition 18. (*Theorem*).

Of all the straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest.



Given :—AB a st. line and PN ⊥ AB drawn from **P**, a point outside it.

Required :—To prove that PN is smaller than any other line drawn from P to AB.

Construction :—Draw any other line from P to meet AB at Q.

Proof :—The ext. ∠PNA > int. opp. ∠PQN.

But ∠PNQ = ∠PNA being rt. angles.

∴ ∠PNQ > ∠PQN.

∴ PQ > PN.

Hence PN is less than PQ.

Similarly PN may be proved to be less than any other st. line drawn from P to meet AB.

∴ PN is the shortest of all such lines.

Q. E. D.

Note.—The distance between a point and a straight line is the shortest line that can be drawn from the point to the line. Hence the perpendicular from a given point to a given line, is called the distance of the point from the line.

Exercises.

1. If two oblique lines are drawn from a point making angles with the perpendicular from the same point to a given st. line, they are equal.

2. Of the two oblique lines drawn from a point to a st. line on the same side of the perpendicular, that one will be greater which makes the greater angle with it.

3. Show that the sum of the three sides of a triangle is greater than the sum of its three altitudes.

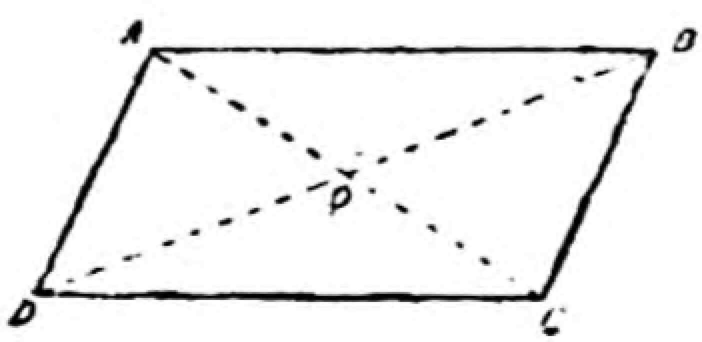
4. Two and only two oblique lines of given length can be drawn from a given point to a given st. line. Will the construction be always possible ?

5. In the base BC of an isosceles triangle ABC a point P is taken. Prove that AP is less than AB or AC. In case point P is taken on the base produced AP is greater than AB or AC.

6. A circle cannot intersect a st line in more than two points.

Proposition 19. (*Theorem*).

The opposite sides and angles of a parallelogram are equal ; each diagonal bisects the parallelogram and the diagonals bisect each other.



Given :—A $\parallel m$ ABCD, having diagonals AC and BD intersecting at O.

Required :—To prove that
(1) AB=DC, BC=AD.
(2) $\angle A=\angle C$, $\angle B=\angle D$.
(3) BD bisects the $\parallel m$ so also does AC.
(4) AO=OC, and BO=OD.

Prove :—$\because$ AB $\parallel$ DC.
$\therefore \angle ABD=\angle BDC$. (Alternate $\angle$s.)
Similarly $\because$ AD $\parallel$ BC,
$\therefore \angle ADB = \angle DBC$. (Alternate $\angle$s.)
Now in the $\triangle$s ABD, BDC,

$$\because \begin{cases} \angle ABD=\angle BDC. \\ \angle ADB=\angle DBC. \\ \text{BD is common.} \end{cases}$$

$\therefore$ the $\triangle$s are congruent.
Hence (1) AB=DC ; BC=AD.
(2) $\angle A=\angle C$; similarly $\angle B=\angle D$.
(3) BD bisects the figure ; similarly AC bisects the figure.
Again, in the $\triangle$s AOB and DOC

$$\because \begin{cases} \angle ABO=\angle ODC & \text{(AB} \parallel \text{DC.)} \\ \angle AOB=\angle DOC & \text{(Vert. opp. } \angle\text{s.)} \\ AB=DC & \text{(Opposite sides of a } \parallel m) \end{cases}$$

$\therefore$ the $\triangle$s are congruent.
Hence (4) AO=OC ; BO=OD.

Cor. I. A rectangle has all its angles right angles.

Cor. 2. Parallel straight lines are always equidistant from one another.

Cor. 3. The sides of a square are all equal and its angles are all right angles.

Exercises.

1. *A quadrilateral is a parallelogram, if*

   (*i*) *one pair of opposite sides are equal and parallel.*

   (*ii*) *Pairs of opposite sides are equal.*

   (*iii*) *Pairs of opposite angles are equal.*

   (*iv*) *The diagonals bisect each other.*

(Converse of Proposition 19).

2. From a point within, without or on one of the sides of a parallelogram draw a line bisecting the parallelogram.

Hint.—Join the given point to the middle point of one of the diagonals.

3. If one pair of adjacent sides of a parallelogram be equal, prove that the parallelogram is a rhombus.

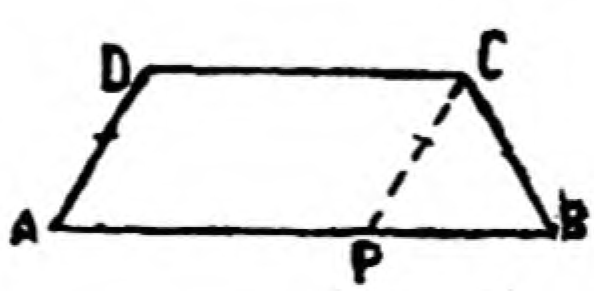
4. *The diagonals of a rectangle are equal.*

5. If the diagonals of a parallelogram are equal, it is a rectangle.

6. A diagonal of a parallelogram is equidistant from the outlying angular points.

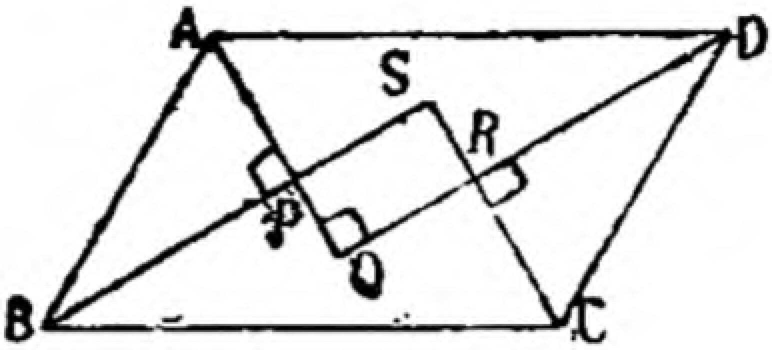
7. The diagonals of a square are equal and bisect each other at right angles.

8. If AD, BC are the equal sides of an isosceles trapezium ABCD, prove that $\angle A = \angle B$, $\angle C = \angle D$ and $AC = BD$.



9. The opposite angles of an isosceles trapezium are supplementary. (Bombay, Matric)

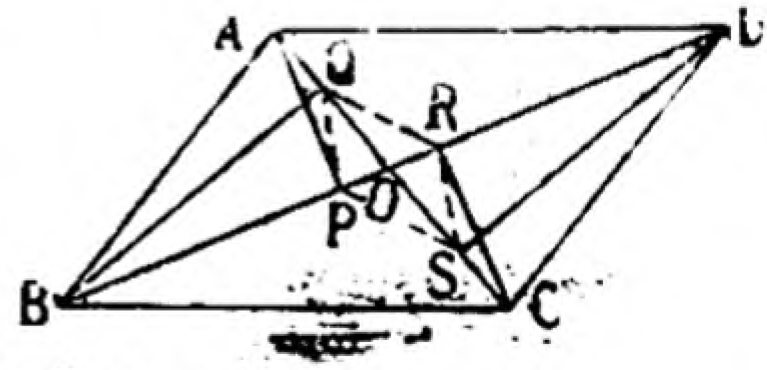
10. The bisectors of the four angles of a parallelogram enclose a rectangle.



11. If the bisectors of the angles of a quad. enclose a rect. Show that the quad. is a parallelogram. (Converse of Ex. 10).

12. If the bisectors of the angles of a quadrilateral form a rectangle, the quadrilateral is a parallelogram.

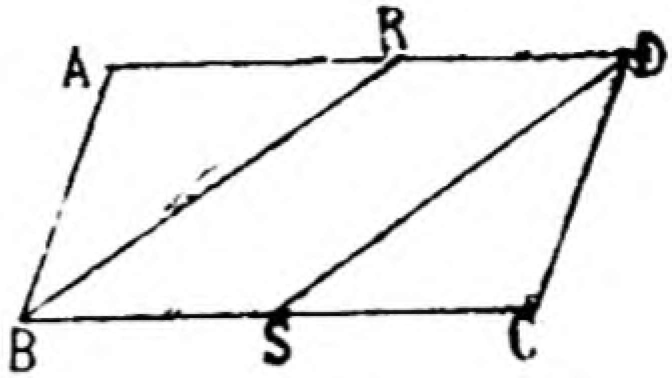
13. The feet of the perpendiculars drawn from the angular points of a $\|$ m on the diagonals are the vertices of a parallelogram.



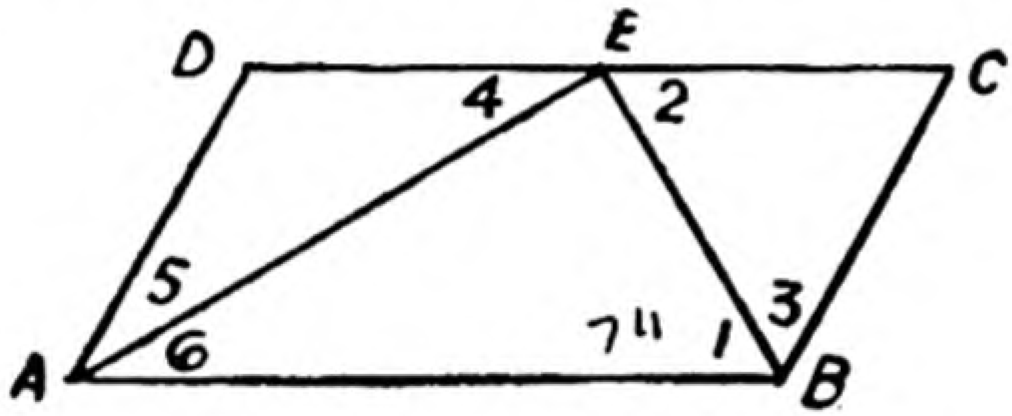
Hint :—In $\|$ m ABCD, AP, CR $\perp$ on BD ; BQ, DS $\perp$ on AC ; $\triangle$s AOP, COR are congruent. $\therefore$ OP=OR, similarly, OS=OQ $\therefore$ PQRS is a $\|$ *m*.

14. *The straight lines joining the mid-points of the opposite sides of a parallelogram are parallel to the other pairs of parallel sides.*

15. *ABCD is a parallelogram and AD and BC are bisected at R and S. Show that BRDS is a $\|$ m.*



16. ABCD is a parallelogram, BE, the bisector of ∠B bisects CD. Prove that AE bisects ∠A.

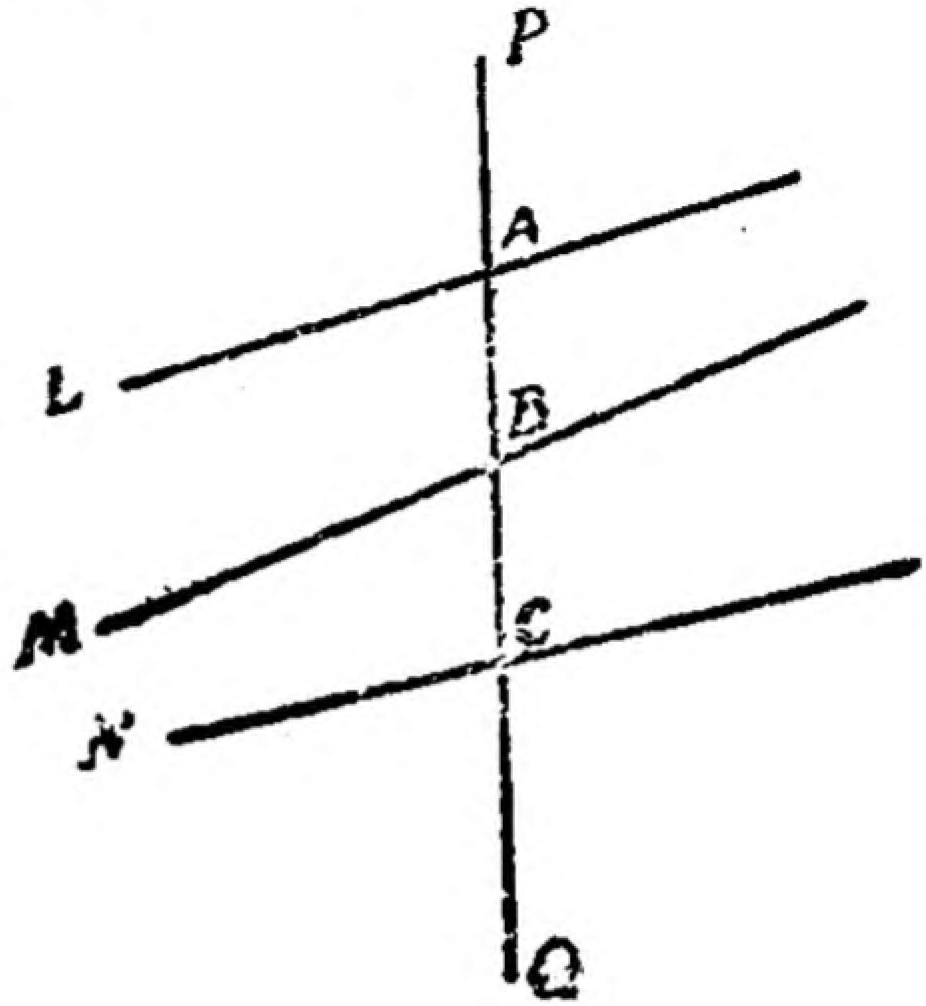


∵	∠1=∠2	[Alt. ∠s]
	∠1=∠3	[Given]
∴	∠2=∠3	
∴	CB=CE	
But	CB=AD	[opp. sides of a ‖ *m*]
∴	CE=AD	
But	CE=DE	
∴	AD=DE	
∴	∠4=∠5	
But	∠4=∠6	[Alt. ∠s]
∴	∠5=∠6	

Q. E. D.

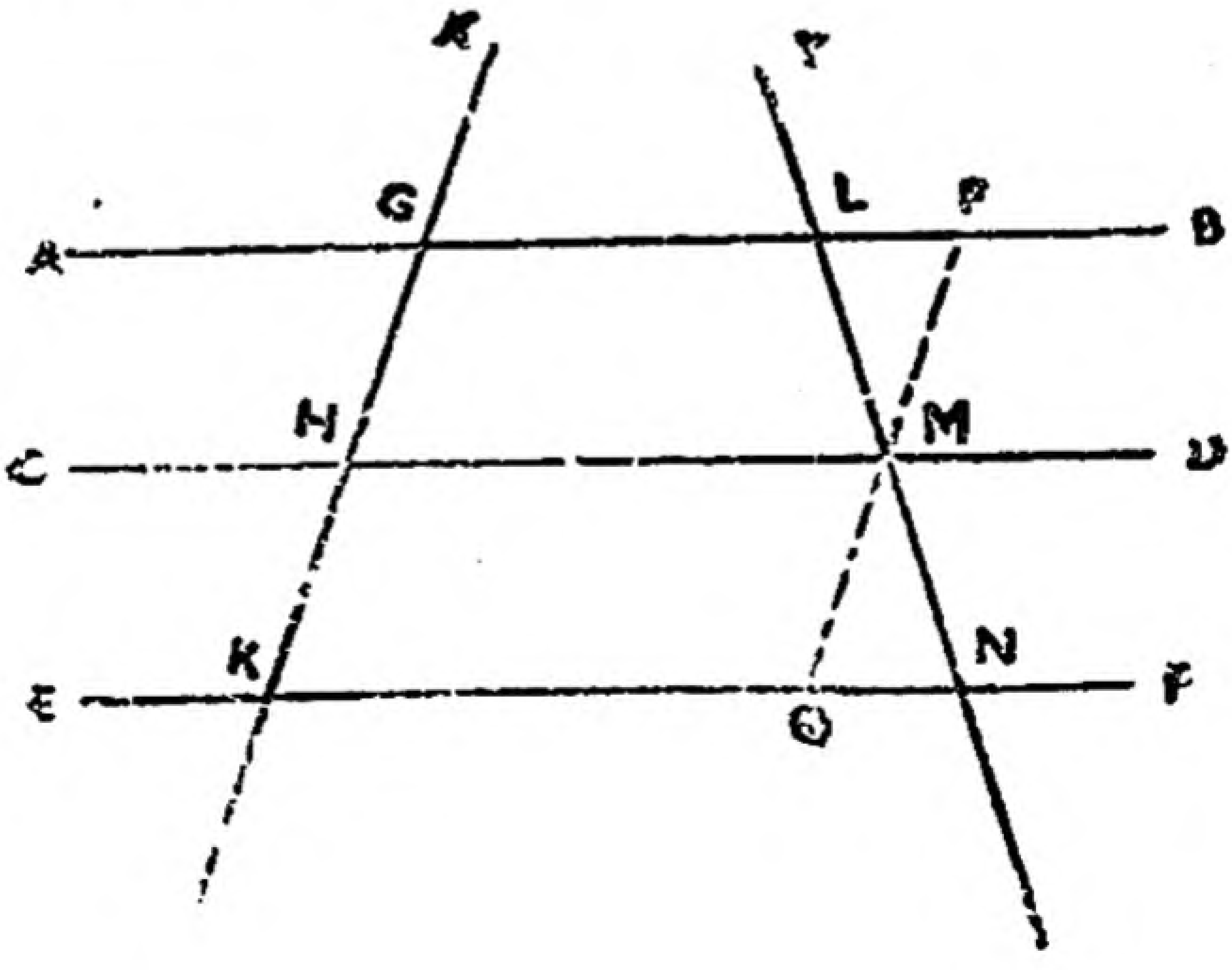
Def. When a transversal cuts two or more straight lines, it is divided into several portions. Any portion that lies between any two of those lines is called an **intercept** made by those lines on the transversal.

For instance PQ cuts lines L, M, and N at the points A, B and C and thus is divided into several portions such as PA, AB and CQ. Now the portion AB lying between the two lines L and M is called the intercept made by those two lines on the transversal.



Proposition 20. (*Theorem*).

If three or more parallel st. lines make equal intercepts on one transversal, they shall make equal intercepts on any other transversal also.



Given :—AB, CD and EF are three parallel st. lines which make equal intercepts GH and HK on the transversal X.

To prove :—They will make equal intercepts on Y also, *i.e.,* LM=MN.

Const :—Through M draw PMQ ∥ X.

Proof :—Since PM ∥ GH (Const.)

and GP ∥ HM (Hyp.)

∴ PGHM is a parallelogram.

Hence PM=GH (Opposite sides)

Similarly MQ=HK

But GH=HK (Hyp.)

∴ PM=MQ

Now in the two triangles LPM and MQN

PM=MQ (proved)

∠LMP=∠QMN (ver. opp.)

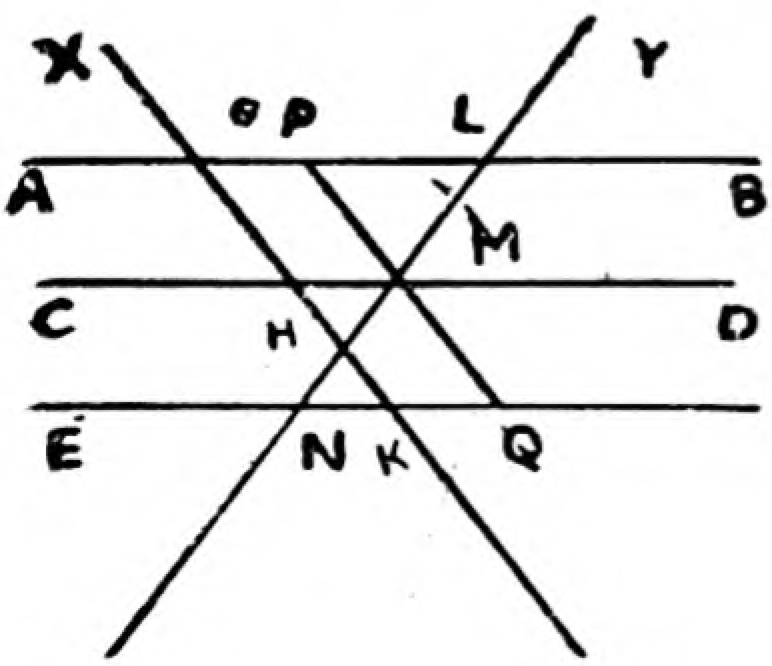
∠LPM=∠MQN (Alt.)

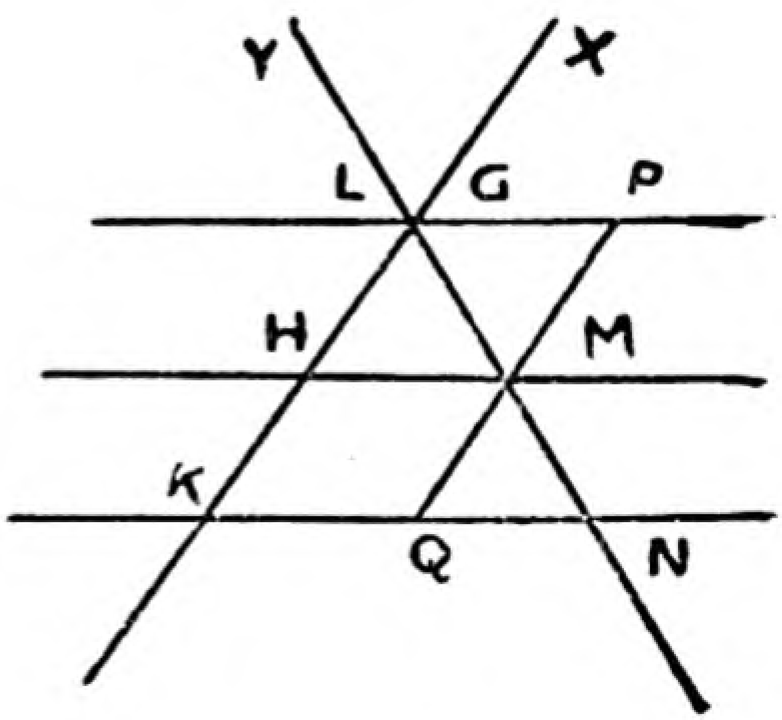
∴ △s are congruent.

Hence LM=MN.

Q. E. D.

Note :—The proposition can also be similarly proved when the transversals are situated as shown in the adjoining figures.





Method of Proof. The proposition has been proved by applying the theorem about congruent triangles.

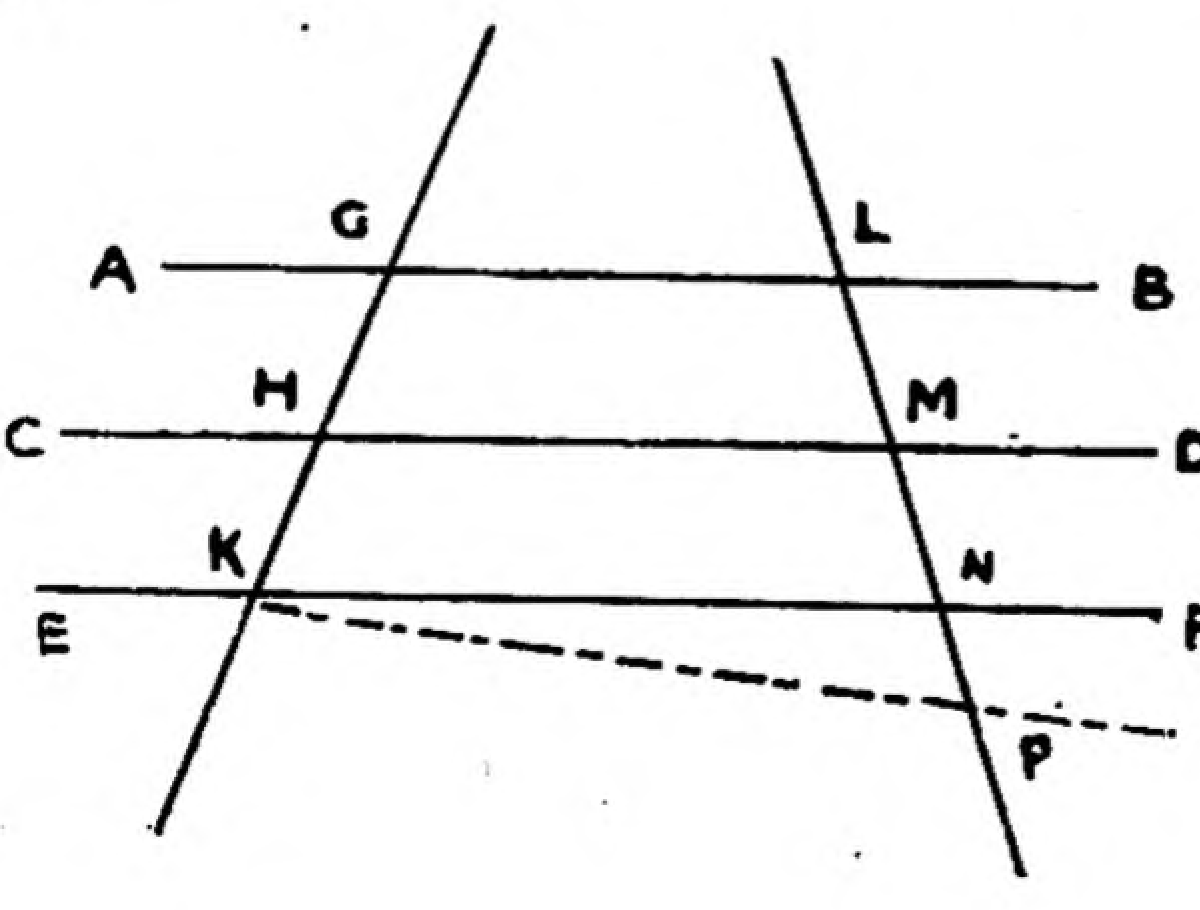
Application. As is evident from the theorems dependent upon it, the proposition has a very vast application. For one thing, it gives us a method of proving or making lines equal, while its converse is used to prove lines parallel.

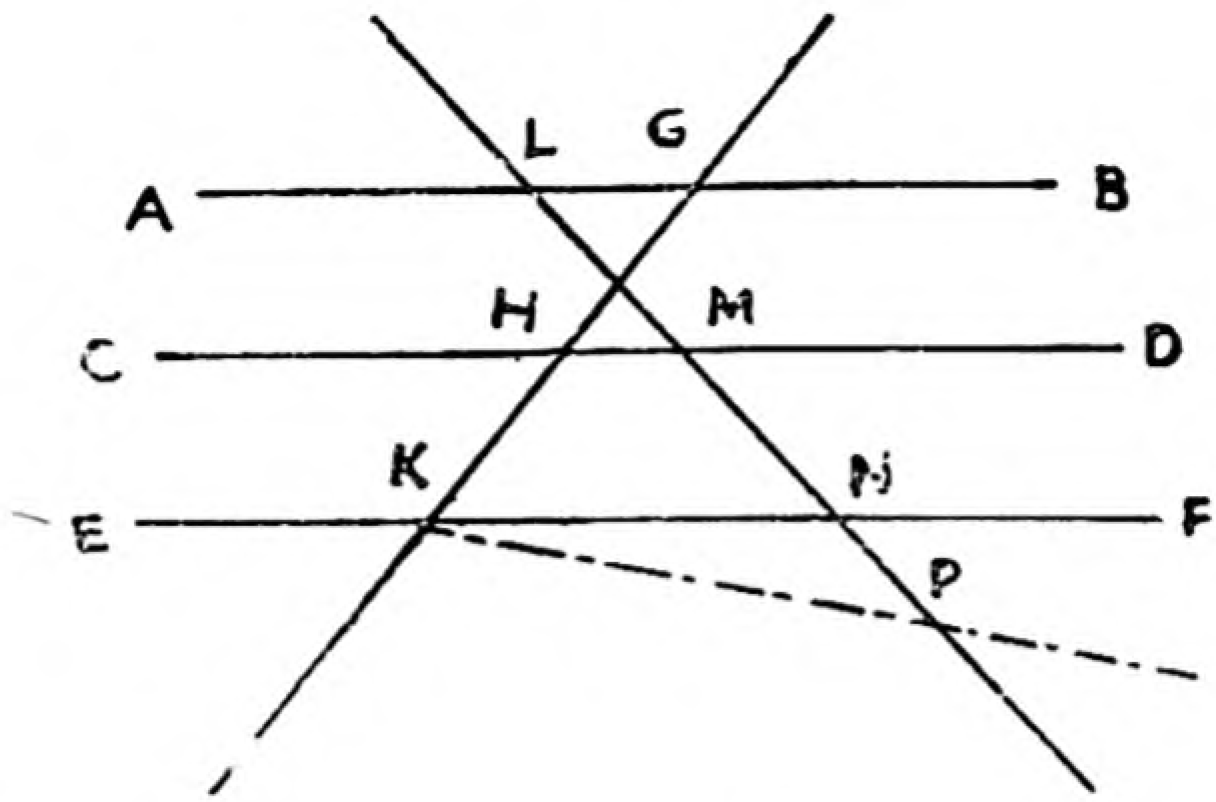
SOME STANDARD THEOREMS CONNECTED WITH PROPOSITION 20.

Theorem A.

(Converse of Prop. 20).

If three straight lines, of which two are parallel to each other, make equal intercepts on any two transversals drawn across them, the third straight line is also parallel to the other two.





Given. AB, CD and EF are 3 st. lines such that AB ‖ CD, GH=HK and LM=MN.

To prove. EF is also parallel to AB or CD.

Const. If EF is not parallel to AB or CD, draw KP‖AB cutting the transversal LM at P.

Proof. AB, CD and KP are 3 parallel st. lines which make equal intercepts GH and HK on GK.

∴ They shall make equal intercepts on LMP, also, that is LM=MP.

But LM=MN. (Hyp.)

∴ MN=MP, but this is impossible because a part cannot be equal to the whole.

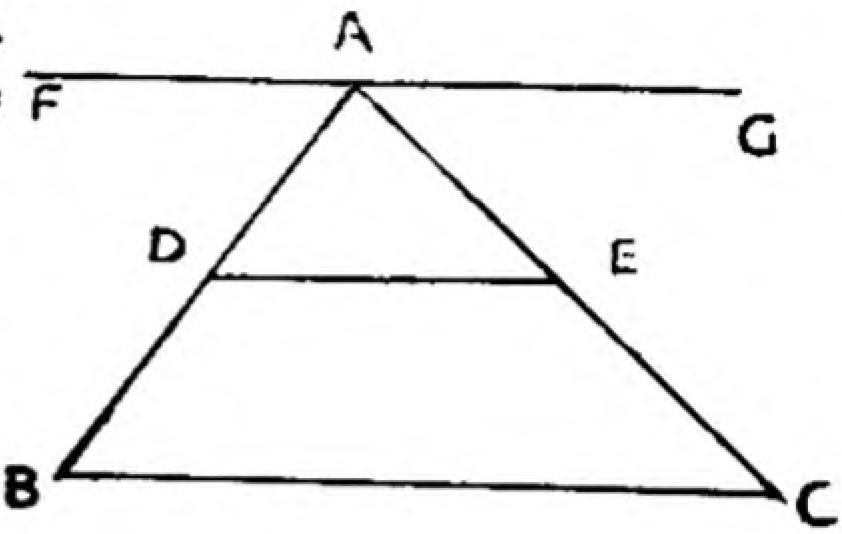
Hence our supposition is incorrect.

∴ EF ‖ AB or CD.

Q. E. D.

Theorem B.

The st. line drawn through the middle point of one side of a triangle parallel to the second side bisects the third side.



Given. ABC is a $\triangle$. DE $\parallel$ BC through D, the middle point of AB.

To prove. AC is bisected at E.

Const. Through A draw FAG $\parallel$ DE.

Proof. FG, DE and BC are 3 parallel straight lines which make equal intercepts AD and DB on AB. They shall make equal intercepts on AC also.

i.e. AE=EC

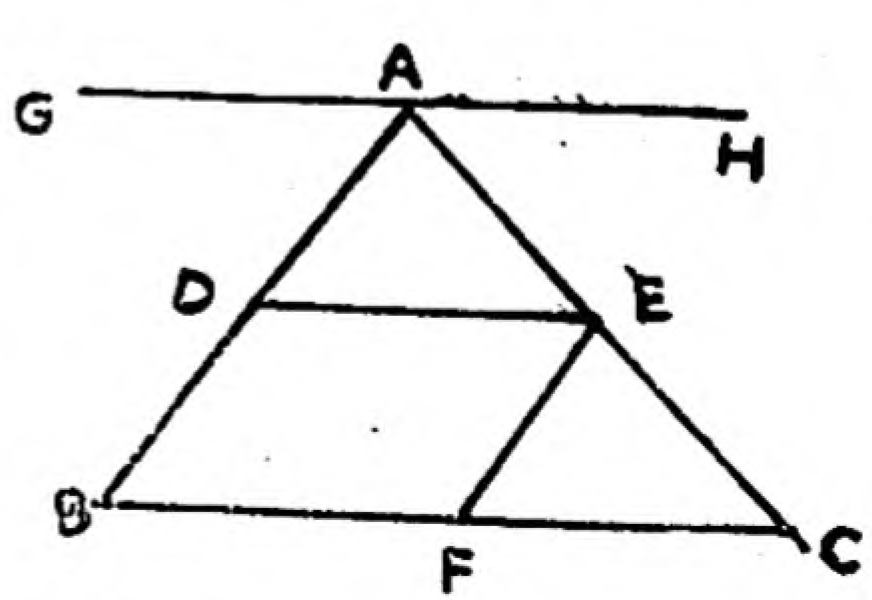
or AC is bisected at E.

Q. E. D.

Theorem C

(Converse of Theorem B.)

The straight lines which join the middle points of two sides of a $\triangle$ is parallel to the third side and is half of it.



Given. ABC is a triangle and DE is a st. line joining D and E, the middle points of AB and AC.

To prove DE $\parallel$ BC and DE$=\frac{1}{2}$BC.

Const. Through A draw GH $\parallel$ BC, and through E draw EF $\parallel$ AB.

Proof. GH, DF and BC are three st. lines such that

(*a*) GH $\parallel$ BC and (*b*) AD=DB and AE=EC

$\therefore$ DE $\parallel$ BC. (Theorem A)

$\because$ EF $\parallel$ AB through the middle point E of AC. (Const.)

$\therefore$ F is the middle point of BC or BF=$\frac{1}{2}$BC

But BF=DE (opposite sides of the $\parallel^{gm}$

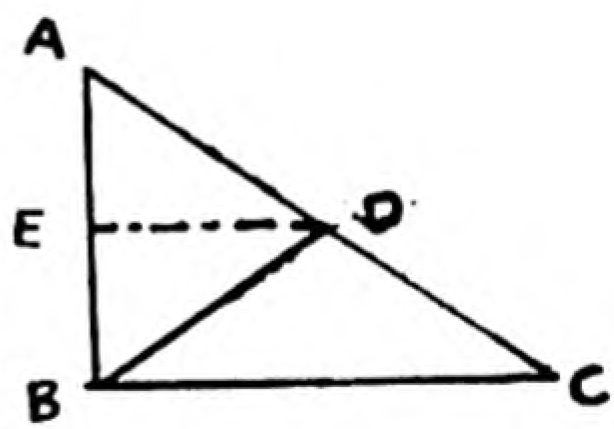
$\therefore$ DE=$\frac{1}{2}$BC.

Q. E. D.

Theorem D.

The median bisecting the hypotenuse of a rt. $\angle$d $\triangle$ is half the hypotenuse.

Given. ABC is a rt. $\angle$d $\triangle$ and BD is the median bisecting the hypotenuse AC.



To prove BD=$\frac{1}{2}$AC.

Const. Draw DE $\parallel$ BC, meeting AB in E.

Proof. Since DE $\parallel$ BC through D, the point of AC.

$\therefore$ E is the middle point of AB.

and $\angle$DEA=$\angle$CBA (corresp.)

=one rt. $\angle$ (Hyp.)

In the two $\triangle$s ADE and DEB.

AE=EB (proved)

DE=DE (common)

Contained $\angle$AED=con. $\angle$BED (each being a rt. $\angle$).

$\therefore$ Δs are congruent

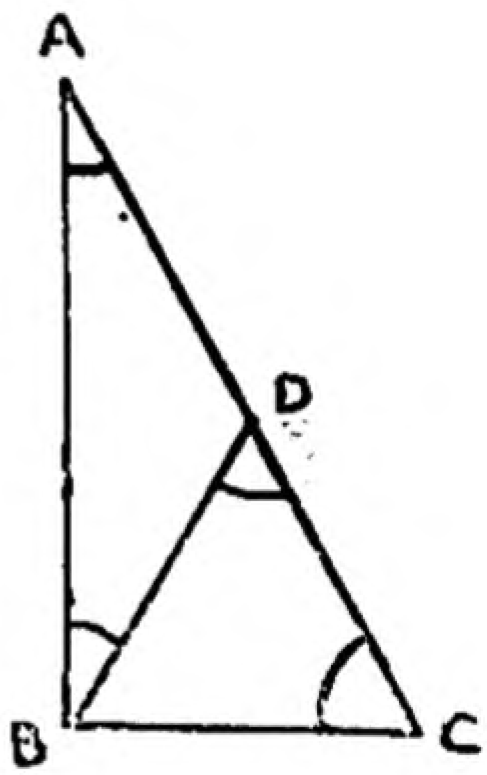
Hence BD$=$DA

But DA$=\frac{1}{2}$AC

$\therefore$ BD$=\frac{1}{2}$AC. Q. E. D.

Theorem E.

If one acute angle of a rt. $\angle$d $\triangle$ be double the other, the hypotenuse is double the shorter side.



Given. ABC is a rt. $\angle$d $\triangle$, right angled at B in which $\angle$C$=2\angle$A.

To prove AC$=$2BC.

Const. Draw BD, the median bisecting the hypotenuse AC.

Proof. BD$=\frac{1}{2}$AC$=$AD

$\therefore$ $\angle$A$=\angle$ABD

$\therefore$ Ext. $\angle$BDC$=\angle$A$+\angle$ABD

$=2\angle$A$=\angle$C (Hyp.)

$\therefore$ BD$=$BC

But BD$=\frac{1}{2}$AC (Theorem D)

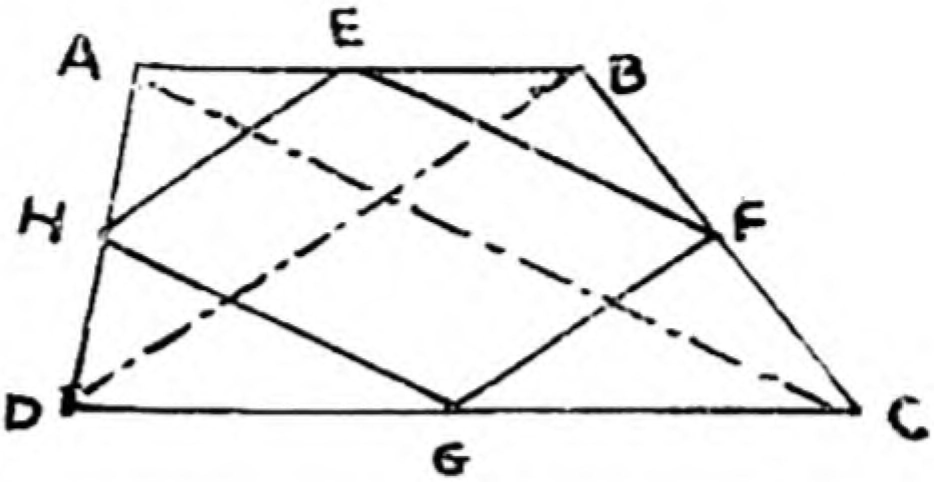
$\therefore$ BC$=\frac{1}{2}$AC

Or AC$=$2BC

Q. E. D

Theorem F.

The straight lines joining the middle points of the adjacent sides of a quadrilateral form a parallelogram.



Given. ABCD is a quadrilateral ; H, E, F, and G are its middle points.

To prove. EFGH is a parallelogram.

Const. Join AC and BD.

Proof. In the $\triangle$ ABD, EH joins the middle points of AB and AD.

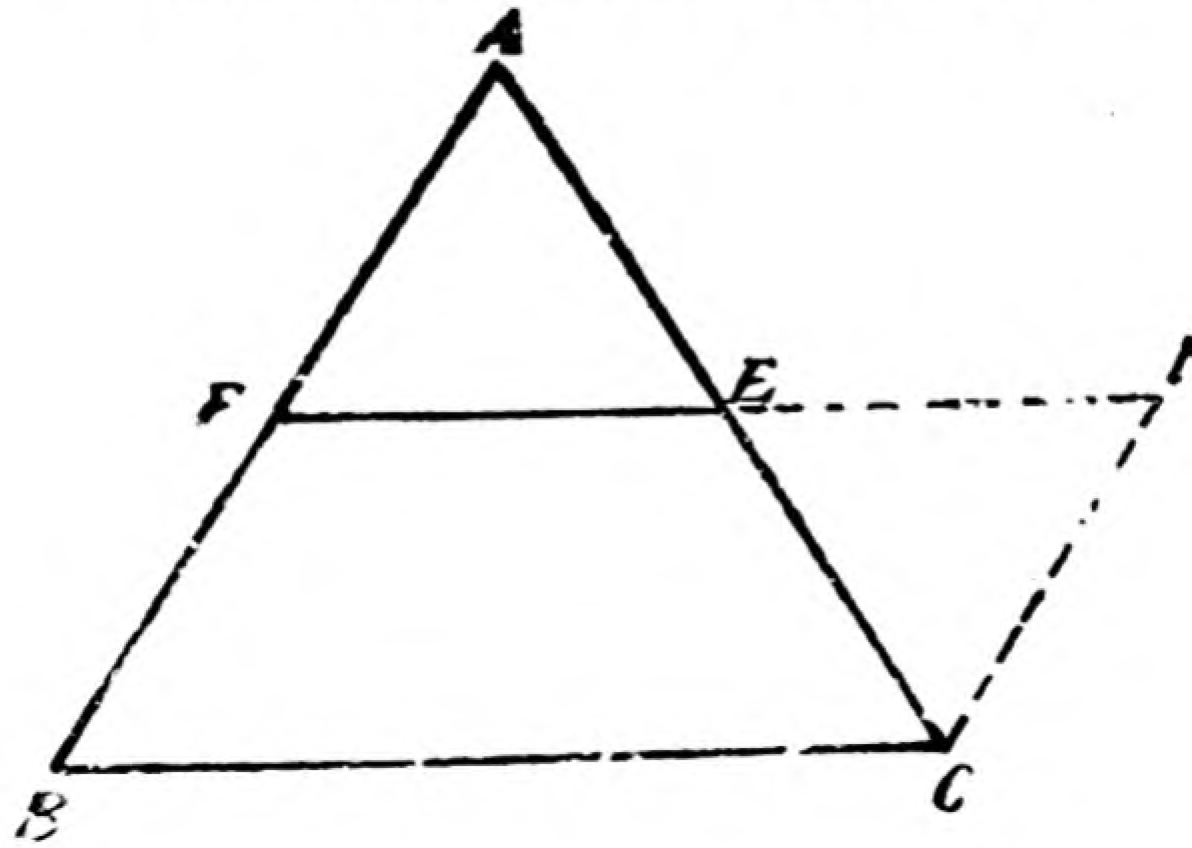
$\therefore$ EH $\parallel$ BD. Similarly FG $\parallel$ BD.

Hence EH $\parallel$ FG.

Similarly EF $\parallel$ HG.

$\therefore$ EFGH is a parallelogram. **Q. E. D.**

Alternative Proof of Theorem C.



Given :—FE, joining the mid-points of the sides AB, AC of $\triangle$ABC.

Required :—To prove that (1) EF $\parallel$ BC.

(2) EF$=\frac{1}{2}$BC.

Construction :—Through C draw CK $\parallel$ BA cutting FE, produced at K.

Proof :—In the $\triangle$s AEF and CEK,

$$\therefore \begin{cases} \angle FAE = \angle ECK & (\because AB \parallel CK) \\ \angle AEF = \angle CEK & (\text{Vert. opp. } \angle s). \\ AE = EC & (\text{Given}). \end{cases}$$

$\therefore$ $\triangle AEF \equiv \triangle CEK$

$\therefore$ FE$=$EK$=\frac{1}{2}$FK.

Also CK$=$FA$=$BF, and CK $\parallel$ BF.

$\therefore$ FK and BC are $\parallel$ and equal.

$\therefore$ FE $\parallel$ BC and $=\frac{1}{2}$BC.

Q. E. D.

Exercises.

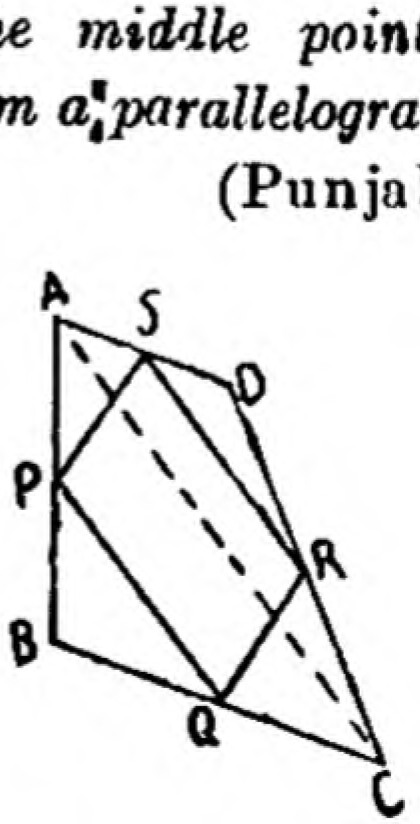
1. All the st. lines drawn from the vertex of a $\triangle$ to points in the base are bisected by the straight line joining the middle points of the sides of the $\triangle$.

2. D, E, F are the middle points of the sides BC, CA, AB of a $\triangle$ABC. Prove that BFED, CEFD, AEDF are parallelograms.

3. *The st. lines joining the middle points of the adjacent sides of a quadrilateral form a parallelogram.*

(Punjab, 1918).

Hint. PQ $\parallel$ AC and$=\frac{1}{2}$AC.



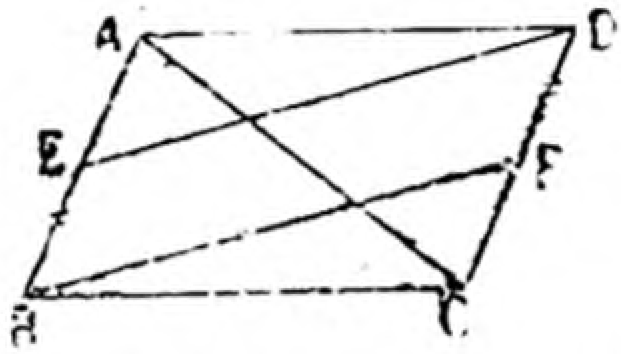
4. *The straight lines joining the middle points of the opposite sides of a quadrilateral bisect each other.*

(Bombay Sch. Final).

Hint.—Apply Ex. 3 above.

5. ABCD is a parallelogram, E and F are the mid-points of AB and CD respectively; prove that BF and ED trisect AC. (Bombay, 1924).

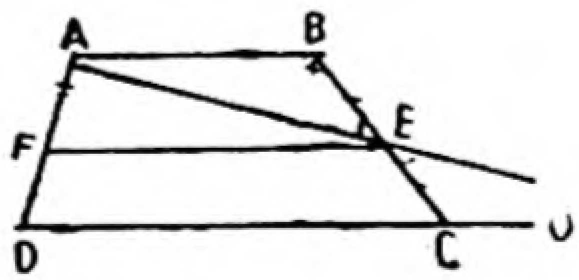
Hint.—DF and BE are = and ∥. ∴ BF ∥ ED and they trisect AC (Prop. 20 cor. 1).



6. *The st. line joining the mid-points of the non-parallel sides of a trapezium is parallel to the parallel sides and is equal to half the sum of parallel sides.*

(Calcutta, 1887)

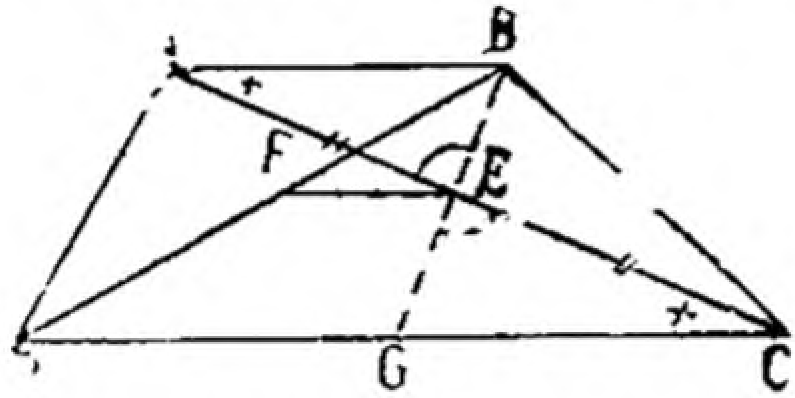
Hint.—E and F are mid-points of BC, AD (non = ∥ sides) of trapezium ABCD; then EF is ∥ AB or CD, and $=\frac{1}{2}$ (AB+CD), join AE and produce it to meet DC produced in G. △s ABE and CEG are congruent. ∴ CG=AB, and AE=EG and DG=DC+AB and EF ∥ and $=\frac{1}{2}$ DG (Prop. 20 Cor. 2).



7. *The st. line joining the mid-points of the diagonals of a trapezium is* ∥ *to the parallel sides and is equal to half the difference of those sides.*

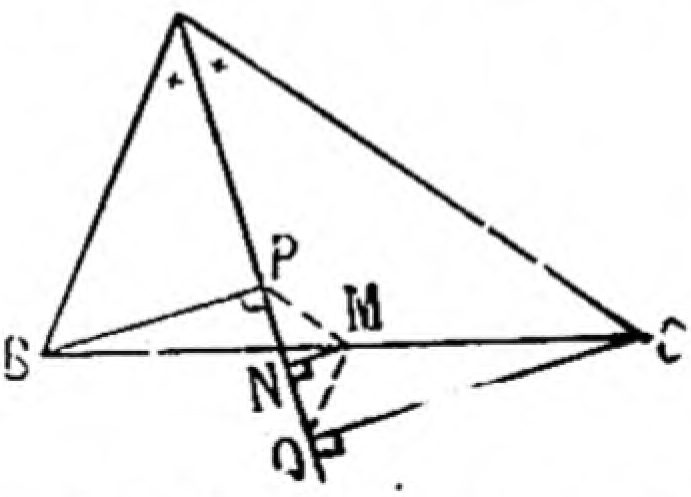
Hint.—In trapezium ABCD (AB ∥ CD), E, F, mid-points of AC and BD. Then EF ∥ DC and $=\frac{1}{2}$(DC−AB); join BE and produce it to meet CD in G. △s ABE and CEG are congruent.

∴ BE=EG and AB=CG ∴ DG=DC=CG=DC −AB. ∴ EF ∥ DG and $=\frac{1}{2}$ DG. (Prop. 20 Cor. 2).



8. From the extremities of the base of a △ perpendiculars are drawn to the bisector of the vertical angle; prove that the feet of the perpendiculars are equidistant from the mid-point of the base.

Hint.—Let P, Q be the ft. of ⊥s from B and C the ends of the base of △ABC, upon the bisector of the vert. angle; join P and Q to M the mid-point of BC; let MN be ⊥ to the bisector. Then CQ, MN, BP are ∥ s. Intercept BM=intercept CM (given) ∴ PN=QN. (Prop.20) ∴ △s MNP and MNQ are congruent (Prop. 3) ∴ PM=QM.

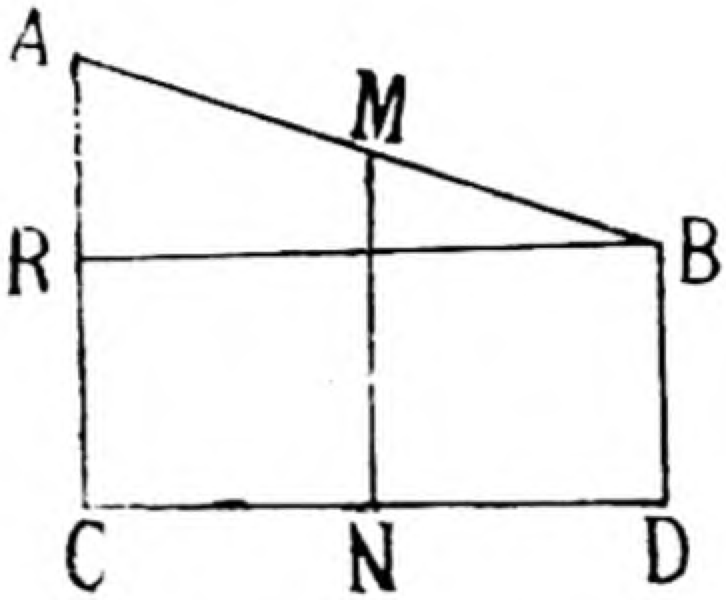


9. In any triangle ABC, BP and CQ are perpendiculars on any line through A, and M is the middle point of BC. Show that MP=MQ.

(Bengal, Eur. Sch.)

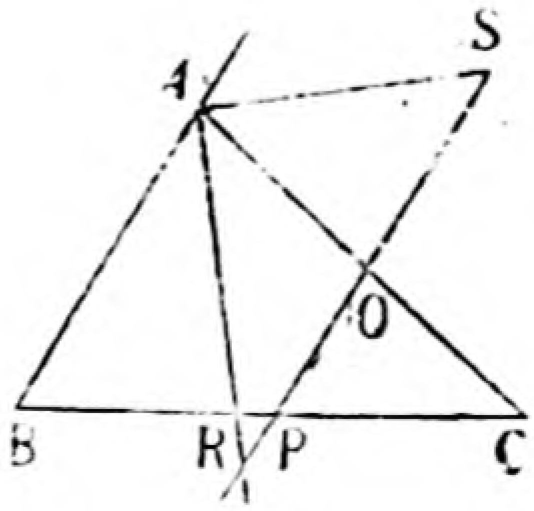
Hint.—Pr oof same as that of Ex. 8.

10. AB and CD are two non-intersecting st. lines. Prove that the sum of the ⊥s from A and B on CD is equal to twice the perpendicular from the middle point of AB on CD. (Bombay, 1909).



11. If through the middle point of the base of a △ a st. line is drawn parallel to one of the sides, prove that its intercept on the internal and external bisectors of the vertical angle is equal to the other side. (Madras, F. E.)

Hint.—Through P, the mid-point of base BC of a △ ABC, suppose PS drawn ∥ to AB cutting AC in Q and internal and external bisectors of vertical ∠A in R and S. Then Q is the mid-point of AC (Prop. 20 Cor. 1). Also SQ= QA=RQ (Prop. 13 Ex. 8) ∴ RS=2 AQ=AC.



Some useful instructions for practical work.

Before attempting any practical work, students should carefully note the following instructions. If they are followed properly, they will not only ensure neatness and accuracy of drawing and measurement, but will also make the work considerably easier and improve it greatly both in quality and quantity.

1. Rule all lines from *left to right.*

2. Make every line of the *same* thickness throughout.

3. Make some distinction between the given lines, found lines, lines of construction and lines required for the proof. It would be advisable to have given lines *thick*, found lines *thicker*, construction lines *fine* and lines required for the proof *dotted.*

4. In order to draw fine lines sharpen the pencil and compass leads to a fine point.

5. Show *all construction lines* very carefully.

6. Hold your compasses by the head only and before using them see carefully that *the two arms are equal in length.*

7. Of the two sets of numbers marked on the protractor, *take care to use the right one.*

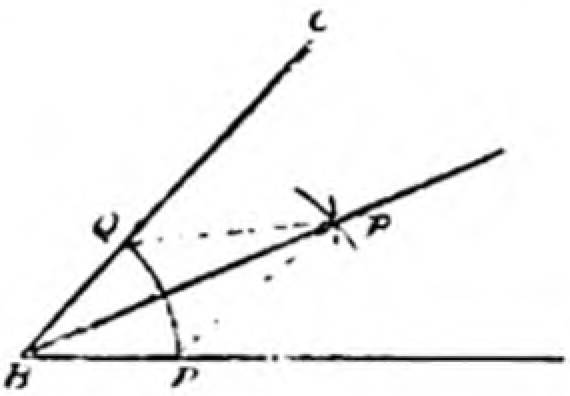
8. When taking lengths by means of the dividers open out their arms by pressing them from within until their points are at a slightly greater distance than that required and then press the arms together.

9. When applying the dividers to the scale, hold them flat to avoid injury to the scale by the points.

10. Always tesc the accuracy of your drawing by actual measurement.

Proposition 21. (*Problem*).

To bisect a given angle



Given :—$\angle ABC$.

Required :—To bisect it.

Construction :—1. With centre B and any radius draw an arc PQ, cutting BA and BC at P and Q respectively.

2. With centres P and Q and any convenient radius draw arcs intersecting at R.

Join BR. Then BR bisects $\angle ABC$.

Proof .—Join Pr and QR.

In the $\triangle PBR$ and QBR

$$\because \begin{cases} PB = QB & \text{(Const).} \\ BR \text{ is common} & \\ RP = RQ & \text{(Const).} \end{cases}$$

$\therefore \quad \triangle PBR \equiv \triangle QBR$

So $\angle PBR = \angle QBR$

$\therefore$ BR bisects $\angle B$.

Q. E. F.

Note—Any convenient radius means that the radii PR and QR should be equal but long enough to enable the arcs to intersect at R This will always be the case when their length is greater than half PQ.

Exercises.

1. Draw an angle of 60° and divide it into two equal parts ; check the result by measurement.

2. Draw an angle of 128° and divide it into 4 equal parts. Hence show that an angle can be divided into 2^n equal parts where n is *any* integer, odd or even.

3. Bisect a straight angle. Show by measurement that each half of the angle is a right angle.

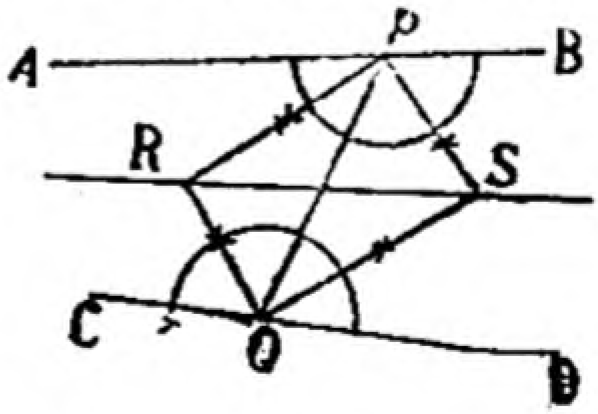
4. Divide an angle into two parts such that one part is $\frac{1}{3}$ of the other.

5. *Divide an angle into two parts in the ratio of* 3 : 5.

6. Draw the bisectors of the angles of a given triangle. (Note that they meet in a point).

7. Find a point on a given straight line equidistant from two given intersecting lines.

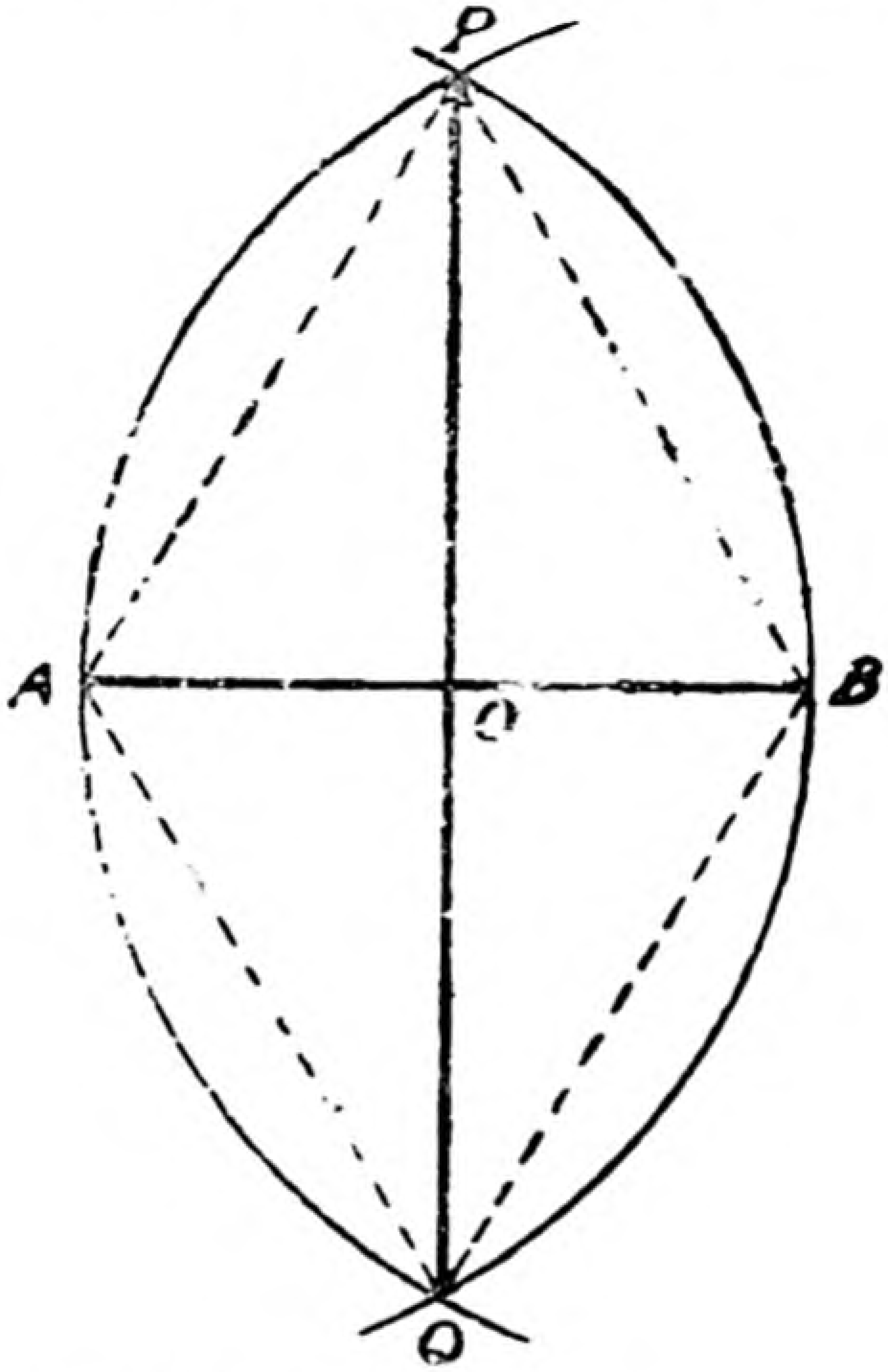
8. *Draw the bisector of the angle between two given st. lines AB and CD where pt. of intersection is inaccessible.*



Hints—Draw any st. line cutting AB in P and CD in Q. Bisect ∠s APQ, CQP by lines meeting in R. Bisect ∠s BPQ, DQP by lines meeting in S. Join RS. Then RS is the reqd. bisector.

Proposition 22 (*Problem*)

To bisect a given straight line.



Given :—A st. line AB.

Required :—To bisect it.

Construction :—With A and B as centres and any nvenient radius draw arcs intersecting at P and Q.

Join PQ cutting AB at O.

Then AB is bisected at O.

Proof :—Join AP, BP, AQ and BQ.

In the $\triangle$s PAQ and PBQ

$$\because \begin{cases} PA=PB & \text{(Const.)} \\ AQ=BQ & \text{(Const.)} \\ QP \text{ is common.} \end{cases}$$

$\therefore$ $\triangle$s are congruent, and $\angle APQ = \angle BPQ$.

Again in $\triangle$s APO and BOP

$$\therefore \begin{cases} AP = BP & \text{(Const.)} \\ PO \text{ is common.} & \\ \angle APO = \angle BOP. & \text{(Proved).} \end{cases}$$

$\therefore$ The $\triangle$s are congruent.

Hence $AO = OB$.

Q. E. D.

Note.—Take the radius AP greater than half AB, otherwise the arcs will not intersect.

Note.—The $\angle$s at O are rt. $\angle$s hence this problem gives the **perpendicular bisector** of AB.

Exercises

1. Divide a line 4″ long into two equal parts.

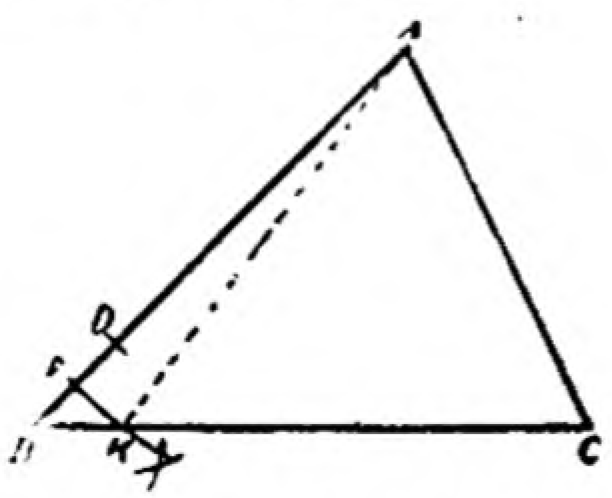
2. Divide a line 3″ long into two parts so that one part is three times the other. Find by measurement the length of each part.

3. Find graphically the value of $\frac{3}{4}$ of 7·2 cm.

4. Draw the perpendicular bisectors of the sides of a given $\triangle$.

5. Draw the medians of a given triangle.

6. Find a point K in the base BC of a triangle ABC such that $AK = \frac{1}{2}(AB + AC)$.



In the $\triangle ABC$ let AB be $>$ AC.

From AB cut off AD=AC.

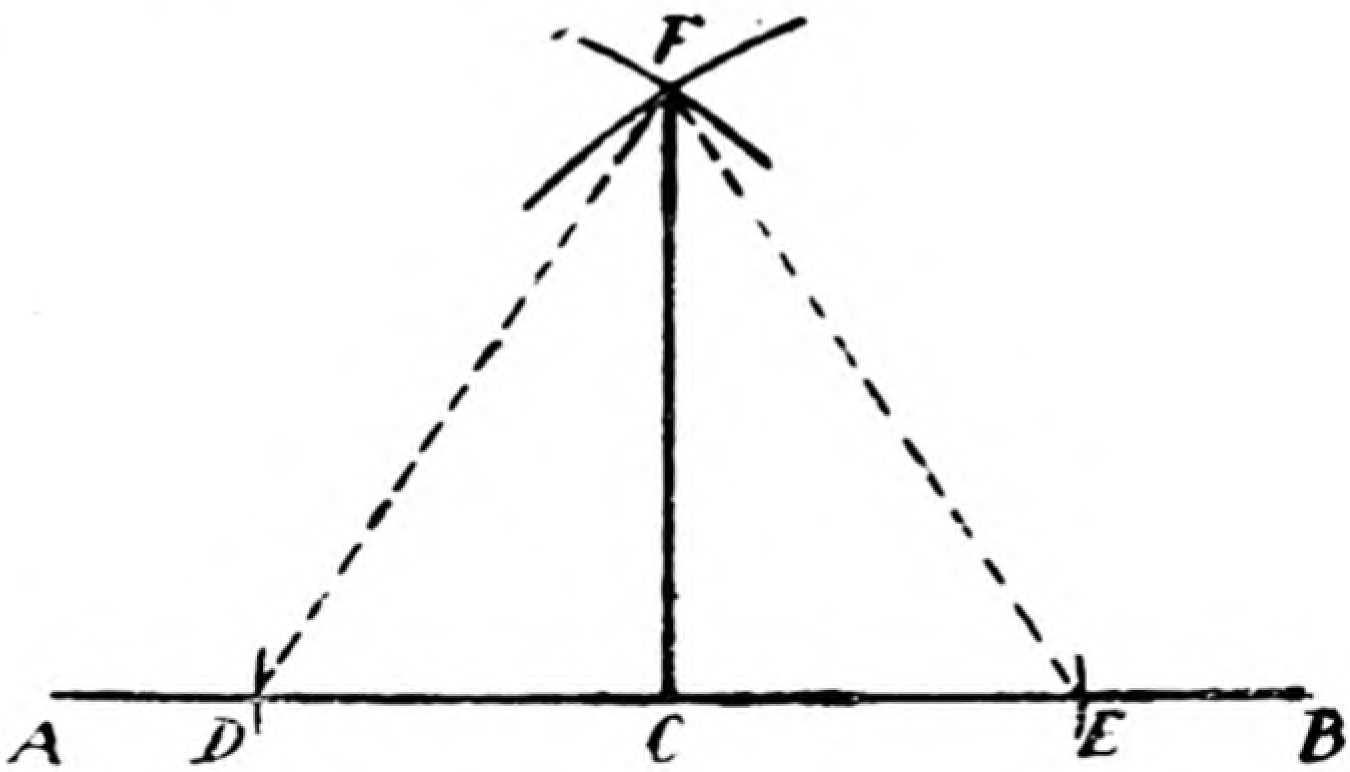
Bisect BD at F.

With centre A and radius equal to AF draw an arc cutting BC at K. Then K is the required point.

Proof :—(AB+AC)=(AF+FB+AF−FD).
=2AF=2AK. Hence, etc.

Proposition 23. (*Problem*)

To draw a perpendicular to a given st. line, at a given point in the line.



Given :—AB a st. line and C a point in it.

Required :—To draw a st. line through C ⊥ to AB.

FIRST METHOD.

Construction :—1. With centre C and any radius mark off equal distances CD and CE on AB.

2. With centres D and E and radius greater than DC, describe arcs intersecting at F.

3. Join FC.

Then FC is the required perpendicular.

Proof :—Join DF and EF.

In the △ DCF and ECF.

$\because \begin{cases} DC = EC. & \text{(Const.)} \\ CF \text{ is common.} \\ FD = FE \text{ (radii of equal circles.)} \end{cases}$

∴ The △s are congruent.

Hence ∠DCF=∠ECF.

Being adjacent supp. angles, each one is a right angle.

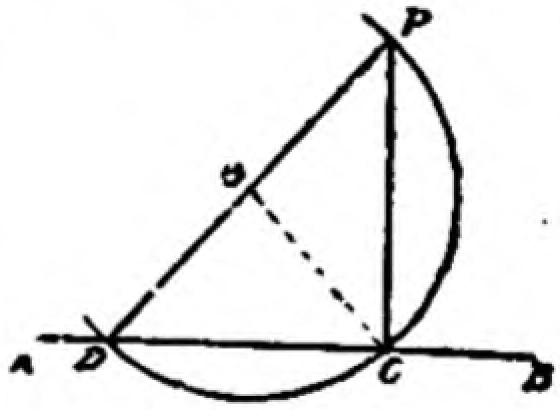
∴ FC ⊥ AB.

Q. E. F.

This method should preferably be adopted when the pt. is well within the line from both ends.

SECOND METHOD.

Construction :—Take any point O outside th given line. Join OC. With centre O and radius OC describe a circle cutting AB at D. Join DO and produce it to cut the circle at P.



Join PC. Then PC is the required ⊥

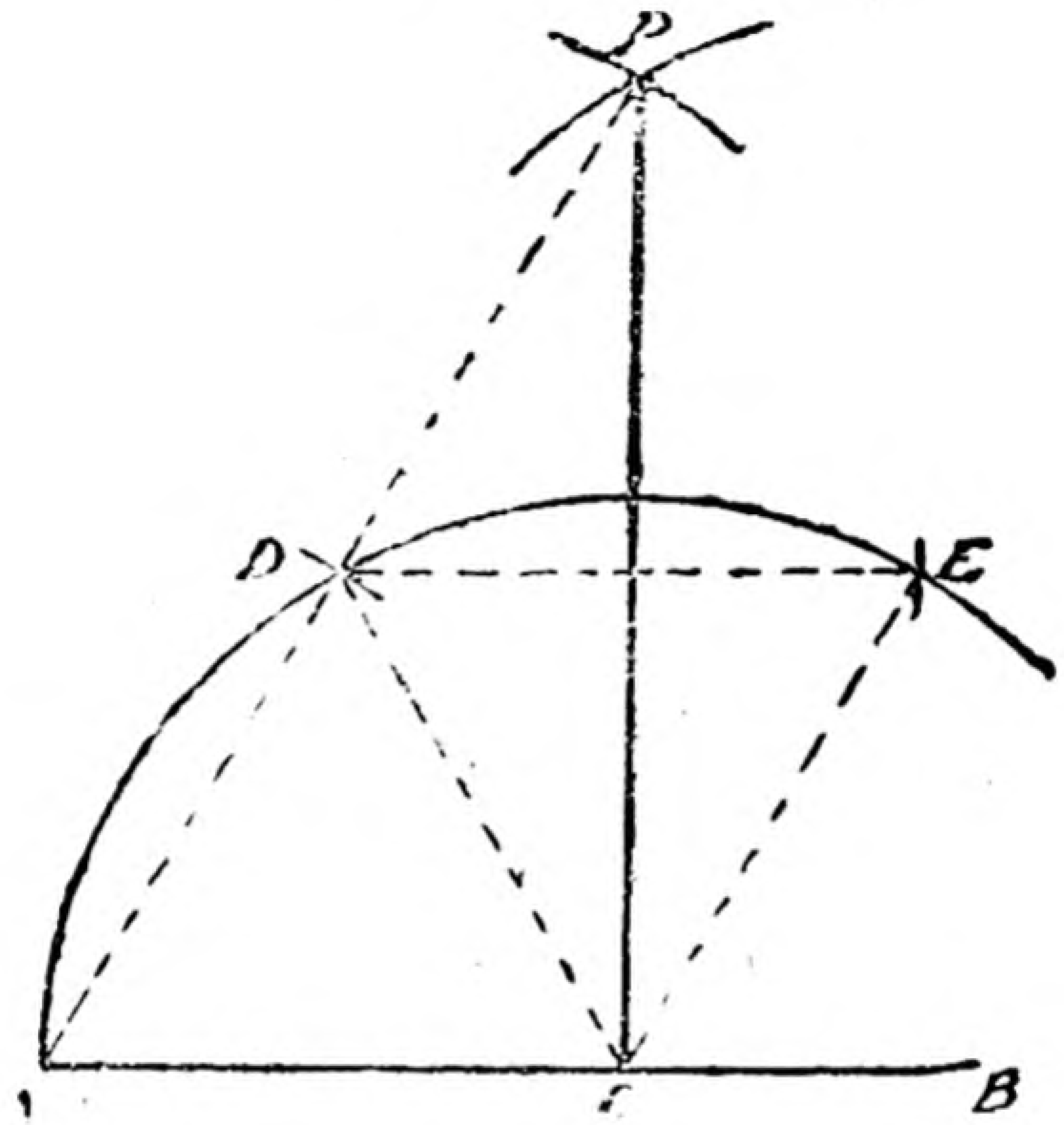
Proof :—∵ PCD is in a semi ⊙,

∴ It is a rt. ∠.

Hence PC ⊥ AB.

THIRD METHOD.

Construction :—With centre C and any radius



draw a ⊙ ADE cutting AB at A. With centre A and the same radius draw an arc cutting the ⊙ ADE at D.

With centre D and the same radius draw an arc cutting the ⊙ ADE at E.

With D and E as centres and the same radius draw arcs intersecting at P.

Join PC. Then PC is the required $\perp$.

Proof :—$\triangle$s ACD, DCE are equilateral.

$\therefore$ $\angle ACD = \angle CDE = 60°$.

Also $\angle DCP = \angle ECP$ and $DC = DP$. (Const.)

Now $\angle CDE + \angle EDP = 120°$.

$\therefore$ each of $\angle$s DPC and DCP is $30°$.

$\therefore$ $\angle ACP = \angle ACD + \angle DCP = 60° + 30° = 90°$ or PC is $\perp$ to AB.

Note.—This last method is more practicable when the given point is near one end of the line.

Exercises.

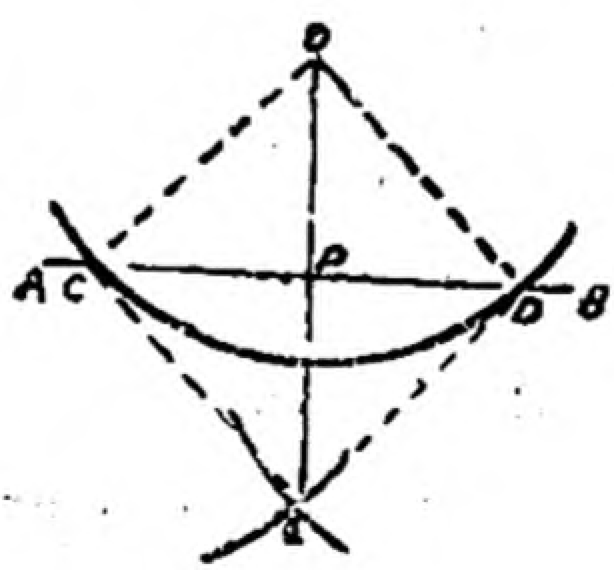
1. Draw a straight line 3″ long and erect a $\perp$ to it at a point which lies on it at a distance of 1 2″ from one of its ends.

2. Draw a st. line at right angles to a given line 5 cm. in length, at one of its extremities without producing the given line.

3. *Draw angles of* 45°, $22\frac{1}{2}$°, $67\frac{1}{2}$° *and* 135° with only ruler *and compasses.*

Proposition 23. (*Problem*).

To draw a straight line perpendicular to a given straight line from a point outside it.



Given :—The st. line AB and a point O outside it.

Required :—To draw a $\perp$ from O to AB.

FIRST METHOD.

Construction:—1. With First Method any convenient radius describe an arc cutting AB at C and D.

2. With centres C and D and any radius describe arcs intersecting at E.

3. Join OE cutting AB at P.

Then OP is the required perpendicular.

Proof:—Join OC, OD, EC and ED.

In the $\triangle$s COE, DOE.

$$\because \begin{cases} CO=DO & \text{(Const.)} \\ EC=ED & \text{(Const.)} \\ OE \text{ is common.} & \end{cases}$$

$\therefore$ The $\triangle$s are congruent.

Hence $\angle COE=\angle DOE$

Again, in the $\triangle$s COP, DOP

$$\because \begin{cases} CO=DO & \text{(Const.)} \\ OP \text{ is common} & \\ \angle COP=\angle DOP & \text{(Proved).} \end{cases}$$

$\therefore$ The $\triangle$s are congruent.

Hence $\angle CPO=\angle DPO$.

But they are adjacent supp. angles. Hence each is a rt. angle.

$\therefore$ $OP \perp AB$

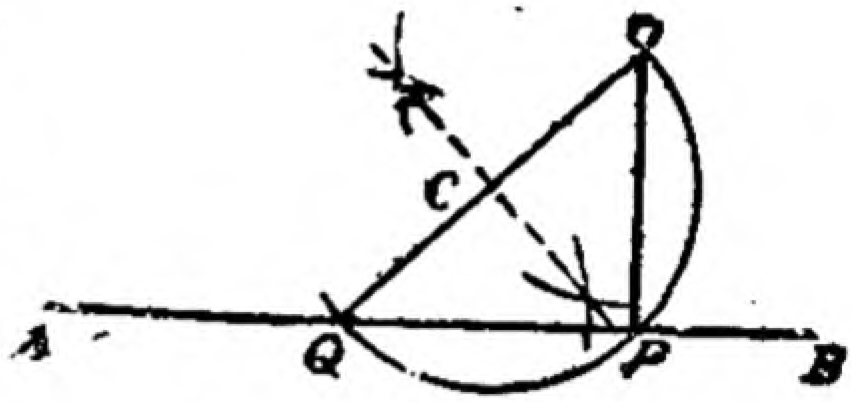
Q. E. F.

SECOND METHOD.

Construction :—Take a point Q in AB.

Join OQ. Bisect OQ at C.

With centre C and radius CO describe a circle cutting AB in P and Q. Join OP.



Then OP is the required perpendicular.

∴ ∠OPQ is in a semi ⊙

Proof :—∵ ∠OPQ is in a semi-⊙

∴ It is a rt. angle.

Hence OP ⊥ AB.

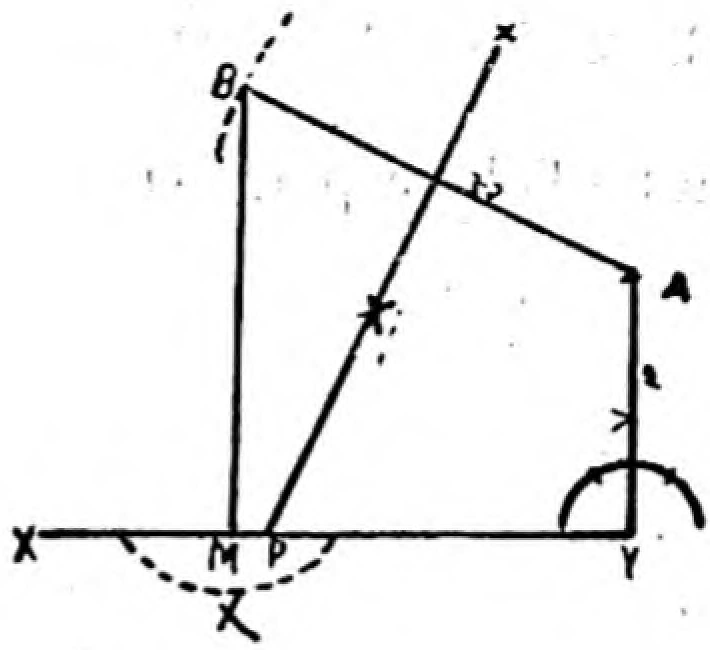
Q. E. F.

Exercises.

1. Draw perpendiculars from the vertices of a given triangle on the opposite sides. (Observe that the three ⊥s meet in a point.)

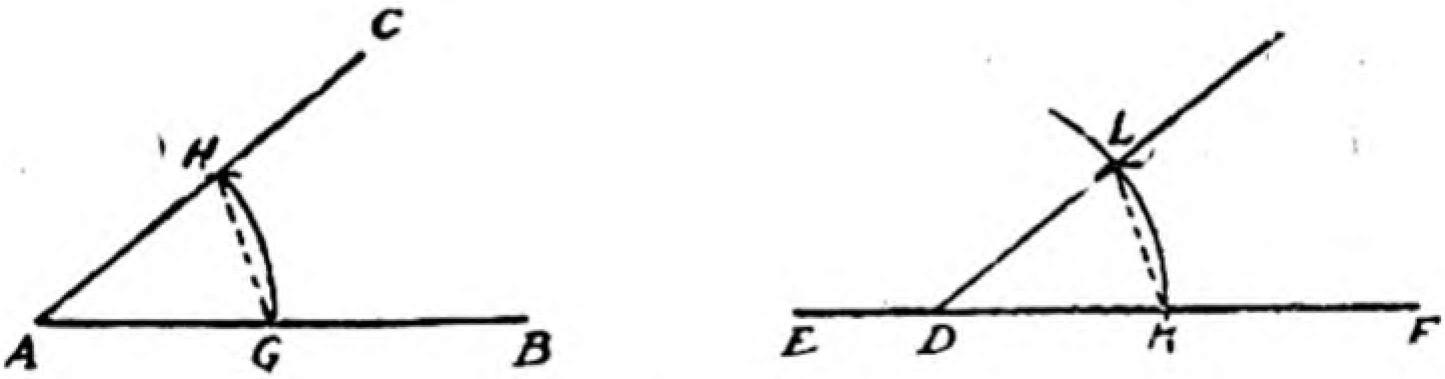
2. Draw two intersecting straight lines OP and OQ 5·4 cm. and 4·6 cm. long. Join PQ. Find by measurement the length of the perpendicular from O on PQ.

3. A and B are two points 3·2" apart. Distance of A from a given line XY is 2·1". Find a point in XY equidistant from A and B; also measure the distance of B from XY.



Proposition 25. (*Problem*).

At a given point in a straight line to make an angle equal to a given angle.



Given :—An angle BAC, a st. line EF and a point D in it.

Required :—To construct at D an $\angle$ equal to $\angle$BAC.

Construction :—1. With centre A and any radius describe an arc GH cutting AB, AC at G and H respectively.

2. Join GH.

3. With centre D and the same radius describe an arc KL cutting DF at K.

4. With centre K and radius equal to GH describe an arc cutting the arc KL at L.

5. Join DL.

Then KDL is the required angle.

Proof :—Join KL.

In the $\triangle$s GAH and KDL.

$$\because \begin{cases} GA = KD & \text{(Const.)} \\ AH = DL & \text{,,} \\ HG = LK & \text{,,} \end{cases}$$

$\therefore \quad \triangle GAH \equiv \triangle KDL.$

Hence $\angle A = \angle D$.

Q. E. F.

Exercises

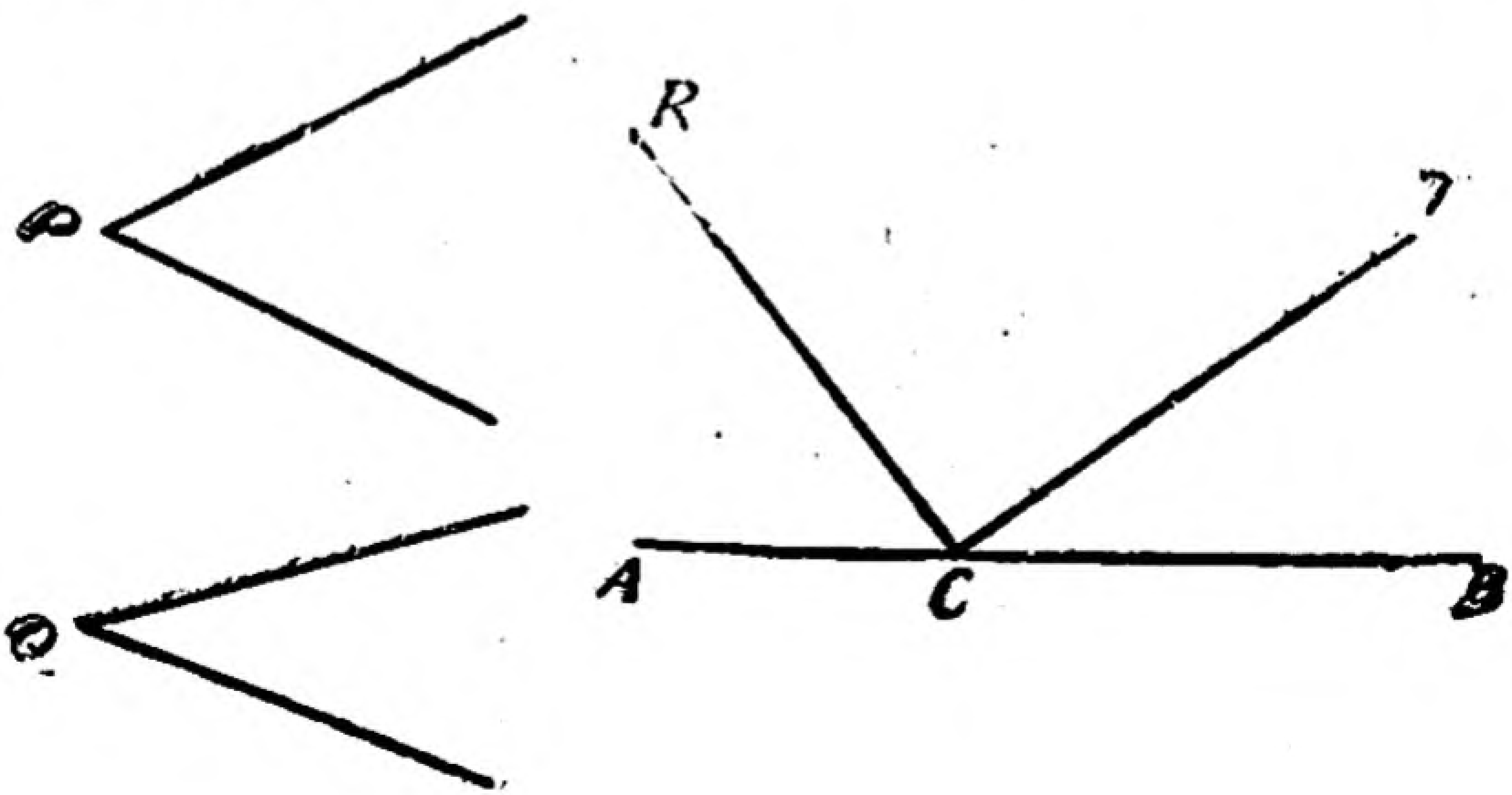
1. On a straight line 5·7 cm. long make a triangle equiangular to a given triangle.

. At one point of a given straight line make an angle equal to the supplement of a given angle.

3. In the base (produced, if necessary) of a triangle, find a point equidistant from the vertex and one of the other corners.

4. Show how to divide a right-angled triangle into two isosceles triangles.

5. *Construct the third angle of a triangle when any two are given.*



Let the two given angles be P and Q. At any point C in a line AB make angles RCA and TCB equal respectively to the given angles P and Q.

The sum of the angles of a triangle being equal to two right angles, $\angle P + \angle Q +$ third angle $= 180°$, also

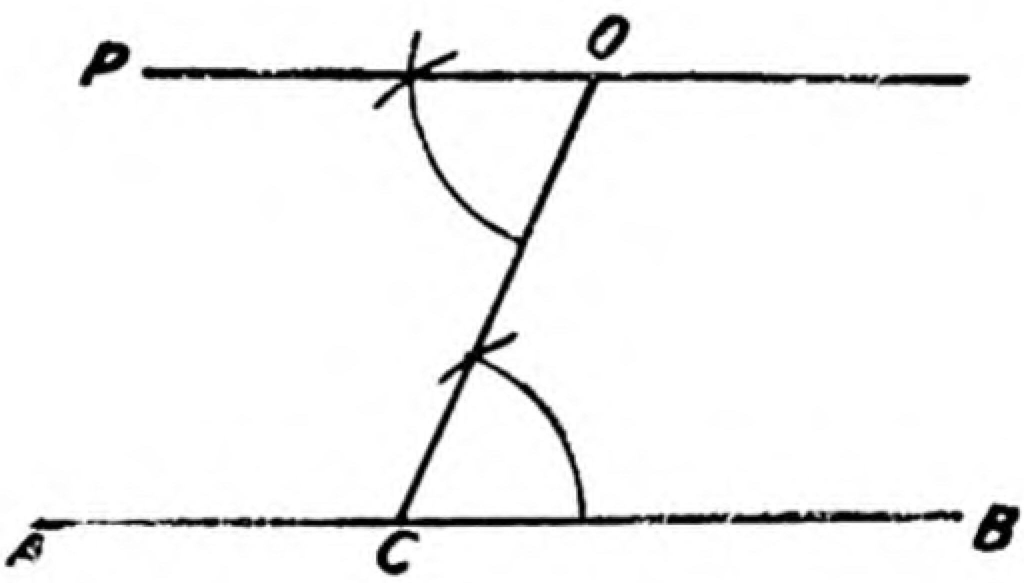
$$\angle RCA + \angle TCB + \angle RCT = 180°.$$

But $\angle P = \angle RCA$, $\angle Q = \angle TCB$.

$\therefore$ Third angle $=$ RCT.

Proposition 26 (*Problem.*)

Through a given point to draw a straight line parallel to a given straight line.



Given :—A point O and a st. line AB.

Required :—To draw through O, a st. line $\parallel$ to AB.

Construction :—1. Take any point C in AB.

2. Join CO.

3. At O in CO make $\angle COP = \angle OCB$.

Then PO $\parallel$ AB.

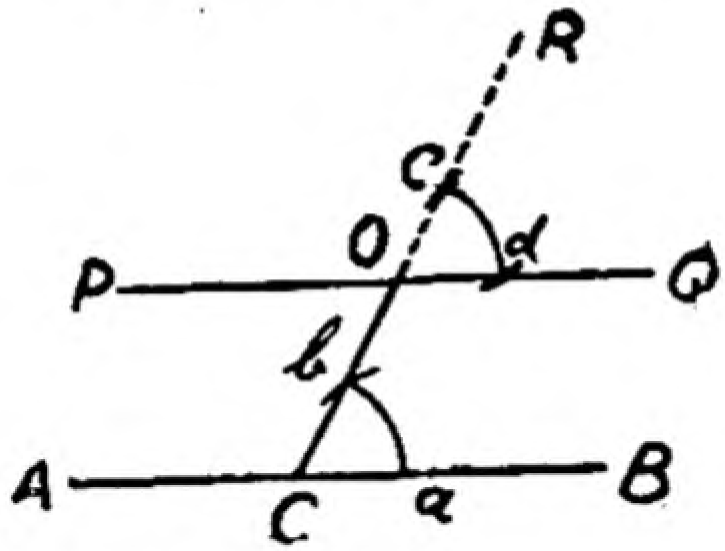
Proof :—$\angle COP = \angle OCB$. (Const.)

But these are alternate angles.

$\therefore$ PO $\parallel$ AB.

Q. E. F.

Alternative method.



(1) Take any pt. C in AB.

(2) Join CO and produce it to R.

(3) With C as centre and any radius draw an arc cutting AB in a and CO in b.

(4) With O as centre and the same radius draw an arc cutting OR in c.

(5) With c as centre and radius$=ab$, draw an arc cutting the previous arc in d.

(6) Join Od and produce it both ways to P and Q. Then PQ $\parallel$ AB.

Proof :—$\angle$ROQ$=\angle$OCB. (Const.)

But these are corresponding angles.

$\therefore$ PQ $\parallel$ AB.

Q. E. F.

Exercises.

1. From the angular points of a given triangle draw lines parallel to the opposite sides.

2. From a point K between two straight lines making an angle of 60° with each other, draw lines parallel to and meeting the intersecting lines. See what figure is formed.

3. In the triangle ABC, find the middle point of AB and through this point draw a line ‖ to the base BC meeting the other side at L. Show by measurement that AL and LC are equal.

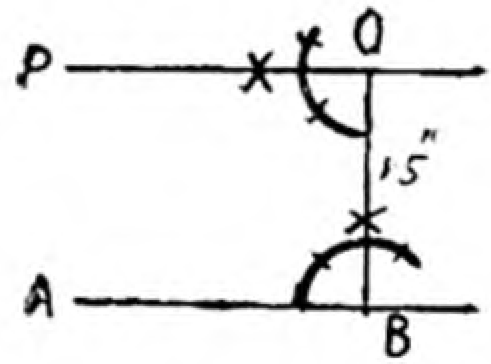
4. In a triangle ABC draw a straight line ‖ to AC and meeting AB, BC in D, E respectively, so that DE $=$AD$+$CE.

Hint :—Draw bisectors of $\angle$s A and C meeting at K. Through K draw ‖ to AC.

5. Draw a straight line through a given point P making an angle of 45° with a given st. line XY.

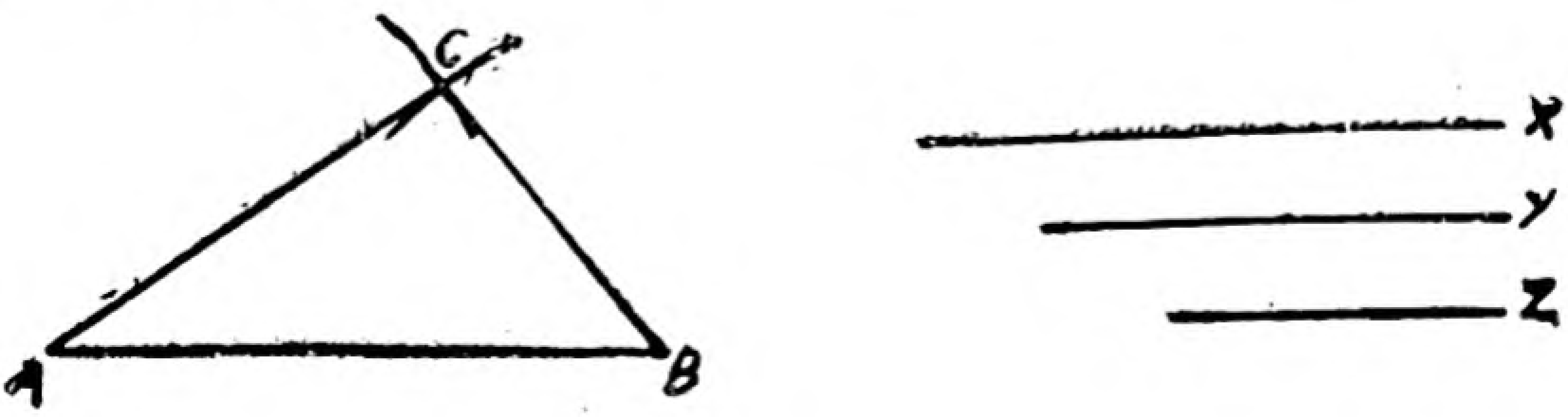
6. Through a point within a given angle draw a st. line meeting the two arms so that it may be bisected at the point.

7. Draw a st. line parallel to a given st. line and at a distance of 1·5″ from it.



Proposition 27 (*Problem*).

To construct a triangle having given the lengths of the three sides.



Given :—The three lengths X, Y, Z.

Required :—To construct a triangle having its sides equal to X, Y, Z, respectively.

Construction :—1. Draw AB=X.

2. With centre A and radius=Y describe an arc.

3. With centre B and radius= Z describe an arc intersecting the first arc at C.

4. Join CA, CB.

Then ABC is the required triangle.

Q. E. F.

Note :—The construction would fail if two of the lengths are together equal to or less than third.

Exercises.

1. Construct a $\triangle$ whose sides are 3·2″, 2·9″ and 4·5″.

2. Describe an equilateral triangle on a st. line 6·5 cm. long.

3. On a given line PQ describe an isosceles $\triangle$ each of whose equal sides is double of PQ.

(Punjab, 1906).

4. Make a triangle whose sides are 5·3″, 2·7″ and 7·1″, and draw the perpendicular bisectors of its sides.

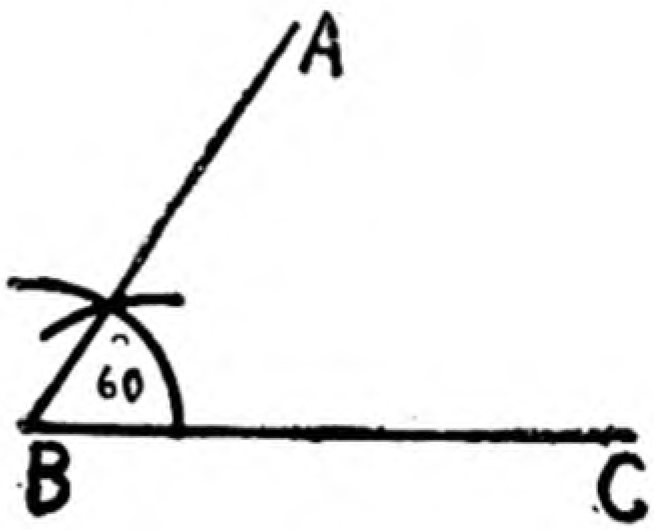
5. Construct a triangle with sides 3·5″, 2·8″ and 5·7″ and draw its medians.

6. Draw a triangle having its sides 2·5″, 3·1″ and 5″. Bisect the greatest angle and draw the smallest altitude.

7. Draw a triangle ABC with sides 3 cm., 4 cm. and 5 cm. Construct a second triangle so that A, B, C are the middle points of its sides.

(Punjab, 1917).

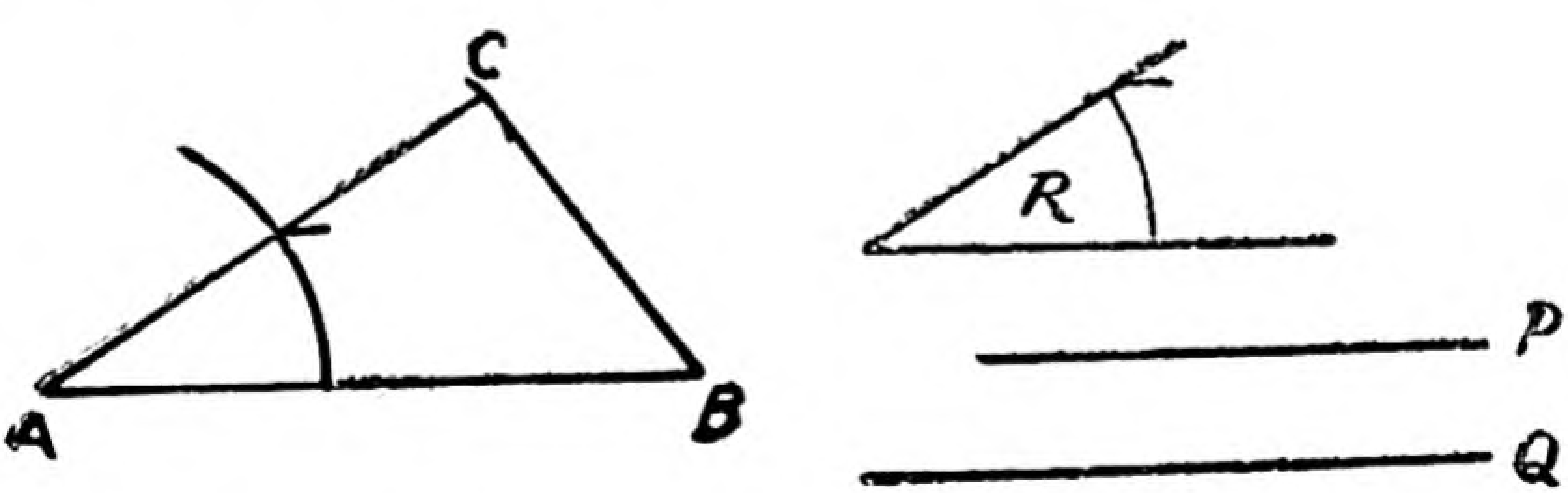
8. With ruler and compasses construct angles of 60°, 30°, 120°, 16°, 75°.



9. *Trisect a right angle.*

Proposition 28 (*Problem*)

To construct a triangle having given two sides and the included angle.



Given :—Two lengths P and Q and an ∠R.

Required :—To construct a △ having two of its sides equal to P and Q respectively and the included angle=∠R.

Construction :—1. Draw a line AB equal to Q in length.

2. At A in AB make $\angle BAC = \angle R$.
3. Cut off AC=P.
4. Join BC.

Then ABC is the required triangle.

Q. E. F.

Exercises.

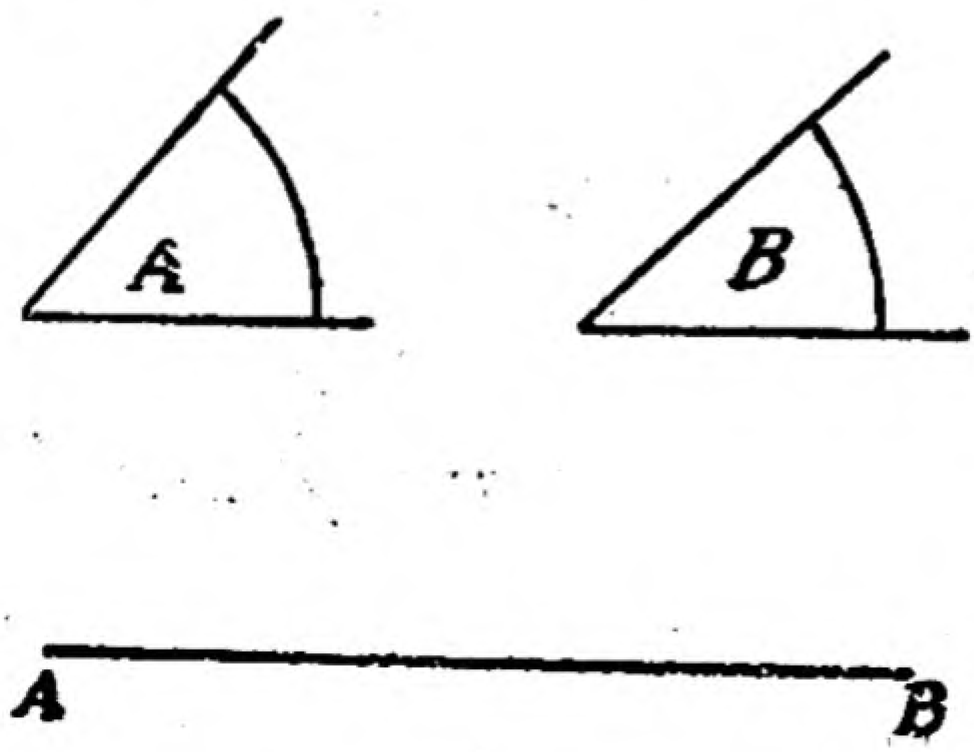
1. Construct a triangle ABC being given AB= 7·5 cm., BC=5·7 cm. and angle B=45°.

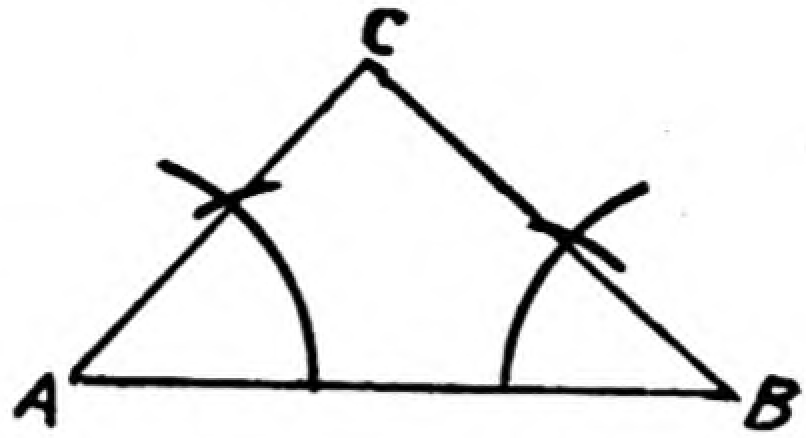
2. Draw an isosceles $\triangle$ having each of its equal sides equal to 8·7 cm. and the vertical angle 75°. Drop a $\perp$ from the vertex upon the base and measure it.

3. Draw a rt. angled $\triangle$ whose two sides are 1·2″, 1·9″.

Proposition 29 (*Problem*)

To construct a triangle having given one side and two of the angles.





Given :—The side AB and two of the angles say $\angle$A and $\angle$B.

Required :—To construct a triangle having these parts.

Construction :—1. Draw a st. line AB equal in length to the given line.

2. At the point A in AB make $\angle BAC = \angle A$.
3. At the point B in BA make $\angle ABC = \angle B$.

Then ABC is the required $\triangle$.

Q. E. F.

Note :—When any two angles of a $\triangle$ are given, the third can be easily found. So two angles being known, we can take the angles adjacent to the given side in every case,

Exercises.

1. Construct triangles from the data given below.

(*i*)	AB=8·3 cm.,	$\angle A = 45°$,	$\angle B = 75°$.
(*ii*)	AC=4·7″.	$\angle A = 60°$,	$\angle B = 45°$.
(*iii*)	BC=7 cm.,	$\angle C = 90°$,	$\angle B = 60°$.

2. On a straight line 7·3 cm. in length construct an isosceles triangle whose vertical angle is of 30°.

3. Draw an isosceles $\triangle$, each of whose base $\angle$s is of 45° and one of the equal sides is 2·6″ long.

4. Draw a rt. ∠d △ whose base $=2{\cdot}3''$ and the base ∠ $=60°$.

5. Construct a triangle ABC, having given AB$=4''$, ∠C $=30°$ and ∠A$=75°$; also draw a triangle containing four times the area of △ABC.

Hint :—From the angular points of △ABC draw lines ∥ to the opposite sides. The triangle thus formed is the required triangle.

6. Draw a triangle PQR in which QR$=4{\cdot}7$ cm., ∠Q$=45°$, ∠R$=105°$; also construct a triangle whose area is one-fourth of that of the △PQR.

Hint :—Join the middle points of the sides of △ PQR.

Proposition 30 (*Problem*)

To construct a triangle being given two of its sides and the angle opposite to one of them

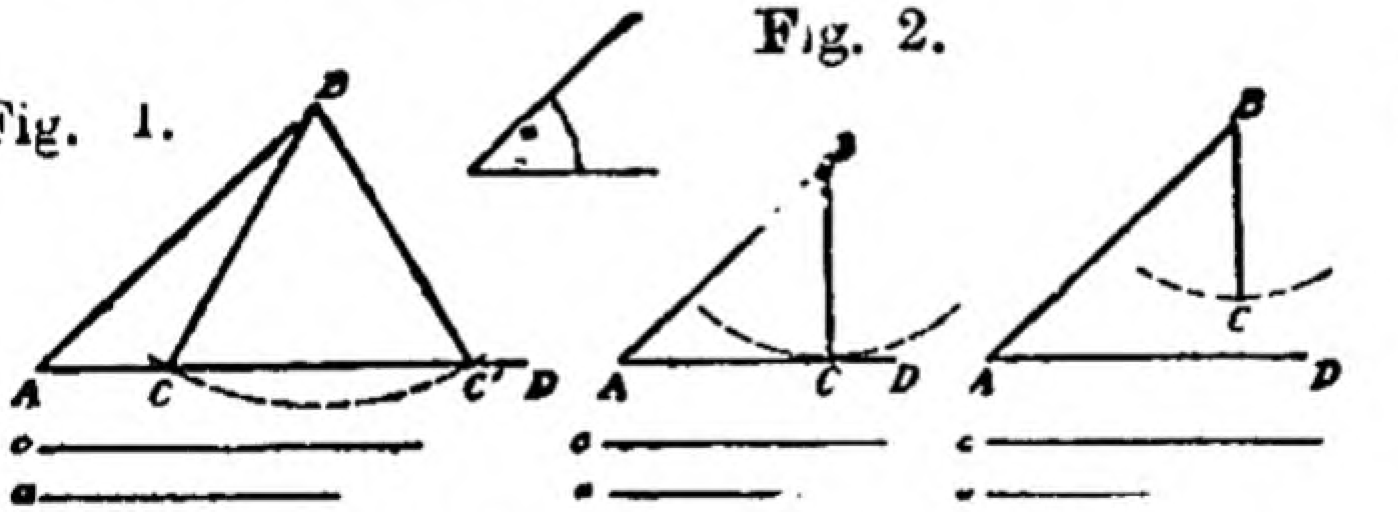


Fig. 1. Fig. 2. Fig. 3

Given :—Two sides a, c and ∠A opposite to one of them (say a).

Required :—To construct a triangle having these parts.

Construction :—

1. Draw an ∠DAB$=$∠A.
2. Cut off AB$=$c *i.e.,* the side other than the one opposite to the given angle.
3. With centre B and radius equal to a, describe a circle.

Now three cases will arise, if a $<$ c ;

Case 1. When the circle cuts AD in C′ and C as in Fig 1.

Join BC and BC′.

Then both the △s ABC and ABC′ have the given parts and are the required triangles.

Case 2. When the circle only touches AD at C, as in Fig. 2.

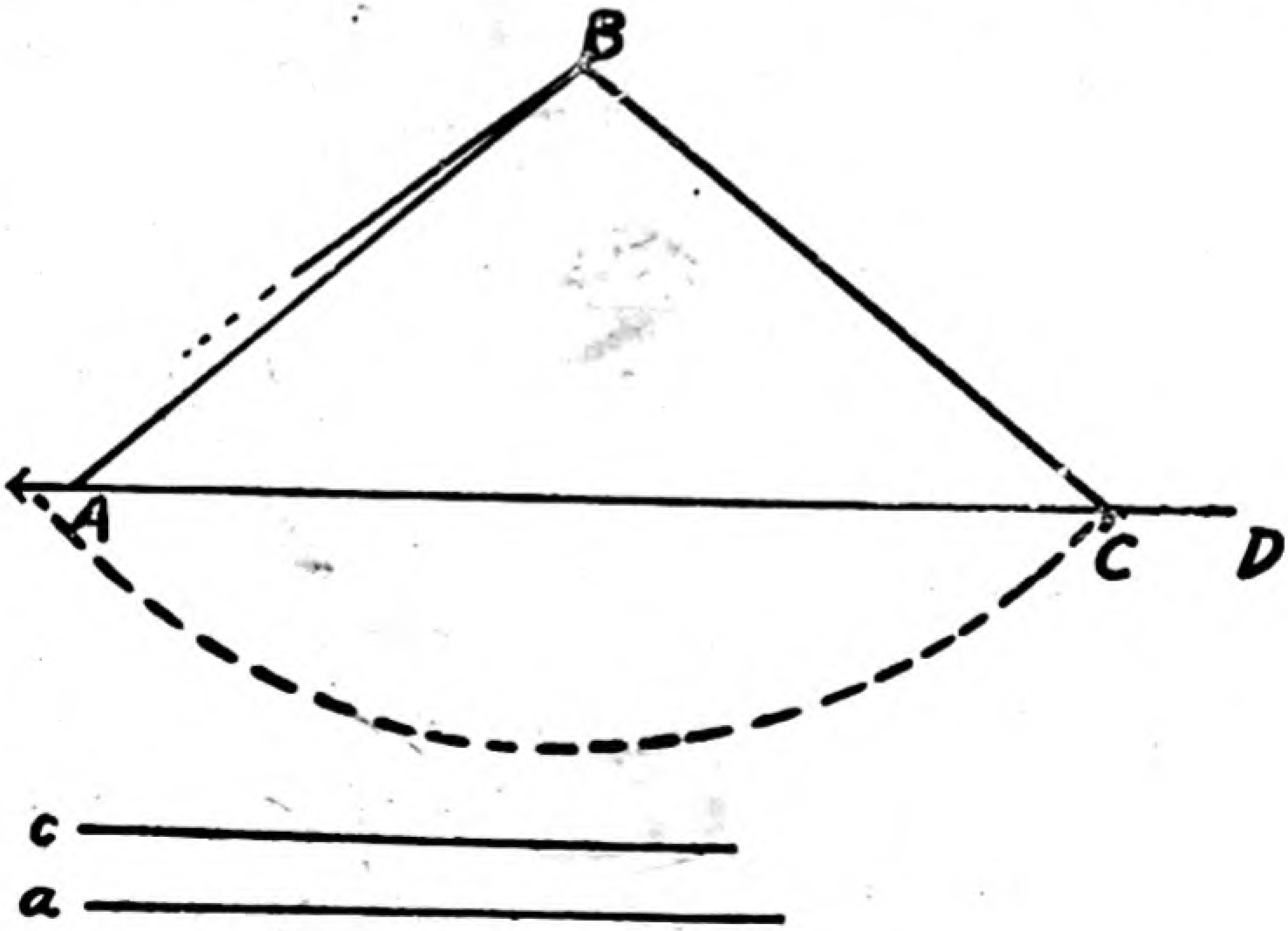
Join BC. Then the △ABC is the required triangle (rt. ∠d. at C).

Case 3. When the circle neither cuts nor touches AD as in Fig. 3.

There is no triangle in this case.

Note—Case 1 is known as the **ambiguous case** as in this case two triangles are formed with the given parts.

Note—Two more cases of this proposition are possible according as, of the two given sides a and c, $a=c$ or $a>c$.



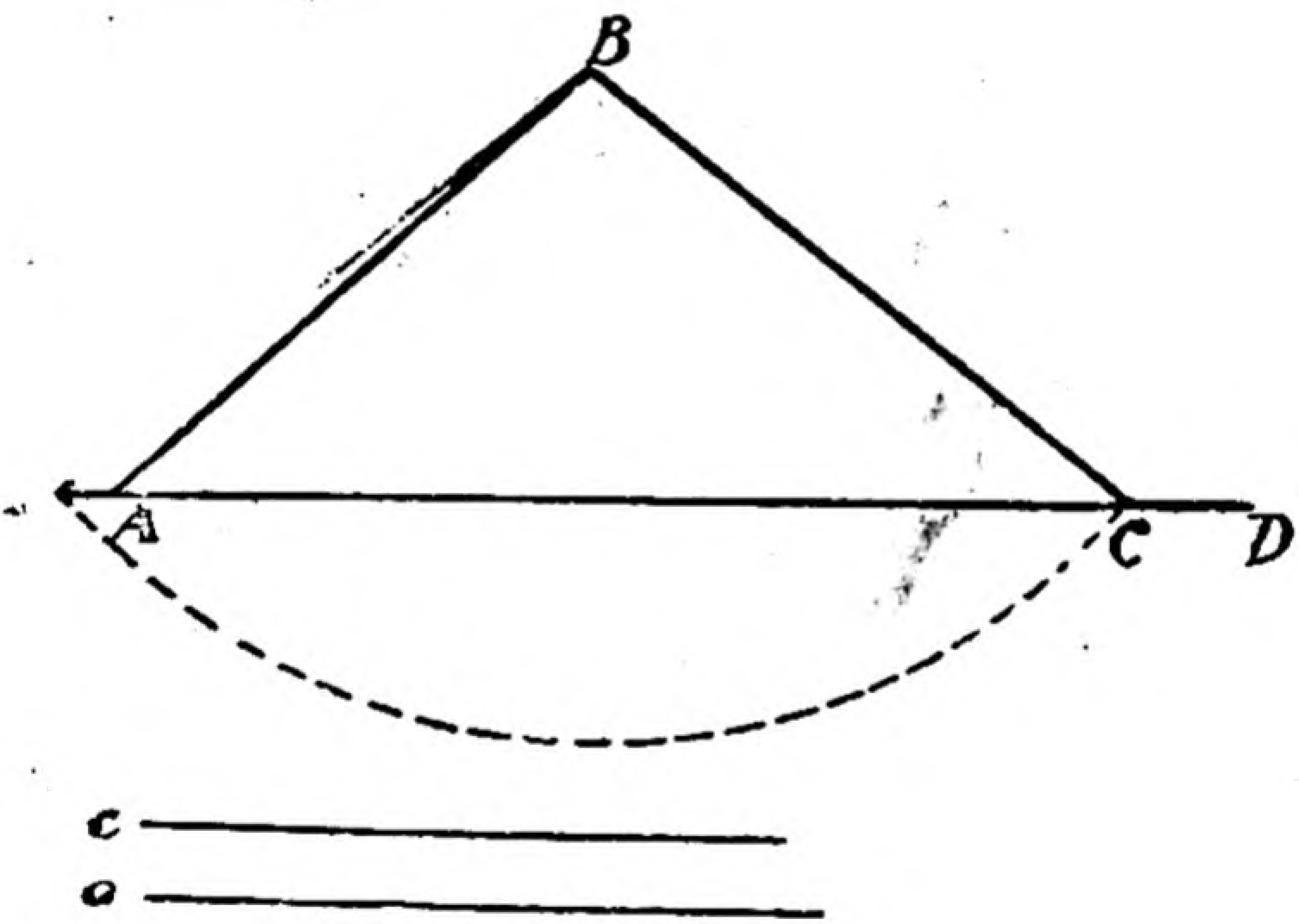
Case. 1. When $a=c$.

The arc described with B as centre and a radius equal to side a will cut the line AD at the points A and C. Join BC.

The triangle formed is ABC in which AB=BC.

In this case also *only one triangle* is formed with the given data and that is an *isosceles* triangle.

Case 2. When $a>c$.



In this case the arc described with centre B and radius equal to a will cut the line AD at C.

Join BC.

Then ABC is the only $\triangle$.

Note.—If $\angle$A is a rt. $\angle$ or an obtuse $\angle$, there is no ambiguity, *i.e.*, one and only one $\triangle$ can be drawn.

Exercises.

1. Construct $\triangle ABC$, with the following data :—

(i) $a=2{\cdot}6$ cm., $\angle A=30°$, $c=6{\cdot}4$ cm.
(ii) $a=3{\cdot}2$ cm., $\angle A=30°$, $c=6{\cdot}4$ cm.
(iii) $a=6{\cdot}4$ cm., $\angle A=30°$, $c=6{\cdot}4$ cm.
(iv) $a=8{\cdot}6$ cm., $\angle A=30°$, $c=6{\cdot}4$ cm.
(v) $a=3{\cdot}6$ cm., $\angle A=30°$, $c=6{\cdot}4$ cm.

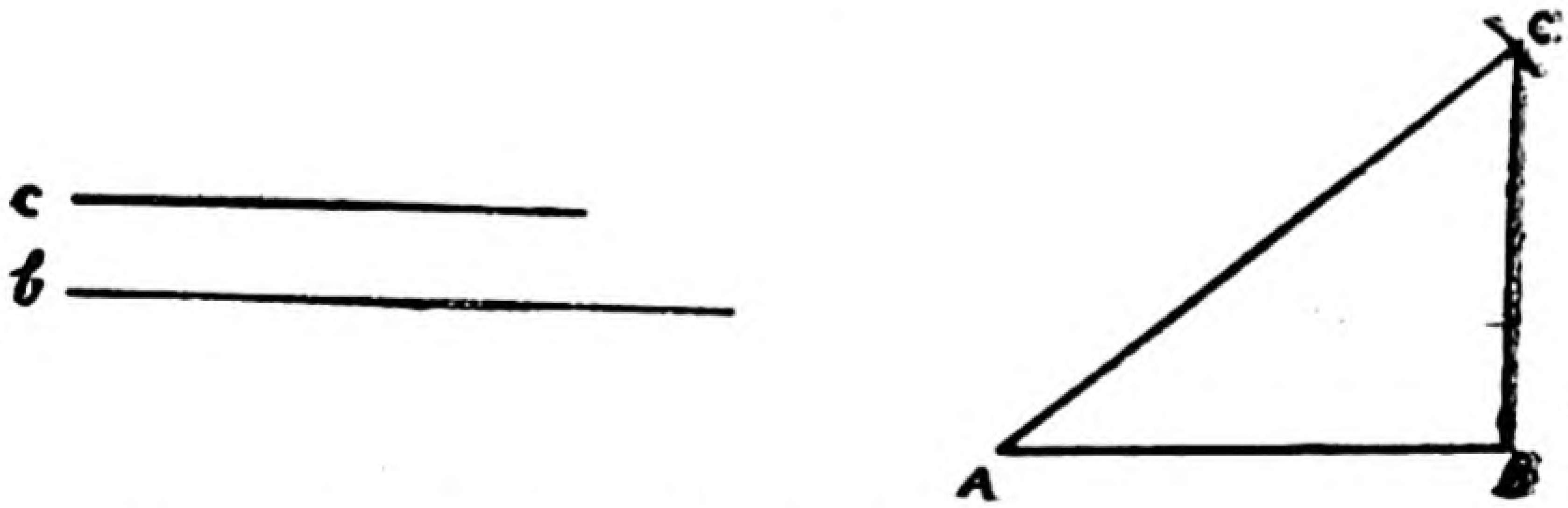
Give the nature and number of solutions in each case.

2. Draw a triangle ABC in which $b=5{\cdot}3$ cm., $c=6{\cdot}8$ cm. and $\angle B=45°$. Can more than one $\triangle$ have these parts ? If so, measure and compare the angles.

3. Construct an isosceles $\triangle$ being given the length of one of its sides, and one of the $\angle$s at the base.

Proposition 31. (*Problem*)

To construct a right-angled triangle being given the hypotenuse and one of the sides.



Given :—Two lengths b and c.

Required :—To construct a rt. angled triangle having one side equal to c, and the hypotenuse equal to b.

FIRST METHOD.

Construction :—1. Take a line $AB=c$ in length.

2. At B draw BC at rt. angles to AB.

3. With centre A and radius equal to b describe an arc cutting BC at C.

4. Join AC.

Then ABC is the required triangle.

Proof :—Obviously in the $\triangle ABC$, $AB=c$, $AC=b$, and $\angle B$ is a rt. angle.

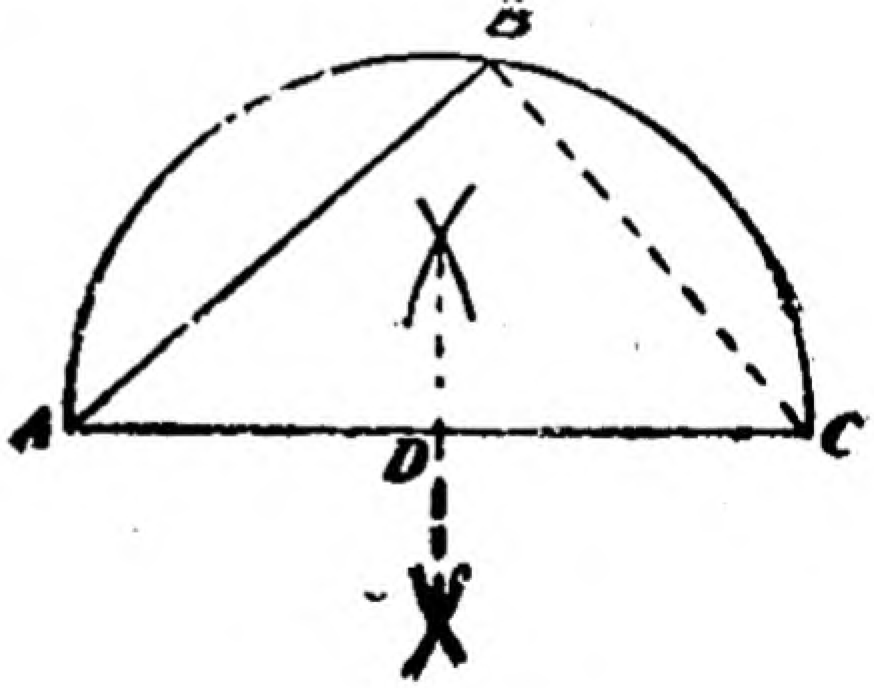
Q. E. F.

SECOND METHOD.

Construction :—Draw a st. line $AC=b$.

Bisect AC at D.

With centre D and radius DA describe a semi-circle ABC.



With centre A and radius c describe an arc cutting the semi-circle at B. Join BC.

Then ABC is the required $\triangle$.

Proof :—$\because$ $\angle ABC$ is in a semi-circle.

$\therefore$ it is a rt. angle.

Hence ABC is the required rt. $\angle$d $\triangle$ having $AC=b$ and $AB=c$.

Exercises.

1. Construct rt. ∠d △s with the following data:—

(*i*) hyp.=3″, Base=2·4″.

(*ii*) hyp.=3.1″, one of the acute ∠s=30°.

(*iii*) hyp.=3″, median bisecting the hyp.=1·5″.

(*iv*) hyp.=3″. ⊥ from the rt. ∠ to the hyp.=1·1″. How many solutions are possible ?

(*v*) one side=3·1″ and the other side=2·2″.

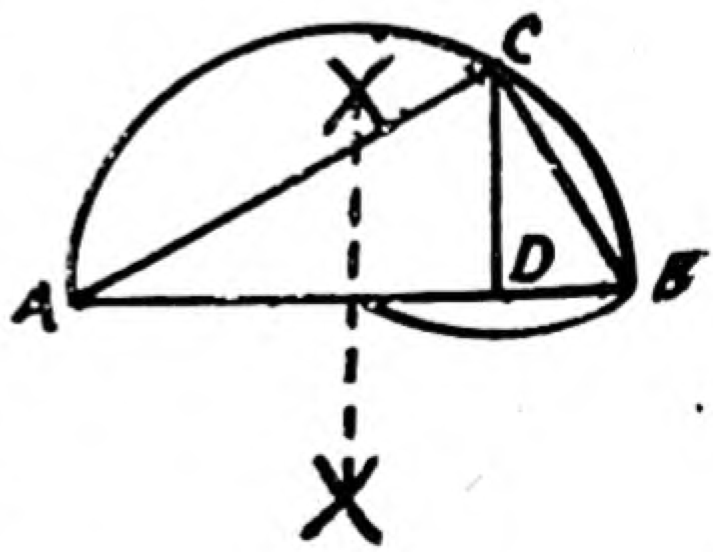
2. *Construct rt. ∠**d** isosceles △ whose hyp. is 4″ long; measure its sides.*

3. Construct a rt. ∠d isosceles △ in which each of the equal sides is 2·3″ long.

4. Draw a rt. ∠d triangle when hyp.=5·6 cm. and one side =3·2 cm.

Draw a ⊥ from the rt. ∠ to the hyp. Measure it. (Punjab, 1929).

Steps of Construction :



1. Take a st. line AB =5·6 cm.

2. On AB as diameter draw a semi-circle.

3. With B as centre and 3·2 cm. as radius make an arc cutting the semi ⊙ at C. Join AC, BC.

Then ABC is the required rt. ∠d △.

By measurement $\perp$ CD$=2{\cdot}62$ cm.

5. *Draw a rt. ∠d △, given that the hyp.* c$=10{\cdot}6$ *cm. and one side* a$=5{\cdot}6$ *cm.*; *measure the 3rd side* b; *and find the value of* $\sqrt{c^2-a^2}$. *Compare the two results.*

6. Two st. roads, which cross at a rt. ∠s at A, are carried over a st. canal by bridges at B and C. The distance between the bridges is 461 yds. and the distance from the crossing A to the bridge B is 261 yds. Draw a plan and by measurement ascertain the distance from A to C.

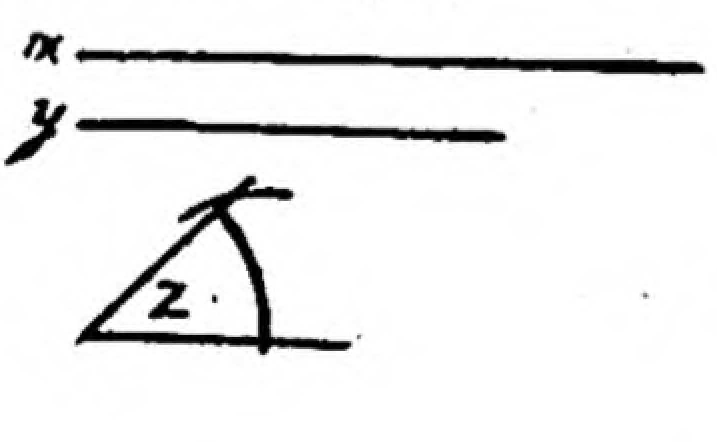
7. Draw △ ABC rt. ∠d at B, and having hyp. AC$=2$ AB. Hence show how to construct an isosceles △ in which each of the equal sides$=$twice the alt. What will be its vertical ∠?

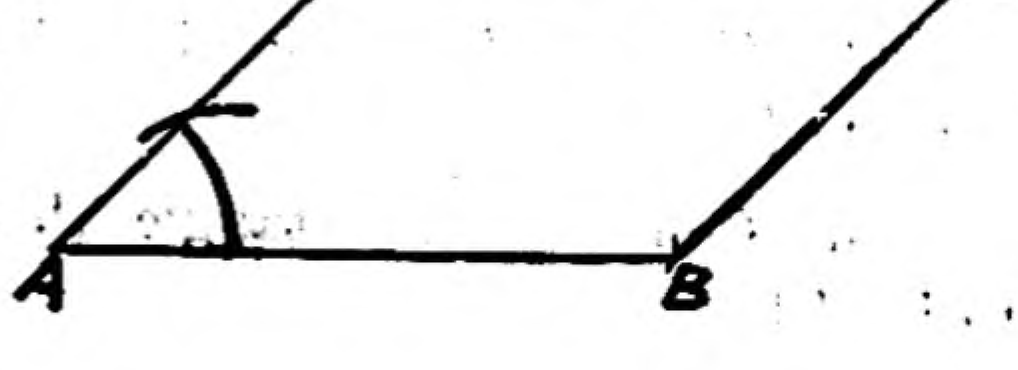
8. *Draw a square on a diagonal of* 6 *cm. Measure its side.*

9. *Construct a rect. in which one side*$=1{\cdot}7''$ *and one diagonal*$=3{\cdot}4''$

Proposition 32. (*Problem*)

To construct a parallelogram, being given two sides and the included angle.





Given :—Two lengths X, Y and $\angle$Z.

Required :—To construct a $\|^m$ having sides equal to X, Y and the included $\angle$ equal to $\angle$Z.

Construction :—

1. Draw an $\angle BAD = \angle Z$.
2. Cut off $AB = X$; and $AD = Y$.
3. With B as centre and radius equal to Y draw an arc.
4. With D as centre and radius equal to X draw an arc cutting the first arc in C.
5. Join BC and DC.

Then ABCD is the required $\|^m$.

Proof :—$\because$ The opposite sides of the quad. ABCD are equal.

$\therefore$ It is a $\|^m$.

$\because$ In $\|^m$ ABCD, $AB = X$, $AD = Y$, and $\angle BAD = \angle Z$ (const.)

$\therefore$ ABCD is the required $\|^m$.

Exercises.

1. Construct a $\|^m$ with sides
   (*i*) 1·4″, 2·7″ and the contained $\angle 75°$.
   (*ii*) 1·9″, 1·9″ and the contained $\angle 60°$.
   (*iii*) 4·5 cm. 6 cm. and the contained $\angle 90°$.
   (*iv*) 1·6″, 1·6″ and the contained $\angle 90°$.

   Give the particular name to each figure.

2. Construct a rhombus having given a side and an $\angle$.

3. Construct a rectangle whose sides are *a*, and *b* units of length.

4. Construct a sq. on a given st. line AB.

5. *Draw a* $\|^{m}$ *PQRS, having given that*

(*i*) *the two adjacent sides PQ, PS are* 6·4 *cm. and* 3·9 *cm. and the diagonal PR* 7·8 *cm. Measure the altitudes of the parallelogram.*

(*ii*) *PQ*=5·4 *cm. and the diagonals PR, QS are* 8·1 *cm, and* 5·9 *cm. long.*

(*iii*) *the diagonals are* 7·8 *cm., and* 10 *cm., and intersect each other at an* $\angle$ *of* 60°.

6. Construct a rhombus ABCD having given

(*i*) *a side*=3·1 *cm. and an* $\angle$=75°.

(*ii*) *a side*=3·1 *cm. and a diagonal*=4·2 *cm.*

(*iii*) *the lengths of the semi-diagonals* 2·2″ *and* 1·8″.

(*iv*) *the lengths of the diagonals* 2·2″ *and* 1·8″.

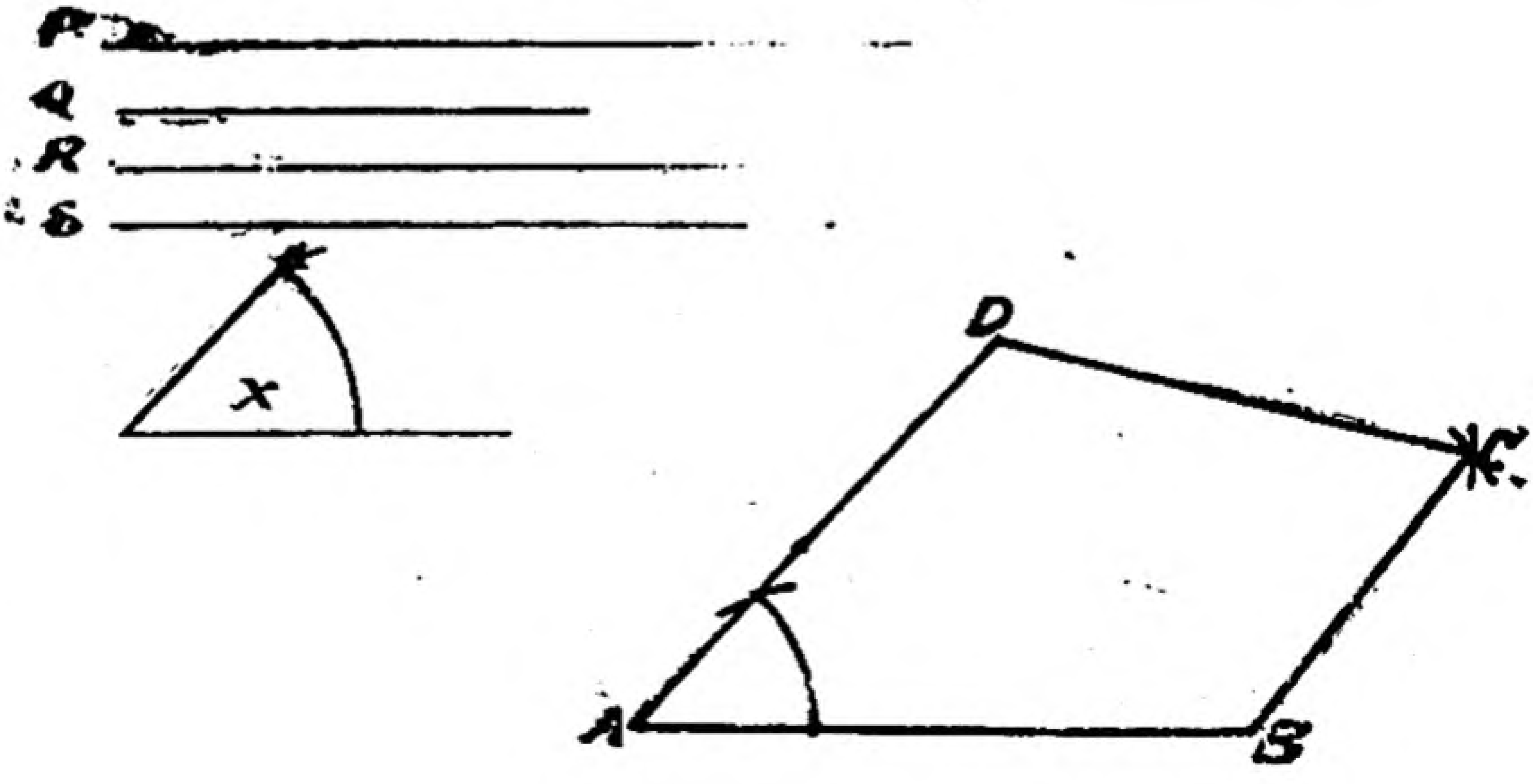
7. Construct a $\|^{m}$ ABCD, being given

(*i*) AB=6·7 cm., the $\angle$A=60° and BD=6 cm.

(*ii*) AB=6·7 cm., the $\angle$A=45° and AC=10·5 cm. Examine the number of solutions in each case.

Proposition 33 (*Problem*)

To construct a quadrilateral, having given the lengths of four sides and one angle.



Given :—P, Q, R, and S, the lengths of four sides; and the $\angle X$ to be included between the sides corresponding to P and S.

Required :—To construct a quadrilateral having its sides equal to P, Q, R, and S and the $\angle$ between the first and the fourth sides equal to $\angle X$.

Construction :—1. Draw AB=P.

2. Construct $\angle BAD = \angle X$.

3. Cut off AD=S.

4. With centre B and radius equal to Q draw and arc.

5. With centre D and radius equal to R draw another arc cutting the first one at C.

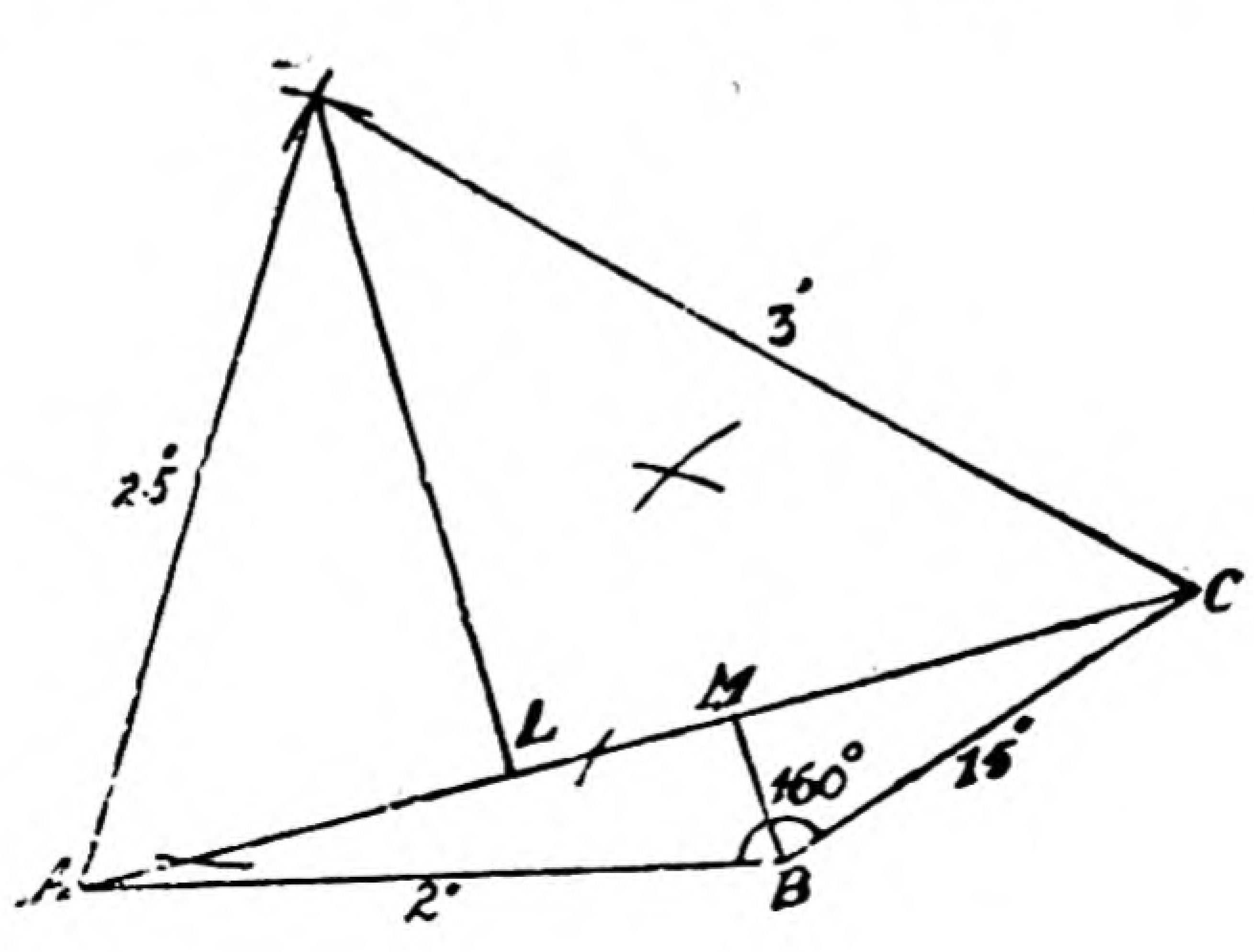
Join CB and CD. Then ABCD is the required quadrilateral.

Q. E. F.

Exercises.

Note.—In each case it would be advisable to begin with a free hand sketch of the required figure.

1. *Construct a quadrilateral ABCD in which AB=2″, BC=1·5″, CD=3″, DA=2·5″ and ∠ABC=160° and measure the distances of B and D from AC.*



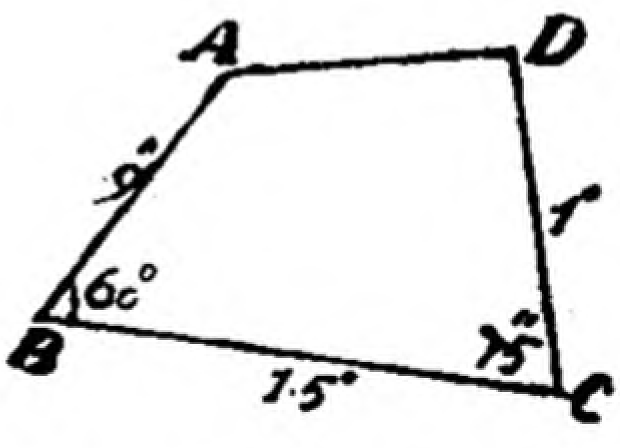
Draw BC=1·5″. At the point B in BC make angle ABC=160° and cut off BA=2″; with A as centre and radius=2·5″ draw an arc; with C as centre and radius equal to 3″ draw another arc. D is the point of intersection of these two arcs. Join DA and DC.

Then ABCD is the required quadrilateral. Drop perpendiculars BM and DL from B and D on AC.

By measurement they are ·3″ and 2″ respectively.

2. *Construct a quadrilateral ABCD, given AB=·9″ BC=1·5″, CD=1″, ∠B=60° and ∠C=75°. Measure the fourth side.*

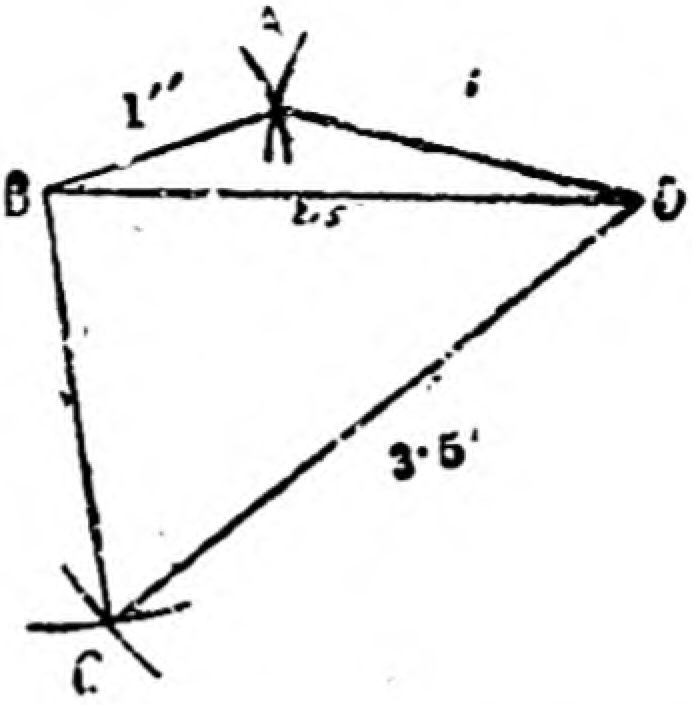
Take BC=1 5″. At B make ∠ABC=60° and at C make ∠DCB=75°. Cut off BA and CD equal to ·9″ and 1″ respectively. Join AD. ABCD is the required quadrilateral.



The fourth side AD=·8″ by actual measurement.

3. *Construct a quadrilateral ABCD, given AB=2·5″, BC=2·1″, ∠A=84°, ∠B=90°, ∠C=72°. Measure BD.*

4. *Draw a quadrilateral ABCD, when AB=1″ BC=3″. CD=3.5″, DA 3″, and BD=2·5″.*



Take BD=2 5″, that is, equal to the given diagonal.

With B as centre and radius equal to 1″ and D as centre and radius equal to 3″ describe two arcs intersecting at A. Join AB and AD.

Again with B as centre and radius =3″ and D as centre and radius =3·5″ draw arcs meeting at C. Join BC and DC.

Then ABCD is the required quadrilateral.

N B.—This construction would have failed if BD had been equal to or greater than the sum of AB and AD or the sum of BC and CD.

5. *Draw the quad. ABCD having given AB=2 4″, BC=·7″, CD=2″, ∠ACD=30°, and ∠ABC= 90°*

6. *Construct a quadrilateral from the following data : AB=2·1″, BC=2·3″, CD=1·1″, ∠B=60° and ∠D=90°.*

7. Construct a quadrilateral ABCD with AB=1·0″, AC=2·0″, ∠ABC=90° and the triangle DAC equilateral.

8. A man walks 100 yds. due East from A to B; then proceeds 120 yds. N. E from B to C and, finally, walks 150 yds. N. W. from C to D. Draw a plan of his walk and find by measurement how far he is now from his starting point. (Scale 1 cm.=20 yds).

(Punjab, 1925).

9. Construct an isosceles trapezium whose ∥ sides are 3·1″ and 2·3″ and each of the non-parallel sides equal to 1·9″.

Hint—Construct first, an isosceles △ with each of the equal sides=1·9″ and the base ·8″ (*i.e*, the difference between the two ∥ sides). Produce the base and cut off a distance =2·3″, and so on.

10. Construct a trapezium whose sides are 10, 4, 7, 6 cm. in length, having the first and the third sides ∥. Measure its diagonals.

11. Construct a trapezium ABCD from the following data: AB=4″. AD=BC=3″, AB and CD being ∥ sides and ∠B=60°.

Miscellaneous Exercises.

1. OA, OB, OC are three radii of a circle equally inclined to one another; show that ABC is an equilateral △.

2. AD, the bisector of the angle A of the △ABC, meets BC in D; CE is ⊥ AD, and is produced to meet AB in F; show that DF=DC.

3. ABCD is a quad. with its vertex A at the centre of a circle and vertices B, C, D on the circumference; prove that ∠C=∠B+∠D.

4. The sum of the sides AB, AC of a $\triangle ABC$ is $>$ twice the altitude AD.

5. Two medians of a $\triangle$ are $\perp$ to the opposite sides ; show that the $\triangle$ is equilateral.

6. In a quad. ABCD, the $\angle$s. B, D are rt. $\angle$s ; if $CB = CD$, show that $AB = AD$.

7. In a scalene $\triangle$ each median is $>$ the corresponding altitude.

8. O is a point inside a quad. ABCD ; show that $OA + OB + OC + OD > AC + BD$.

9. D is any point in the side BC of a $\triangle$ ABC ; show that AD is less than half the perimeter.

10. The st. line which bisects an exterior angle of a $\triangle$ is $\parallel$ the opp. side ; show that the $\triangle$ is isosceles.

11. The bisector AD of the $\angle A$ of the $\triangle ABC$ meets BC in D ; CF is drawn $\parallel$ DA meeting BA produced in F ; show that $AC = AF$.

12. ABCD is a quad in which $AD = BC$ and $\angle D = \angle C$; show that AB $\parallel$ DC.

13. ABCD is a trapezium ; E, F the mid-points of $\parallel$ sides AB, CD ; AG, BH, are drawn $\parallel$ to EF, meeting CD in G, H. Prove that $CH = DG$.

14. ABC, DEF are $\triangle$s such that AB, AC are equal and parallel to DE and DF respectively ; show that BC and EF are equal and parallel.

15. From any point P in the base BC of an isos. $\triangle ABC$, PE is drawn parallel to BA to meet AC in E and PF $\parallel$ CA to meet AB in F ; Prove that $FP = FB$ and that the sum of the sides of the $\parallel^{m}$ AFPE $=$ sum of the equal sides of the $\triangle ABC$.

16. The medians BE, CF of the $\triangle$ABC intersect in G ; H, K, are the mid-points of BG, CG ; show that EFHK is a $\parallel^{m}$.

17. F, E are the mid-points of the sides AB, AC of the $\triangle$ ABC ; Y, X those of AF, AE ; show that YX is $\parallel$ to BC.

18. D, E, F are the mid-points of the sides BC, AC and AB of the $\triangle$ABC ; show that BE and DF bisect one another.

19. P is any point in the side AB of the $\parallel^{m}$ ABCD show that $\angle$CPD=$\angle$ADP+$\angle$BCP.

20. AD is a median of the $\triangle$ABC, G is a point in AD such that AG=2GD ; show that BG produced bisects CA.

21. D and E are the mid-points of the sides AB and AC of a $\triangle$ABC, F is the foot of the $\perp$ from A on BC. Show that the $\triangle$s. ADE and DFE are congruent.

22. In an isosceles trapezium :—

(i) the join of the mid-points of non-parallel sides is $\parallel$ to the $\parallel$ sides.

(ii) the diagonals are equal ; (iii) the base angles are equal ; (iv) the join of the mid-points of parallel sides is $\perp$ to the parallel sides.

23. A st. line cuts the equal sides AB, AC of an isosceles $\triangle$ABC in X, Y and cuts the base BC produced towards C. Prove that AY $>$ AX.

24. The bisector of the $\angle A$ of a $\triangle$ ABC meets BC in D, and BC is produced to E. Prove that $\angle ABC + \angle ACE = 2\ \angle ADC$.

25. If the bisector of an angle of a $\triangle$ bisects also the opposite side, the $\triangle$ is isosceles.

26. From the base of an isosceles $\triangle$ perpendiculars are drawn to the sides; show that the angles made by the perpendiculars with the base are each equal to half the vertical angle.

27. In the quad. ABCD, the lines drawn from A to the other angular points are all equal. Show that the angle BAD is double of the sum of the angles CBD and CDB.

28. In the isosceles $\triangle$ ABC the bisectors of the equal angles, B and C, meet the opposite sides in D and E respectively; prove that BE=CD=DE.

29. The bisectors of the base angles of an isosceles $\triangle$ contain an angle equal to an exterior angle at the base of the $\triangle$.

30. Each angle of a regular polygon is of $160°$; find the number of sides.

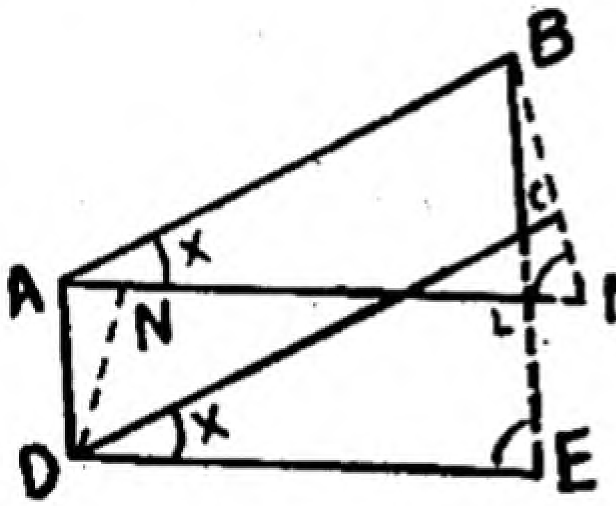
31. The ext. $\angle$ of a regular polygon is of $18°$; find the number of sides.

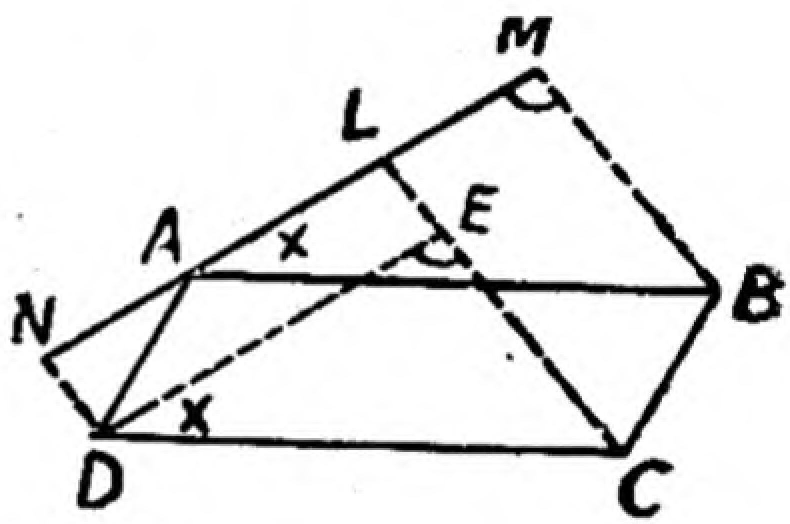
32. The int. $\angle$ of a regular ploygon is 17 times the ext. $\angle$; find the number of sides.

33. *The internal bisectors of the angles of a* $\|^{m}$ *form a rectangle, whose diagonals are parallel to the sides of the original* $\|^{m}$.

34. Prove that if the mid-points of the sides of a rectangle, taken in order, be joined, the figure so formed is a rhombus; and if the mid-points of the sides of a rhombus be similarly joined, the resulting figure is a rectangle.

35. *If through an angle of a parallelogram any st. line is drawn, the perpendicular drawn to it from the opposite angle is equal to the sum or difference of the perpendiculars drawn to it from the two remaining angles, according as the given st. line falls without the ||m or intersects it.*





Hint.—A ||m ABCD, through a line AL is drawn (*i*) without the ||m, (*ii*) intersecting the ||m; CL, BM, DN, are drawn ⊥ to AL. Then (*i*) CL=DN+BM; (*ii*) CL=DN−BM. Draw DE ⊥ CL meeting CL or CL produced in E. DE || AL (both ⊥ CL). Similarly DN || CL ∴ DNLE is a ||m or rect. ∴ EL=DN; ∠MAB=∠EDC (∠s between ||s). ∴ △s ABM and DCE are congruent. 2∠s and one side are equal. ∴ EC=BM ∴ EL+EC=DN+BM or (*i*) CL=DN+BM (*ii*) EL−EC=DN−BM or CL=DN−BM.

36. ABC is a △; the angle ACB is bisected by CE which cuts AB in E; EF is drawn || to BC cutting AC in F and the bisector of the ext. ∠ at C in G; prove that EF=FG.

37. On the equal sides AB, AC of an isosceles △, equilateral △s ADB, ACE, are drawn externally, prove that if CD, DE intersect at O, then OD=OE.

38. D, E are the mid-points of the sides AB, AC of a △ ; BE, CD are produced to F, G, so that EF=EB and DG=CD ; show that FAG is a st. line.

39. P and Q are the mid-points of the sides of an isosceles △ ABC and R the mid-point of the base BC ; prove that PQ, AR bisect each other at rt ∠s.

40. D is the mid-point of the base BC of an isosceles △ABC and E is any point in AC. Prove that the difference of BD and DE is less than the difference of AB and AE.

41. *An angle of a △ is rt., acute or obtuse according as the median drawn through it is=, >* or *<* *half the side it bisects.*

42. ABCD is a ||^{m} ; K, L, M, N, are points on the sides, such that AK=CL and BM=DN ; prove that KMLN is a ||^{m}.

43. In the diagonal AC of a ||^{m} ABCD, points P and Q are taken such that AP=CQ ; show that BPDQ is a ||^{m}.

44. *If the vertical angle of a △ is contained by unequal sides, and if from the vertex the median and the bisector of the angle are drawd, then the median lies within the angle contained by the bisector and the longer side.*

Hint.— Same as Ex. 7, Prop. 15.

45. ABC is a ▽ ; AD, BE are perpendiculars to the opposite sides, produced, if necessary ; prove that

,the perpendicular from the mid-point of AB to DE bisects DE.

Hint.—G. mid-point of AB. In the rt. ∠d △ ABE, EG$=\frac{1}{2}$ AB (Ex. 8, Prop. 12). Similarly DG$=\frac{1}{2}$ AB. ∴ EG=DG. Hence DE is bisected at rt ∠s by the perpendicular from G (Mid. point of AB) by Prop. 14.

46. *If from either extremity of the base of a △, a perpendicular is drawn to the internal bisector of the verticle angle, the st. line joining the middle point of the base to the foot of the perpendicular is equal to half the difference of the two sides.*

If the perpendicular is drawn to the external bisector, the joining line is equal to half the sum of the two sides.

Hint.— (D is the mid-point of base BC). Let BT′, CT be ⊥ to bisector ATT′, then DT$=\frac{1}{2}$(AB – AC)= DT′. Cut off AQ,AQ′=AC, AB. Prove that CTQ, BT′Q′ are st. lines and that T, T′ are their mid-points. Hence DT, DT′ ∥ BQ, CQ′ Also BQ=AB–AC=CQ′ ∵ DT$=\frac{1}{2}$(AB–AC)=DT′. Similarly the second part.

47. *If the vertical angle of a △ is contained by two unequal sides and if from the vertex there are drawn the bisector of the vertical angle, the median and the perpendicular to the base, the 1st. of these lines is intermediate in position and magnitude to the other two.*

Hint.—See hints to Ex. 7, Prop. 15. (In the fig. of that exercise draw AF⊥BC. Now ∠FAC = 90°–∠ACF and ∠BAF=90° ∠BAF. But ∠ACF>∠ABF ∴ ∠BAF> ∠FAC ∴ ∠BAF>∠BAP.

Hence AF lies within the ∠PCA ∴ AP lies between AF and AD. Also ∵ AF⊥BC ∴ AD>AP>AF.

48. Through any point Y in the st. line CD which stands on another st. line ACB, a st. line is drawn parallel to AB, so as to cut the bisectors of the

angles ACD and BCD at X and Z. Prove that XY= YZ.

Hint.—$\angle$YZC=alternate $\angle$ZCB=$\angle$ZCY. $\therefore$ CY= YZ, similarly CY=XY $\therefore$ XY=YZ.

49. If one angle at the base of a $\triangle$ is double the other angle at the base and a perpendicular is drawn from the vertex to the base, prove that the difference between the parts into which the base is divided, is equal to the smaller side of the $\triangle$.

Hint.—In $\triangle$ABC, $\angle$B=2 $\angle$C and AD$\perp$BC. Produce CB to E making DE=DC. Join AE. $\triangle$s ADE and ADC are congruent. $\therefore$ $\angle$AED=$\angle$ACD, but=$\angle$ABC= 2$\angle$ACD $\therefore$ $\angle$ABC=2$\angle$AED $\therefore$ $\angle$EAB=$\angle$ABC$-\angle$AEB =2$\angle$AEB$-\angle$AEB=$\angle$AEB $\therefore$ BA=BE=DE$-$DB=DC $-$DB.

50. In a right angled $\triangle$ the st. line joining the rt. $\angle$ to any point (except the middle point) of the hypotenuse is greater than one part of the hopotenuse and less than the other.

Hint.—In a $\triangle$ABC rt. $\angle$d at C, P is a pt. in AB; then CP$>$AP and less than BP (P being near to A). Take D the mid-point of AB. Join CD. Now $\angle$A=$\angle$DCA ($\because$ CD=$\frac{1}{2}$AB) $\therefore$ $\angle$A>PCA $\therefore$ PC$>$AP. Again BD =CD $\therefore$ $\angle$DCB=$\angle$B $\therefore$ $\angle$DCB=$\angle$B $\therefore$ $\angle$PCB> $\angle$B $\therefore$ BP>CP.

51. D, E, F are the mid-points of the sides BC, CA, AB of a rt. $\angle$d $\triangle$ABC rt angled at C; the perpendicular from C on AB meets DF and EF produced at H and G; prove that AG $\parallel$ BH.

Hint—EF, DF are $\parallel$ to BC, CA. (Prop. 20 Cor. 2) $\therefore$ FF is $\perp$ AC and DF to BC $\therefore$ $\angle$GAB=$\angle$AGE$-\angle$AFE =$\angle$CGE$-\angle$AFE = $\angle$HCB$-\angle$ABC = $\angle$HBC $-$ $\angle$ABC = $\angle$HBA $\therefore$ AG is $\parallel$ BH.

52. D is the mid-point of the base BC of a $\triangle$ABC and BX and CY are drawn perpendiculars upon any st. line drawn through A ; prove that DX=DY.

Hint—See Hints on Ex. 13, Prop. 20.

53. ABCD is an equilateral quad. Prove that the points B, D, and the mid-point of AC are collinear.

54. ABC is an isosceles $\triangle$ having AB=AC. The median AD is produced to E so that DE=AD. E is joined to the middle point of AB and AC by st. lines cutting BC in F and G. Prove that AFEG is an equilateral quadrilateral.

Hint.—L and M are the mid-points of AB and AC, AD bisects $\angle$BAC and is $\perp$ to BC. $\triangle$s ALE, AME are congruent $\therefore$ $\angle$AEL=$\angle$AEM. Again $\triangle$s EFD and EGD are congruent $\therefore$ FD=DG. Also DE=AD and $\angle$s at D are rt. $\angle$s. $\therefore$ Diagonals AE, FG, of quad. AFEG bisect each other at rt. $\angle$s. $\therefore$ AFEG is a rhombus.

55. From the ends of the base of a $\triangle$ perpendiculars are drawn to the opposite sides. Prove that the feet of these perpendiculars are equidistant from the middle point of the base.

56. From the angular points of the squares described on the sides of a right angled $\triangle$, perpendiulars are let fall upon the hypotenuse. Prove that the two extreme perpendiculars are together equal to the hypotenuse.

57. If through the mid-point of the base of a $\triangle$ a st. line is drawn $\parallel$ one of the sides, prove that the portion of this st. line intercepted between the internal and external bisectors of the vertical angles=the remaining side of the $\triangle$.

Hint.—See hints on Ex. 14. Prop. 20.

58. *The sum of the medians of a* $\triangle$ *is greater than* $\frac{3}{4}$ *of the perimeter of the* $\triangle$.

Hint – In $\triangle$ABC, AD, BE, CF are medians which intersect at G. Now BG+CG>BC.

$\therefore \frac{2}{3}$ BE+$\frac{2}{3}$CF>BC (medians trisect at G.) Similarly $\frac{2}{3}$CF+$\frac{2}{3}$AD>AC and $\frac{2}{3}$AD+$\frac{2}{3}$BE>AB. Adding $\frac{4}{3}$(AD+BE+CF)>(BC+AC+AB). Hence (AD+BE+CF)>$\frac{3}{4}$(BC+AC+AB).

59. *M is the middle point of AL, a median of the* $\triangle$ *ABC, BM produced meets AC in N. Prove that AN is half of NC.*

Hint.—Through L draw LR $\parallel$ BN $\therefore$ NC is bisected at R or NR=$\frac{1}{2}$NC. Again MN $\parallel$ LR $\therefore$ AN=NR $\therefore$ AN =$\frac{1}{2}$NC.

60. In the $\|^m$ ABCD, prove that the sum of the perpendiculars from A and C upon any st. line XY without the $\|^m$, is equal to the sum of the perpendiculars from B and D upon the same st. line XY

Hint.—From O the point of intersection of diagonals draw OL $\perp$ XY, then sum of perpendiculars from A and C upon XY=2 OL. (See Ex. 3, Prop. 20).

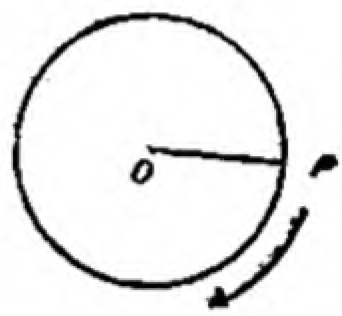
Similarly sum of perpendiculars from B and D upon XY=2 OL.

LOCI

Suppose a donkey is tied to a peg by a rope of given length and it moves about the peg in such a manner that in all its positions the rope remains tightly stretched. The donkey remains always at the same distance from the peg and the only path that

it can describe with the imposed conditions is a circle whose centre is at the peg and whose radius is equal to the length of the rope.

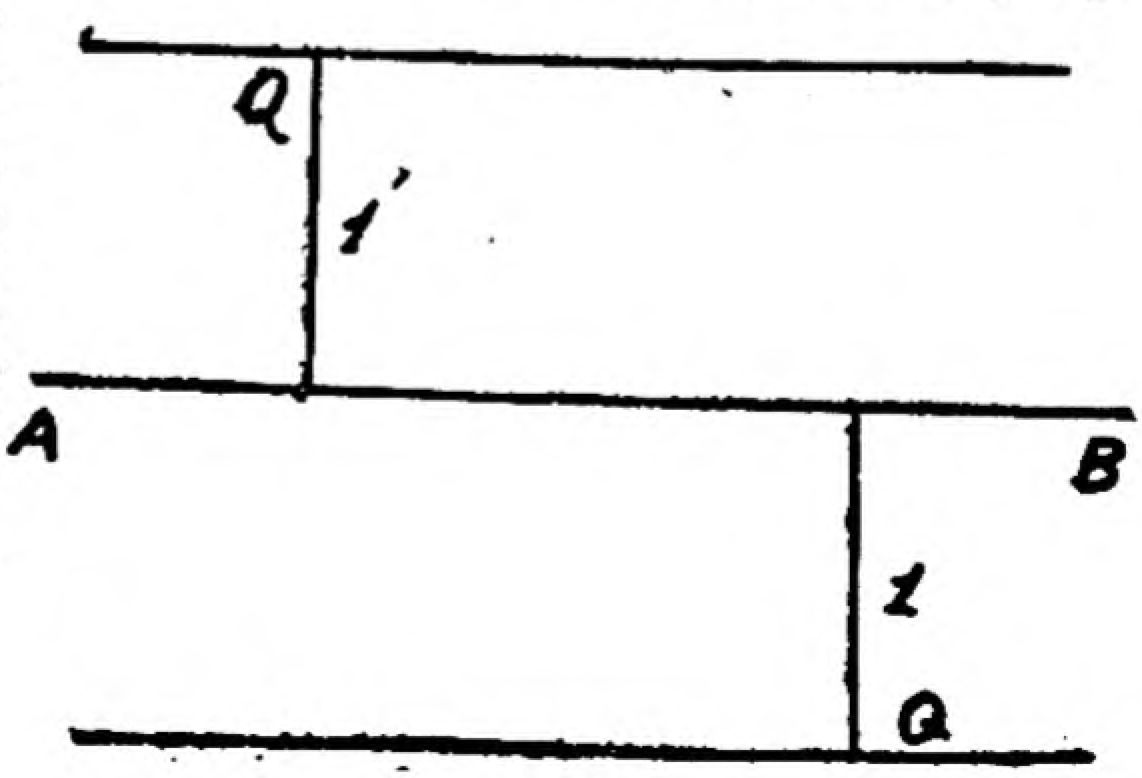
In the same way if a point P moves in a plane and remains at a given distance of 2 cm. from a given point O, its path obviously is a circle whose centre is O and whose radius is equal to 2 cm.



It is evident from the figure that every point on this circle is at the same distance of 2 cm. from O and also all points distant 2 cm. from O lie on the circumference of the circle. It should also be noted carefully that the point P has not been moving freely and at random but under a given geometrical condition, that is *its distance from O is always 2 cm.*

To explain the point further let us take another illustration. Suppose point Q moves so that its distance from a given st. line AB remains unaltered and equal to 1". Obviously its path consists of two straight lines parallel to the given line AB, one lying on either side of it.

Here again it is evident that every point lying on this pair of straight lines is 1" distant from AB and also all points which are at a distance of 1" from AB lie on these two st. lines.



It is obvious from the above illustrations that the path of the moving point in one case is a circle and in the other a pair of st. lines. This circle and the pair of parallel st. lines are called loci (singular locus) of points P and Q respectively. Hence a locus may be defined as follows :—

Locus is the the name given to the path described by a point which moves under some given geometrical condition.

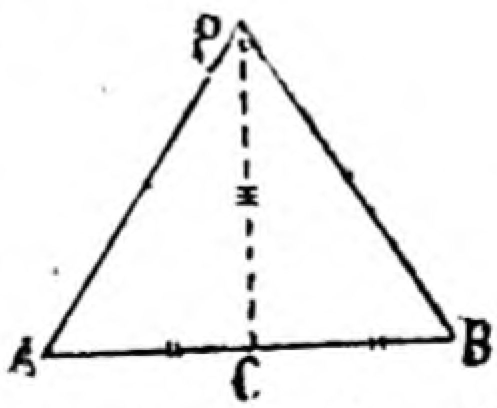
In proving a locus it will, therefore, be necessary to show that—

(*i*) every point satisfying the given condition lies on the supposed locus.

and (*ii*) every point on the supposed locus satisfies the given condition.

Proposition 34. (*Theorem*).

The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.



Given :—Two fixed pts. A and B.

Required :—To prove that (1) P is a point such that PA=PB, then P lies on the perp. bisector of AB and also (2) any point Q on the perp. bisector of AB is equidistant from A and B.

Proof :—(1) Join AB and bisect it at C. Join PC.

In the $\triangle$s ACP, BCP,

$\because$ AC=BC, AP=BP and CP is common.

$\therefore$ The $\triangle$s are congruent, and $\angle$ACP=$\angle$BCP.

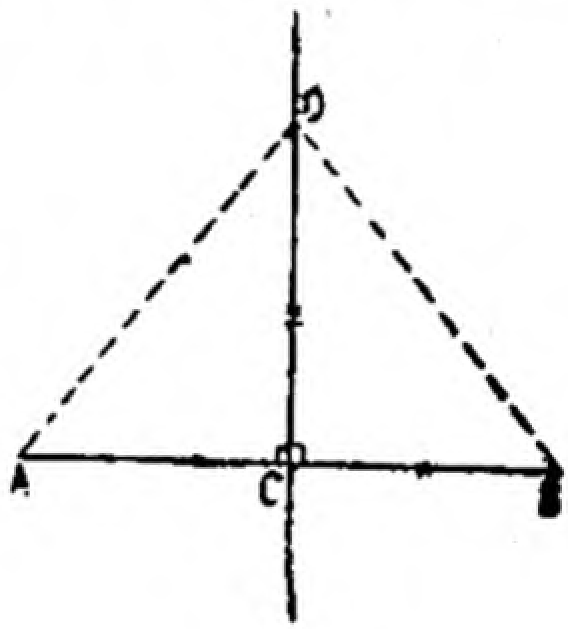
But they are adjacent supp. $\angle$s, $\therefore$ Each is a rt. angle.

Hence PC $\perp$ AB.

$\therefore$ P lies on the perp. bisector of AB.

(2) Erect a perpendicular to AB at C, the middle pt. of AB.

Let Q be any pt. on it. Join QA, QB.



In the $\triangle$s ACQ and BCQ.

$\because \begin{cases} AC=BC & \text{(Const).} \\ CQ \text{ is common} \\ \angle ACQ=\angle BCQ, \end{cases}$

each being a rt. angle.

$\therefore$ The $\triangle$s are congruent, and QA=QB.

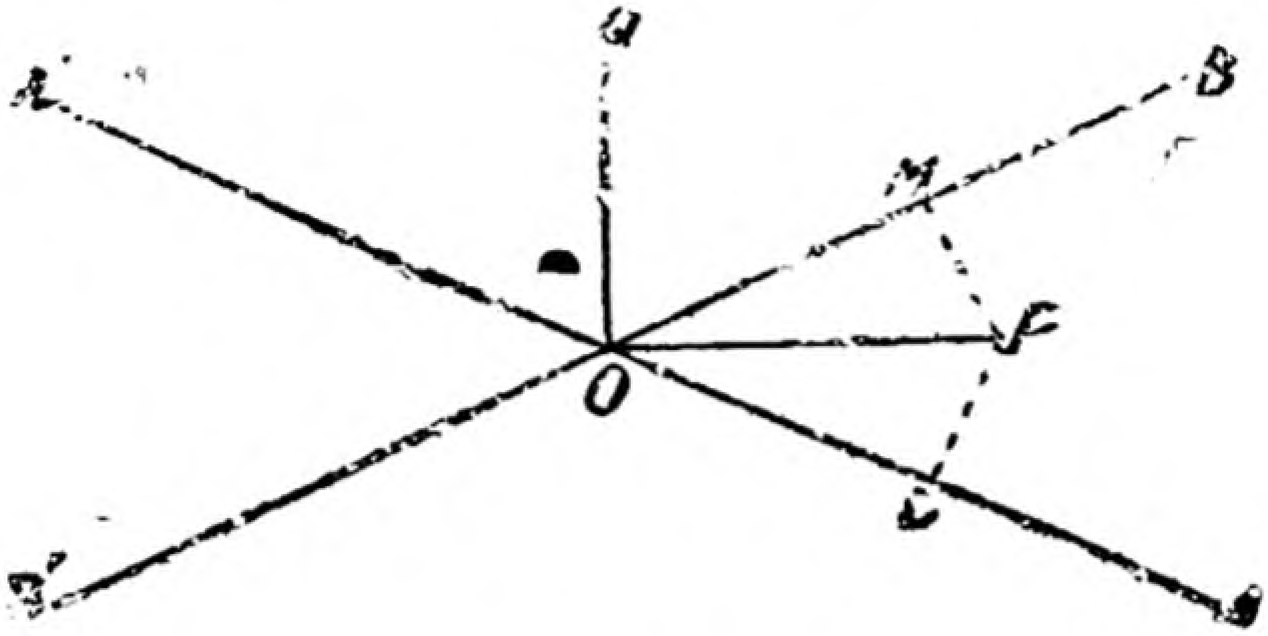
$\therefore$ Q is equidistant from A and B.

Thus, every pt. equidistant from A and B lies on the perp. bisector of AB, and every pt. on the perp. bisector of AB is equidistant from A and B. Hence the perp. bisector of AB is the required locus.

Q. E. D.

Proposition 35 (*Theorem*)

The locus of a point, which is equidistant from two intersecting straight lines, cousists of the pair of straight lines which bisects the angles between the two given lines.



Given :—AOA′ and BOB′ two st. lines intersecting at O.

Required :—To prove that :

(1) If P is any point such that the perp. PL= the perp. PM, then PO is the bisector of ∠AOB, and also

(2) If Q is any point on the bisector of ∠AOB, then the perp. QL=the perp. QM.

Proof :—(1) In the rt. ∠d △s PLO and PMO.

$\because \begin{cases} \text{PL=PM} & \text{(Given)} \\ \text{OP, the hypotenuse, is common.} \end{cases}$

∴ The △s are congruent.

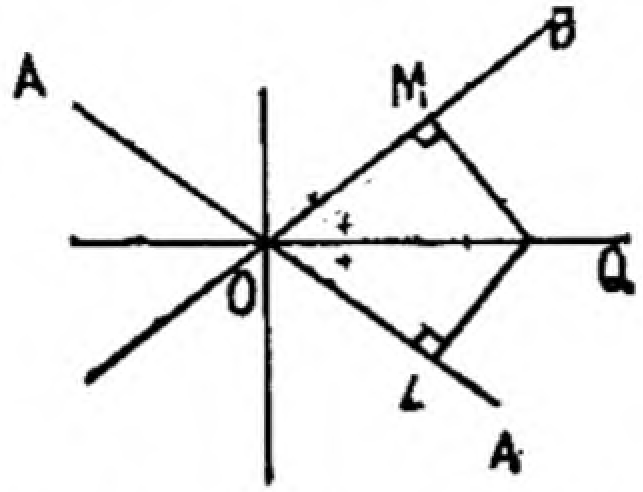
Hence ∠POL=∠POM.

i.e., PO is the bisector of ∠AOB.

(2) In the △s QLO, QMO.

$\because \begin{cases} \angle\text{QLO=QMO. (Being rt. } \angle\text{s.).} \\ \angle\text{QOL=}\angle\text{QOM. (Given).} \\ \text{OQ is common,} \end{cases}$

∴ The Δs are congruent.
Hence $QL = QM$.



i.e., P is equidistant from the given lines.

Thus, every pt. equidistant from two intersecting lines. lies on the bisector of the $\angle$ between them, and every pt. on the bisector of the angle is equidistant from the given lines. Hence the bisector of the angle is the required locus.

Similarly, the bisector of $\angle BOA'$ may also be p·oved to be the locus.

Hence the locus consists of a pair of lines.

Q. E. D.

Exercises.

1. Show that the locus of all points at a given distance from a fixed point is a circle.

2. Prove that the locus of all points at a given distance from a given straight line is a pair of st. lines parallel to the given line.

3. The locus of a point equidistant from two fixed parallel st. lines is a straight line parallel to each of them, lying midway between them.

4. The radius of a circle being given, find the locus of its centre if it rolls along the inside of the circumference of a given circle.

5. Find the locus of the centres of circles passing through two fixed points.

6. Find the locus of the middle points of all the st. lines drawn from a given point to a given st. line.

7. A number of triangles stand on the same base and lie between the same parallels. Find the locus of the middle points of their sides.

8. Prove that the locus of the vertices of isosceles triangles on a given base is the perpendicular bisector of the given base.

9. Find the locus of the centres of all circles which pass through a given point and have their radii equal to a given length.

10. *A line of given length moves with its ends on two fixed stright lines at right angles to one another. Find the locus of its middle point.* (*Calcutta Matric.*)

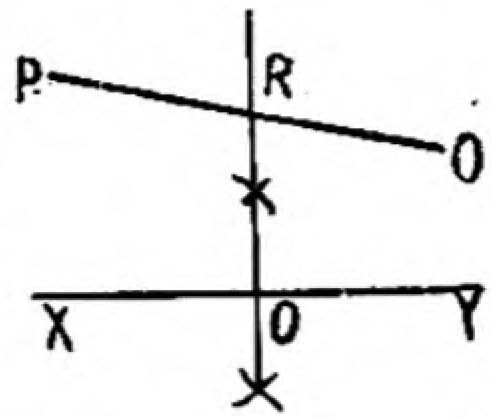
11. *The locus of the vertices of all right-angled △s described on a given straight line as hypotenuse is the circle described on the given straight line as a diameter.*

The Intersection of Loci.

We have seen that when a point is made to satisfy some *condition* or *law*, it can have any position on some *locus* or *path*. When two conditions are imposed, the point is tied down to a limited number of definite positions. These positions are found by the method of "*Intersection of Loci.*"

This method will be best understood from the following examples :—

1 *Find a point in a given st. line PQ equidistant from two given paints X and Y.*



As the required point is equidistant from X and Y, it lies in OR, the bisector of XY.

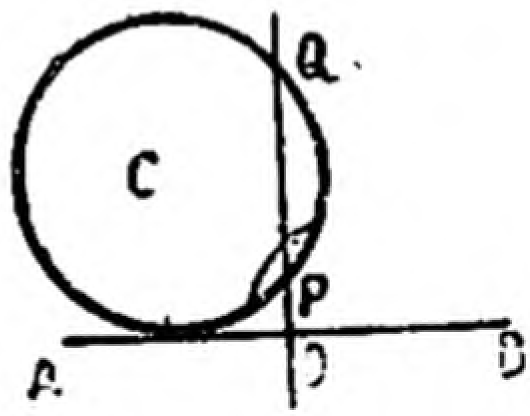
It also lies in PQ.

Hence the point R common to OR and PQ must satisfy both the conditions.

∴ R is the point in PQ equidistant from X and Y.

2. *Find a point equidistant from two points A and B and at a given distance from C.*

As the required point is equidistant from A, B. it lies on the ⊥ bisector OP of AB.



As the required point is at a given distance from C, it lies on the ⊙ drawn with C as centre and the given distance as radius.

Hence the points P, Q common to the ⊥ bisector of AB and the ⊙ must satisfy both the conditions.

∴ P and Q are the required points.

Exercises.

1. Find a point or points at a given distance rom a fixed point and

(*i*) at a given distance fr m a given st. line.

(*ii*) equidistant from two fixed points,

(*iii*) equidistant from two intersecting st. lines.

2. In a given st line find a point or points

(*i*) equidistant from two given points.

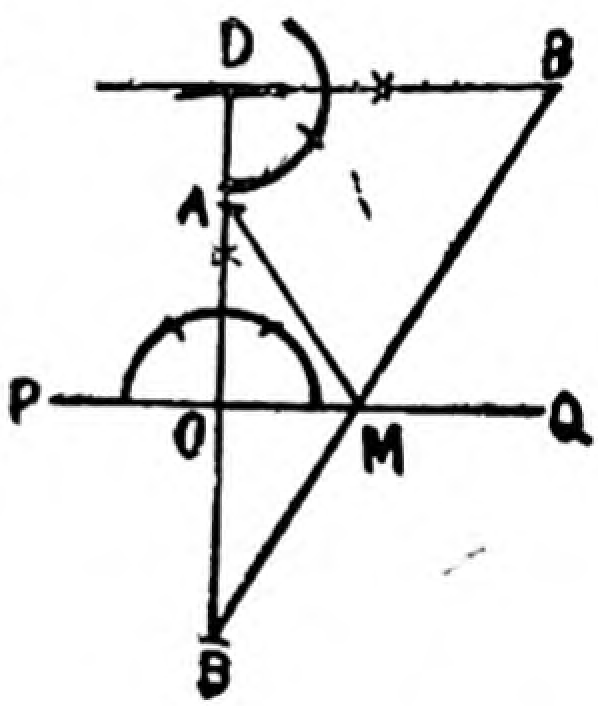
(*ii*) equidistant from two intersecting st. lines.

(*iii*) equidistant from two || lines.

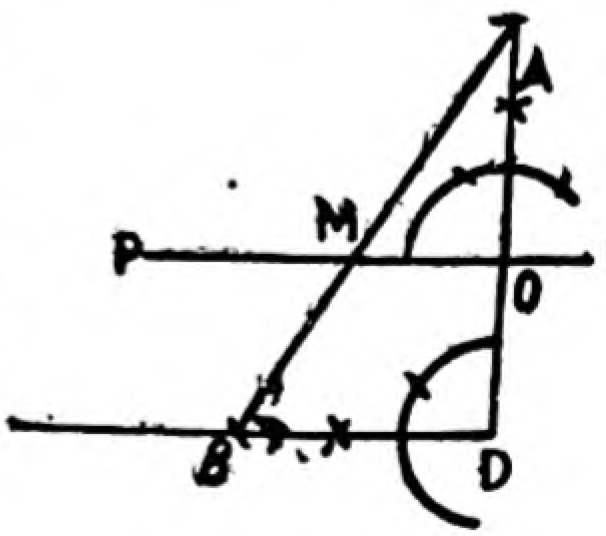
(*iv*) at a given distance from a given pt.

3. Find a point equidistant from three given points P, Q and R.

4. In a given st line PQ find a point A equidistant from two given points X, Y. State when the construction fails.

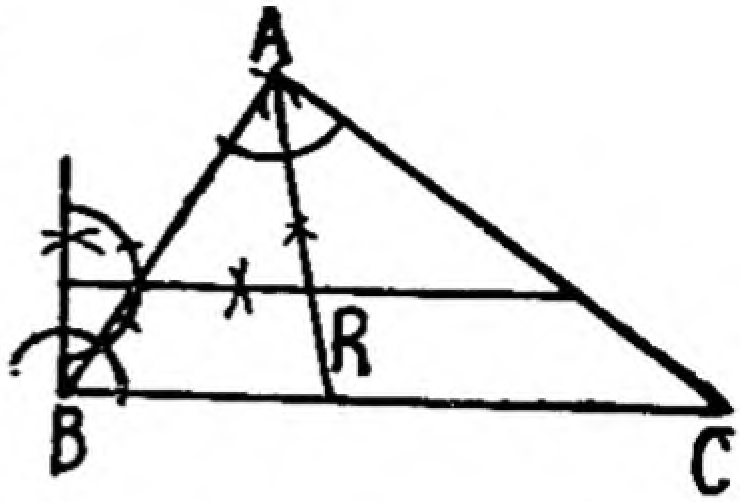


5. A and B are two points 8·5 cm. apart and 2·3″ and 2·3″ from a straight line PQ. Through A and B draw two lines meeting in PQ and also equally inclined to it.

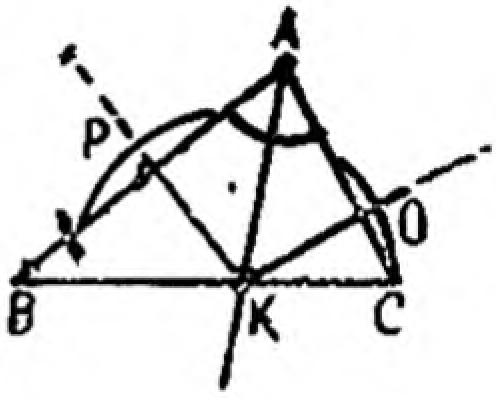


6. Give the construction of Exercise 6 if the points A and B lie on opposite sides of the line PQ.

7. Draw a $\triangle ABC$, having $AB = 1{\cdot}3''$, $BC = 2{\cdot}1''$, $CA = 1{\cdot}7''$. Find the position of a point equidistant from AB, AC and at a distance of $\cdot 9''$ from BC.



8. Draw a $\triangle ABC$, having $AB = 6$ cm., $BC = 5$ cm. and $CA = 4{\cdot}5$ cm. Find a point in BC equidistant from AB and AC. Measure these distances.

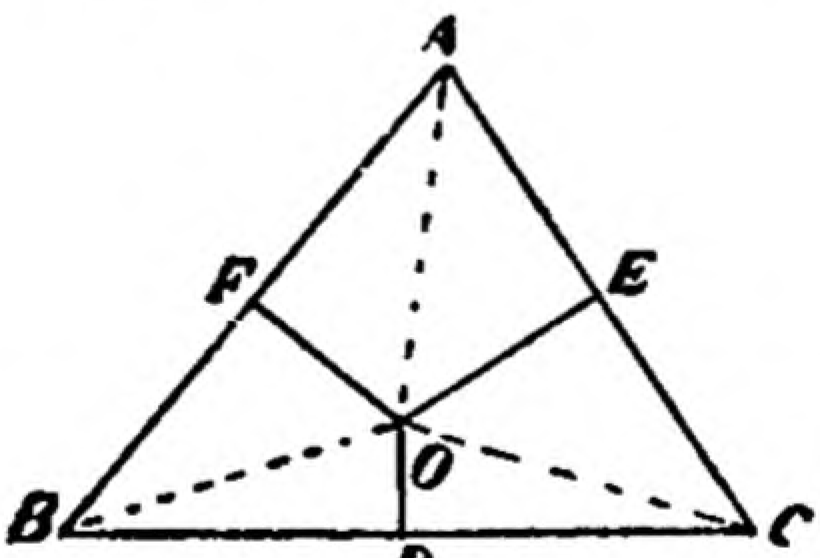


CONCURRENCY OF STRAIGHT LINES

Def.—If three or more lines meet in a point, they are said to be **concurrent**.

Proposition 36 (*Theorem*)

The perpendiculars at the middle points of the sides of a triangle are concurrent.



Given :—D, E and F the mid-points of the sides of a $\triangle ABC$. FO, EO are perp. bisectors of AB, AC respectively. Join OD.

Required :— To prove that OD is per. bisector of BC.

Proof : — ∵ O lies on the perp. bisector of AB,

∴ OA=OB.

Similarly ∵ O lies on the perp. bisector of AC,

∴ OA=OC.

Hence OB=OC.

∴ OD is the perp. bisector of BC.

Hence the three ⊥ bisectors of the sides meet at O. **Q. E. D.**

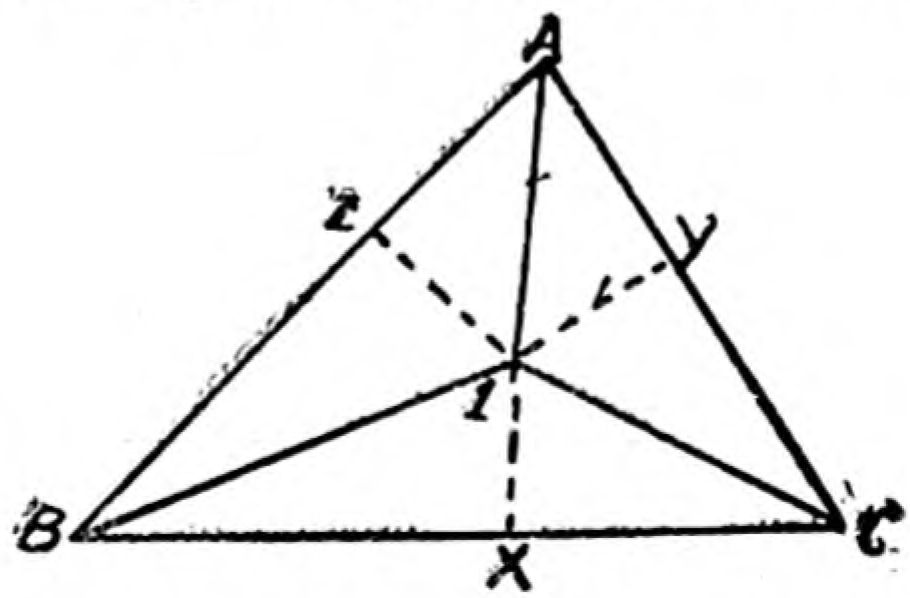
Note 1.—If a circle be described with centre O and a radius equal to OA or OB or OC it will pass through the vertices of the triangle ABC. This circle is called the **Circum circle** of the triangle ABC. The point O is called the **Circum centre** and the radius of the circle is known as **Circum radius**.

Note 2.—This proposition can also be enunciated as follows:—

The perpendicular bisectors of the sides of a △ are concurrent.

Proposition 37 (*Theorem*)

The bisectors of the angles of a triangle are concurrent.



Given :—BI and CI the bisectors of ∠B and ∠C of the △ABC, intersecting at I. Join AI.

Required: —To prove that AI is the bisector of $\angle A$.

Construction:—Draw IX, IY, IZ, $\perp$s to BC, AC, AB respectively.

Proof:—$\because$ The point I lies on the bisector of $\angle B$.

$\therefore \perp IX = \perp IZ$.

Also $\because$ I lies on the bisector of $\angle C$.

$\therefore \perp IX = \perp IY$.

Hence $\perp IY = \perp IZ$.

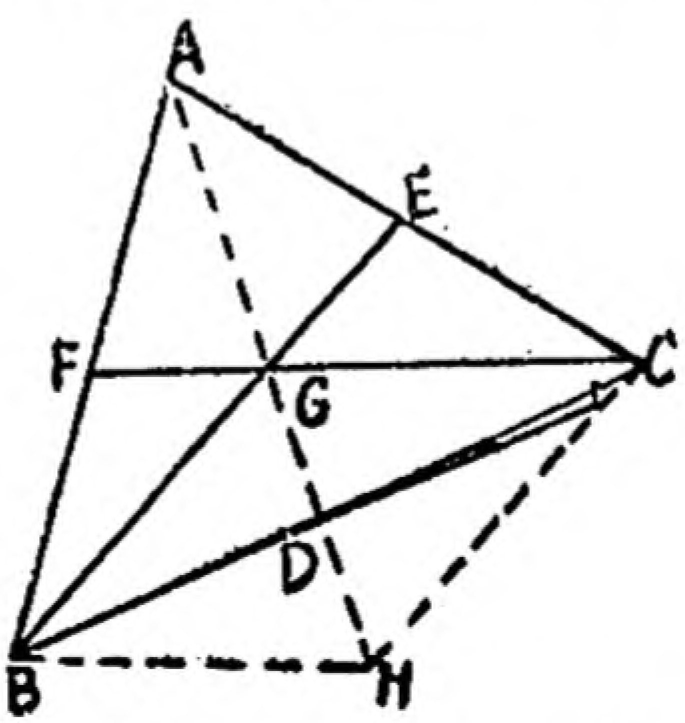
$\therefore$ AI is the bisector of $\angle A$.

Q. E. D.

Note.—If a circle be described with centre I and a radius equal to IX or IY or IZ, it will touch the sides of the triangle at X, Y and Z. This circle is called the **In-circle** of the triangle ABC. The point I is called the **In-centre** and the radius of the circle, the **In-radius**.

Proposition 38 (*Theorem*)

The medians of a triangle are concurrent.



Given:—A $\triangle ABC$. Medians BE and CF intersect at G. Join AG and produce it to meet BC in D.

Required :—To prove that D is the middle point of BC.

Construction :—Produce AD making GH=AG. Join BH, CH.

Proof :—In the $\triangle$ABH,

$\because$ F is the mid. pt. of AB (Given.)

and G is the mid. pt. of AH (Const.)

$\therefore$ FG or GC $\parallel$ BH.

In the $\triangle$ACH,

$\because$ E is the mid. pt. of AC (Given.)

and G is the mid. pt. of AH (Const.)

$\therefore$ GE or GB $\parallel$ CH.

$\therefore$ GBHC is a $\parallel^m$.

But the diagonals of a $\parallel^m$ bisect each other.

$\therefore$ D is the middle pt. of BC.

$\therefore$ the three medians of a $\triangle$ are concurrent.

Q. E. D.

Note —G, the pt. of intersection of the medians of a $\triangle$, is called the *Centroid.*

Cor :—The medians of a $\triangle$ trisect at the pt. of intersection.

In the above fig. we have proved that GBHC is a $\parallel^m$.

$\because$ the diagonals of a $\parallel^m$ bisect each other,

$\therefore$ D is the mid. pt. of GH.

$\therefore$ $GD=\frac{1}{2}GH$

$\therefore$ $GD=\frac{1}{2}AG$ [$\because$ AG=GH Const.]

$\therefore$ $GD=\frac{1}{3}AD$

Similarly we can prove that

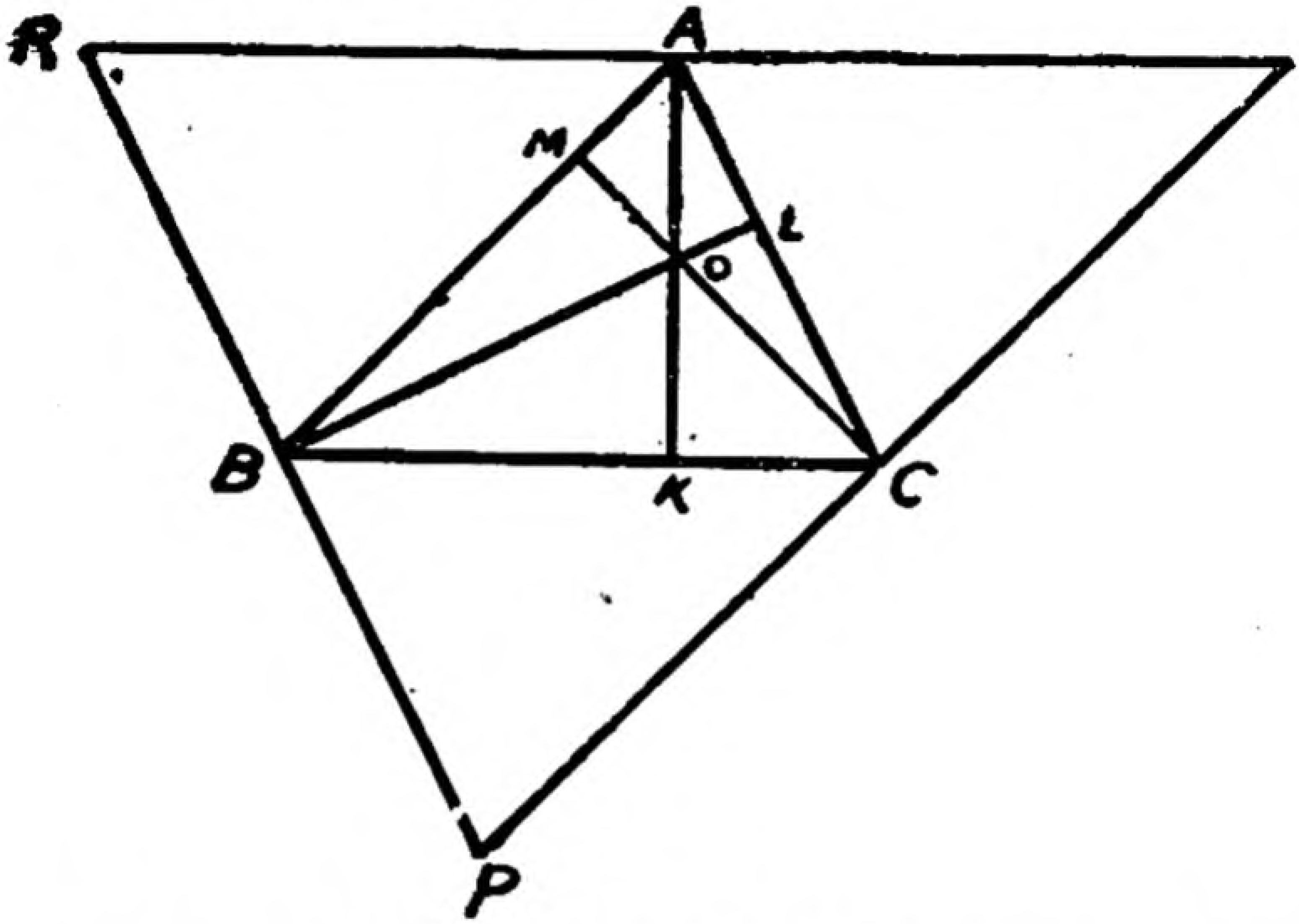
$GF=\frac{1}{3}CF$

and $GE=\frac{1}{3}BE$.

Q. E. D.

Proposition 39 (*Theorem*)

The perpendiculars from the vertices on the opposite sides of a triangle are concurrent.



Given :—AK, BL CM, the three perps. drawn from the vertices A, B and C of $\triangle$ ABC to the opp. sides.

Required :—To prove that AK, BL, CM are concurrent.

Construction :—Through A, B and C draw QR, PR, QP $\parallel$ BC, CA and AB respectively

Proof :—ABCQ is a $\parallel^m$, (Const.)

$\therefore$ AQ=BC.

Similarly ACBR is a $\parallel^m$, (Const.)

$\therefore$ RA=BC.

$\therefore$ RA=AQ. Hence A is the mid-point of RQ.

But RQ $\parallel$ BC and AK $\perp$ BC.

$\therefore$ AK is the perp. bisector of RQ.

Similarly BL is the perp. bisector of RP and CM of PQ.

But the perp. bisectors of the sides of a $\triangle$ are concurrent.

$\therefore$ AK, BL, CM are concurrent. **Q. E. D.**

Note 1.—O, the point of intersection of the $\perp$s from the vertices of a triangle to the opposite sides is called the **orthocentre** of the triangle.

Note 2.—The triangle formed by joining the feet (M, L, K) of the perpendiculars in the figure of the proposition, is called the **Pedal Triangle**. This triangle is sometimes called the *orthocentric triangle*.

Exercises.

1. The circumcentre of an isosceles triangle lies on an altitude of the triangle.

2. O is the circumcentre of the triangle ABC and AD is an altitude, show that $\angle$OAB$=\angle$CAD.

3. O is the circumcentre of the $\triangle$ABC and AD is an altitude ; show that $\angle$OAD$=$difference between the $\angle$s B and C.

4. If the circumcentre of a triangle be equidistant from the sides, the triangle is equilateral.

5. O is the circumcentre of the triangle ABC, in which AB$>$AC ; show that $\angle$OAB $<$ $\angle$OAC.

6. *The st. lines which bisect the sides of a rectangle at right-angles are concurrent.*

7. *Prove that in any triaugle ABC the bisectors of the angle A and the exterior angles B and C are concurrent.*

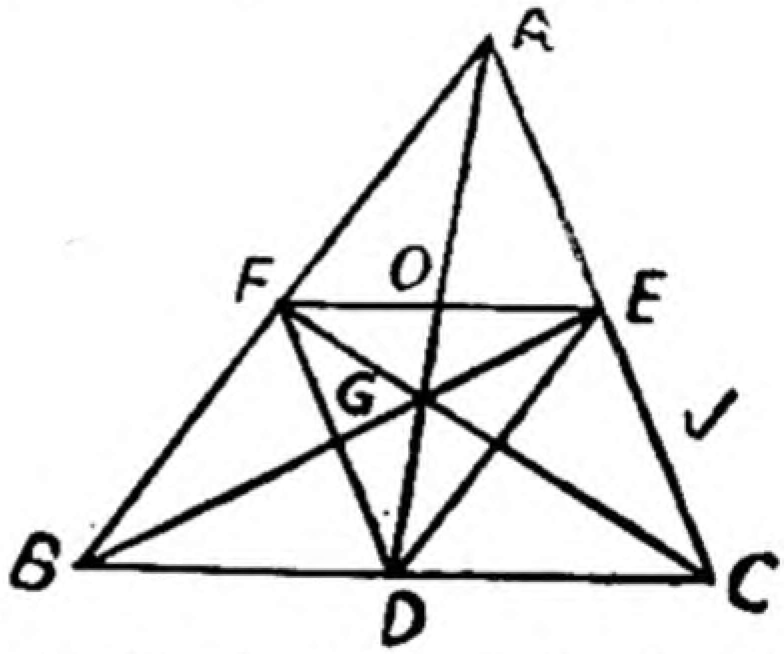
8. If the circumcentre and the incentre of a triangle coincide, the triangle is equilateral.

9. The sides of a triangle subtend obtuse angles at the incentre.

10. From I, the incentre of the triangle ABC, ID, IE, IF are drawn $\perp$s to the sides, show that I is the circumcentre of the triangle DEF.

Hint.—ID=IE=IF $\therefore$ I is the circumcentre of $\triangle$ DEF.

11. If the angles A, B, C be in the descending order of magnitude show that if I be the incentre, IA IB, IC are in ascending order of magnitude.

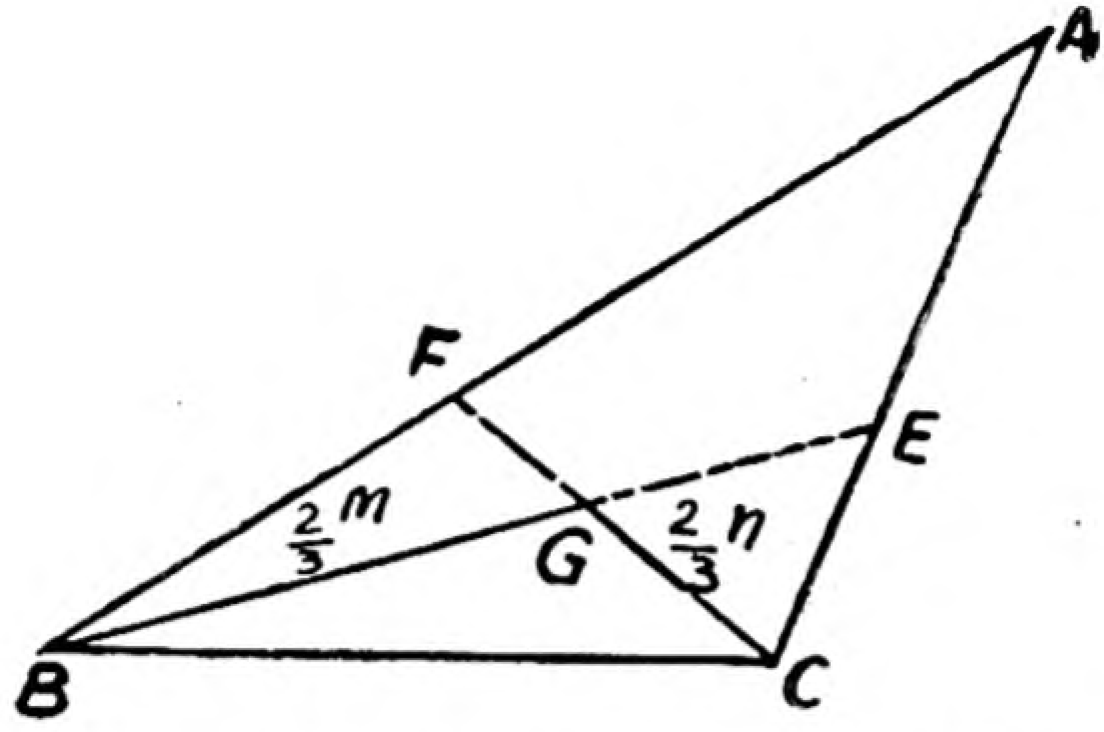


12. *The centroid of a triangle is also the centroid of the triangle formed by joining the mid-points of the sides.*

Hint—In $\triangle$ABC, medians AD, BE, CF, intersect at G. $\therefore$ G is the centroid of $\triangle$ABC, then G is also the centroid of $\triangle$DEF. AFDE is a $\|^{m}$ $\therefore$ AD and EF bisect each other at O. Hence O is the mid-point of EF. $\therefore$ DO is a median of $\triangle$DEF. Similarly DE is bisected in N and DF in M. $\therefore$ medians DO, FN and EM of $\triangle$DEF also intersect at G. $\therefore$ G is the centroid of $\triangle$DEF.

13. BE, CF are medians of the triangle ABC; if BE>CF show that $\angle$BCF<$\angle$CBE.

14. *Given the lengths of one side of a triangle, and of the medians drawn from the ends of that side, construct the triangle.*



Hint.—Let BC be the side and M and N the medians. On BC describe a $\triangle$BGC having BG$=\frac{2}{3}$M and CG$=\frac{2}{3}$N. Produce BG and CG to E and F, so that GE$=\frac{1}{3}$M and GF $=\frac{1}{3}$N. Join CE and BF and produce them to meet at A. Then ABC is the required $\triangle$.

15. *If two medians of a $\triangle$ are are equal, the triangle is equilateral.*

Hint.—In the fig. of Prop. 38 median BE=median CF. Prove that $\triangle$ABC is isosceles, BG=CG ($\frac{2}{3}$ of equal median) GE=FG, ($\frac{1}{3}$ of equal medians) $\angle$CGE$=\angle$BGF $\therefore$ $\triangle$s CGE and BFG are congruent. $\therefore$ EC=BF, $\therefore$ AC =AB.

16. *If three medians are equal, the triangle is equilateral.*

Hint.—Follows immediately from Ex. 15.

17. *Any two medians of a triangle are together greater than the third.*

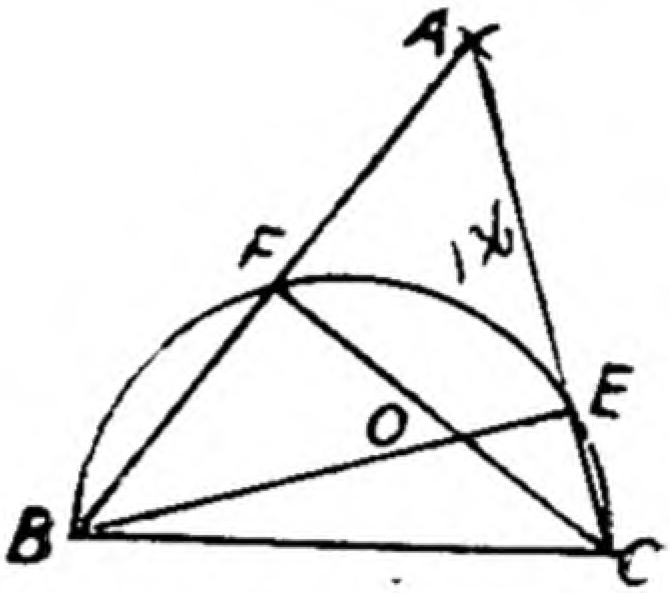
Hint.—In the fig. of Prop. 38; BG+BH>GH. But BG$=\frac{2}{3}$BE, BH=CG$=\frac{2}{3}$CF and GH=AG$=\frac{2}{3}$AD. $\therefore$ $\frac{2}{3}$BE $+\frac{2}{3}$FC$>\frac{2}{3}$AD. $\therefore$ BE + CF>AD.

18. *Prove that the sum of the medians of a triangle is greater than $\frac{3}{4}$ths of the sum of its sides.*

19. The sides of an acute angled triangle subtend obtuse angles at the orthocentre.

20. If O be the orthocentre of the triangle ABC, prove that A, B, C, are the orthocentres of the triangles OBC, OAC, OAB respectively.

21. *If the orthocentre and one side be given, construct the triangle.* (Punjab, 1930)



Hint.—Let BC be the side and O the orthocentre. On BC describe a semi-circle BFEG, join BO and CO, and let them meet the semicircle in E and F. Join BF and CE and produce them to meet at A. Then ABC is the required Δ. $\angle$E and $\angle$F$=90^\circ$ each being $\angle$ in a semi $\odot$.

22. Prove that in an equilateral triangle, the incentre, the circumcentre, the orthocentre and the centroid are coincident.

23. Given the three images of the incentre in the sides of a triangle, construct the triangle.

24. *The distance of each vertex from the orthocentre is twice the $\perp$ distance of the circumcentre from the side opposite to that vertex.*

———

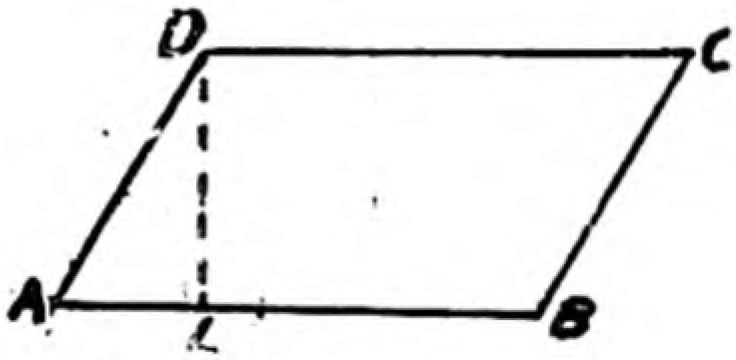
SECTION II

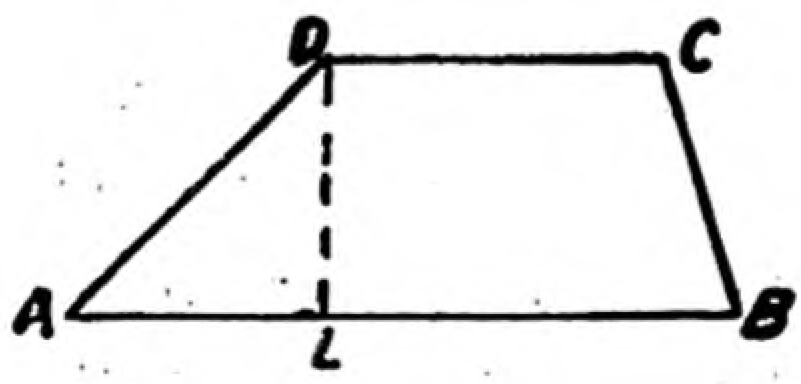
AREA.

Def. The **altitude** or **height** of a parallelogram or a trapezium is the perpendicular distance between a pair of parallel sides.

The side on which a parallelogram may be made to stand is called its **base**. In the case of a trapezium either of its two parallel sides may be looked upon as its base.

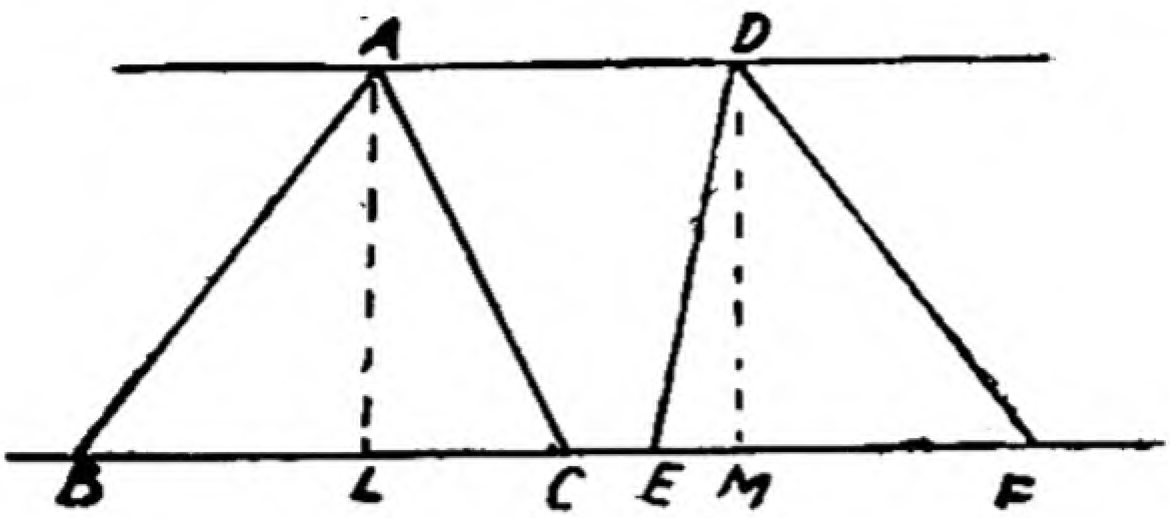
For instance in Fig. 1, AB is the base of the parallelogram ABCD, and DL is the height or altitude.



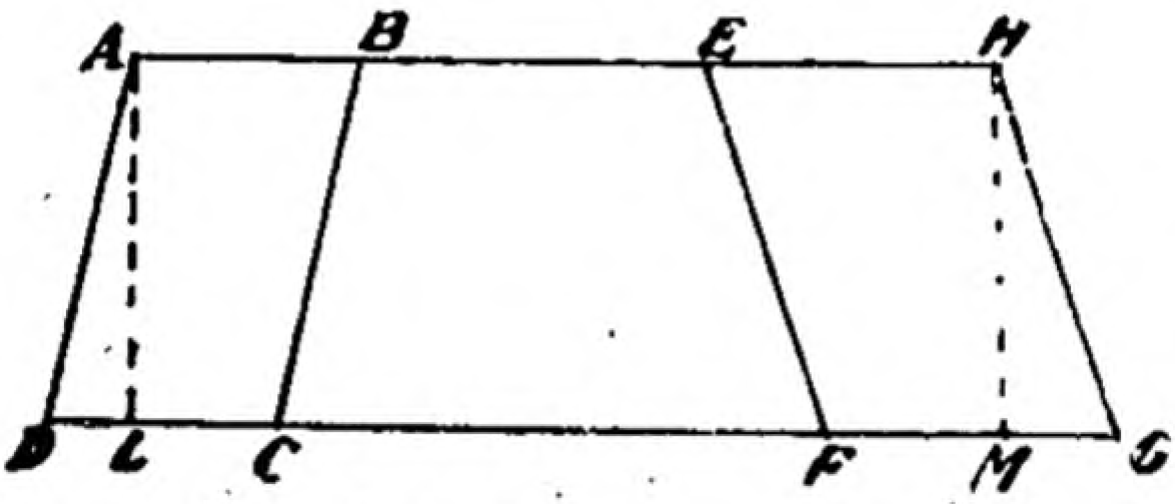


Similarly in Fig. 2 A′ B′ is the base of the trapezium and D′L′ its altitude or height.

Two triangles are said to lie between the *same parallels*, if their vertices lie on one of two parallel straight lines and their bases on the other, as in the following figure.



Two parallelograms are said to lie between the *same parallels*, if their bases lie *along* one of two parallel lines and the sides opposite to the bases along the other, as in the following figure.



Triangles of the same or equal altitudes are between the same parallels.

Let $\triangle$s ABC, DEF, have their bases on the same line BF and AL $\perp$ BC and DM $\perp$ EF. Let AL$=$DM.

To prove that :—AD $\parallel$ BF.

Proof :—$\because$ AL and DM are both $\perp$ BF.

$\therefore$ AL $\parallel$ DM.

Now in quad. ALMD, $\because$ AL $\parallel$ DM and AL$=$DM.

$\therefore$ ALMD is a $\parallel^{m}$.

Hence AD $\parallel$ LM or BF.

Similarly we can prove the same truth about $\parallel^{ms}$.

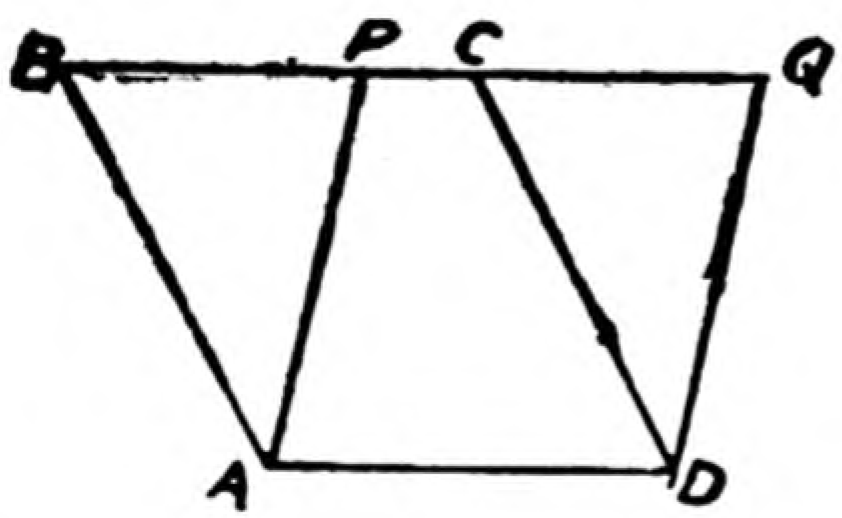
Generally a four-sided figure is named by four letters, one placed at each of the angular points but sometimes for shortness it is called by *only two letters* placed at opposite corners. For instance $\|^{m}$ **ABCD** is often called $\|^{m}$ AC or $\|^{m}$ BD.

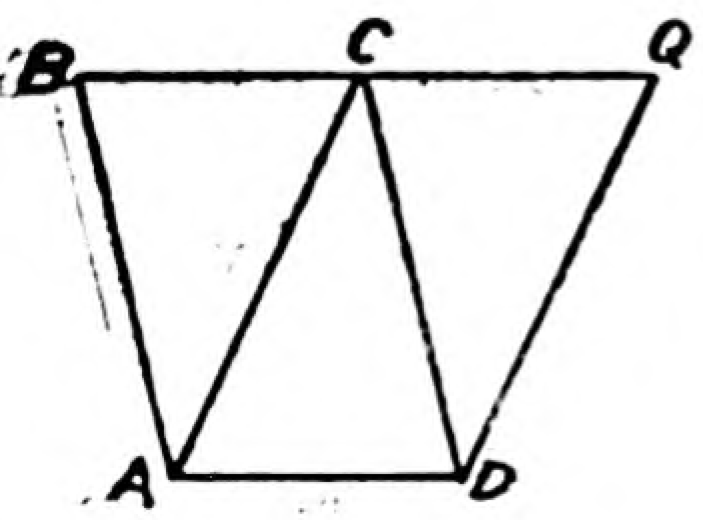
In what follows it is assumed that the student knows what is meant by the Area of a figure, and how to find the area of a rectangle whose length and breadth are given. To refresh his memory he would be well-advised to revise the formula :—

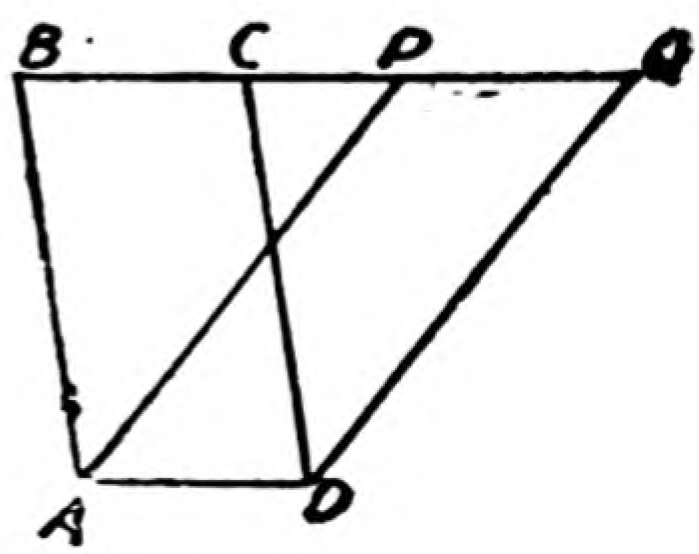
Area of a rectangle = length of the rectangle × breadth of the rectangle.

Proposition 40 (*Theorem*)

Parallelograms on the same base and lying between the same parallels are equal in area.







Given :—$\|^{ms}$ ABCD and APQD on the same base AD and between the same parallels AD and BQ.

Required :—To prove that these two $\|^{ms}$ are equal in area.

In $\triangle$s ABP and DCQ,

$\angle ABP = \angle DCQ$ (AB $\|$ CD)

$\angle APB = \angle DQC$ (AP $\|$ DQ)

and $AB = CD$ (opposite sides of a $\|^m$).

$\therefore$ $\triangle$s ABP and DCQ are congruent and equal in area.

Now figure ABQD = figure ABQD.

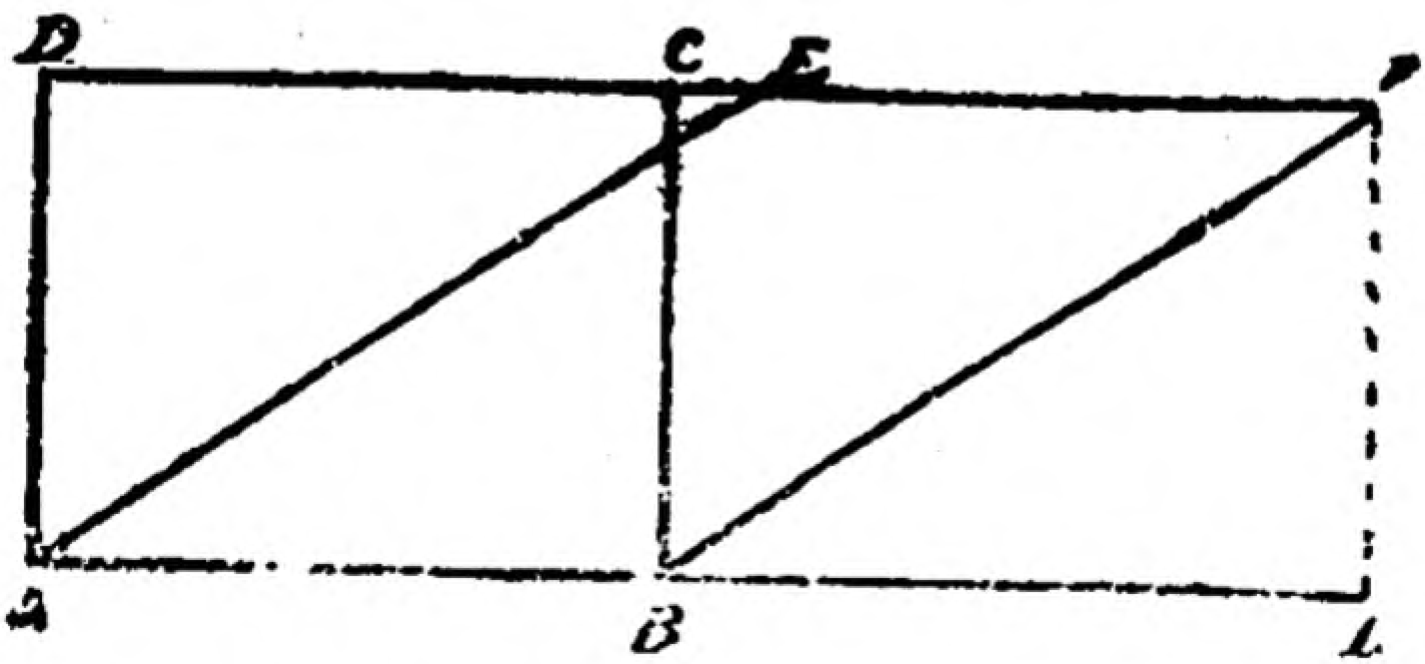
and $\triangle$ ABP = $\triangle$DCQ. (Proved).

$\therefore$ Fig. ABQD $-$ $\triangle$ABP = fig. ABQD $-$ $\triangle$DCQ.

Hence $\|^m$ APQD = $\|^m$ ABCD. **Q. E. D.**

Cor. 1. Parallelograms on the same base and of the same altitude are equal in area.

Cor. 2. The area of a parallelogram is equal to that of rectangle having the same base and height.



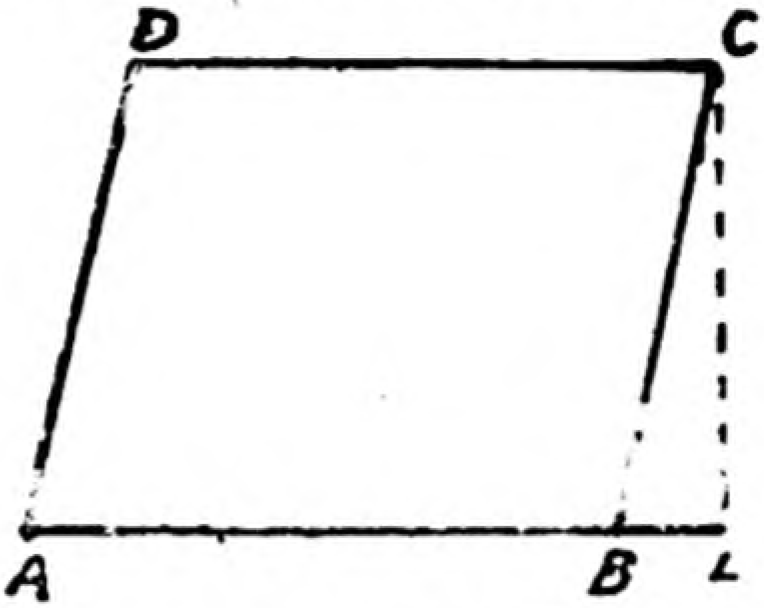
Hint.—Rect. ABCD and $\|^m$ ABFE stand on the same base AB and have the same height.

$\therefore$ Area of $\|^m$ ABFE = area of a rect. ABCD (a rect. is a special $\|^m$.)

Hence area of $\|^{m}$ ABFE $=$ AB $\times$ AD.
$=$ AB $\times$ FL $=$ base of the $\|^{m} \times$ its height.

Hence **the area of a $\|^{m}$ $=$ base $\times$ height.**

Cor. 3. *Parallelograms on equal bases and having the same or equal altitude, are equal in area.*



Given :—ABCD and EFGH two $\|^{ms}$ on equal bases AB and EF and having the same or equal altitudes.

Required :—To prove that they are equal in area.

Construction :—Draw CL and HM $\perp$s to AB and EF respectively.

Proof :—Area of $\|^{m}$ ABCD $=$ base $\times$ height $=$ AB $\times$ CL.

Similarly area of $\|^{m}$ EFGH $=$ EF $\times$ HM.
But AB $=$ EF (given).
Also CL $=$ HM (given).
$\therefore$ AB $\times$ CL $=$ EF $\times$ HM.
or area of $\|^{m}$ ABCD $=$ area of $\|^{m}$ EFGH.

Exercises.

1. Equal parallelograms on equal bases have the same altitude.

2. Equal parallelograms having the same altitude stand on equal bases.

3. Straight lines joining the middle points of the sides of a triangle form with those sides three parallelograms which are equal in area.

4. *The perimeter of a square is less than that of an equivalent ‖ᵐ on the same base.*

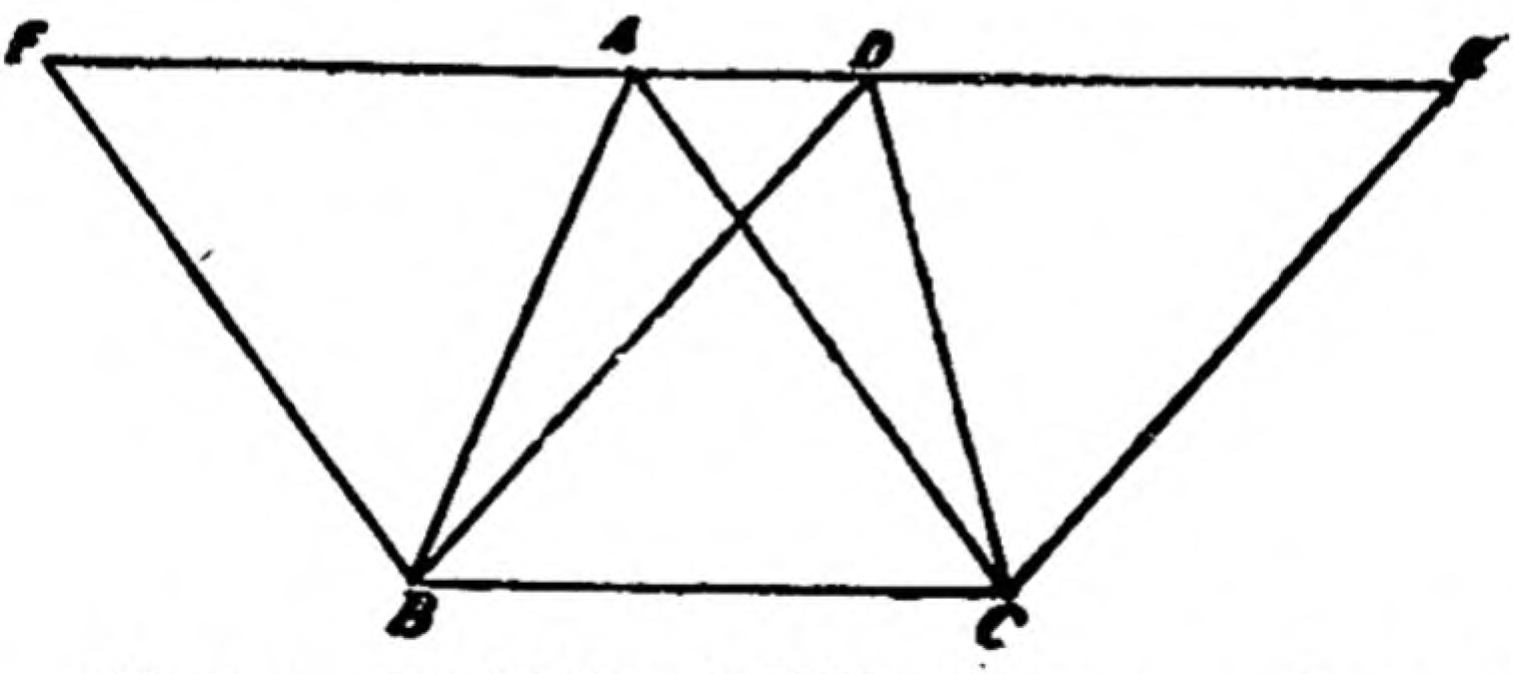
5. If a square and a rhombus stand on the same base, prove that the square has the larger area.

6. A point P, inside the parallelogram ABCD is joined to its angular points ; show that the sum of the △s PAB and CPD is equal to the sum of △s PAD and PBC, each sum being ½ of the parallelogram.

7. Divide a given ‖ᵐ of 3 ‖ᵐˢ of equal area.

Proposition 41. (*Theorem*).

Triangles on the same base and of the same altitude are equal in area.



Given :—△s ABC and DBC on the same base BC and having the same altitude.

Required :—To prove that these triangles are equal in area.

Construction :—Draw CE ‖ BD and BF ‖ CA to meet AD produced at E and F respectively.

Proof :—As the triangles have equal altitudes, they lie between the same parallels *i. e.*, FE ∥ BC.

∴ FE ∥ BC, CE ∥ BD, BF ∥ CA ∴ BCED and BCAF are ∥^{ms}.

Now ∥^{m} BCED ≡ ∥^{m} BCAF. (They stand on the same base and lie between the same parallels.)

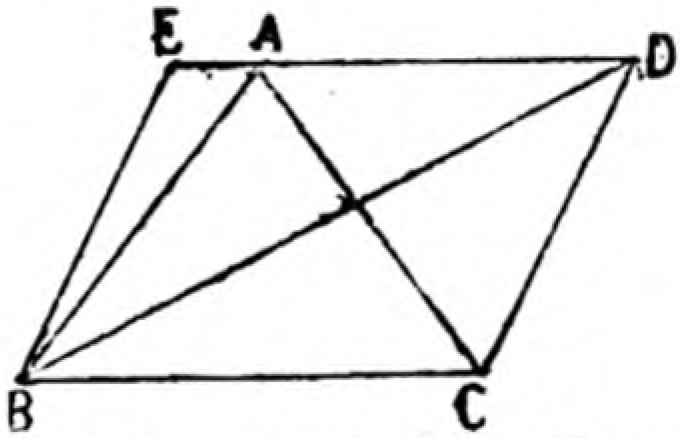
But $\triangle BCD = \frac{1}{2}$ ∥^{m} BCED (Each diagonal bisects a ∥^{m}).

and $\triangle ABC = \frac{1}{2}$ ∥^{m} BCAF (")

∴ $\triangle BCD = \triangle ABC$. **Q. E. D.**

Cor. 1. Triangles on the same base and between the ∥^{s} are equal in area.

Cor. 2. If a parallelogram and a triangle stand on the same base and have the same altitude, the area of the triangle is equal to half that of the parallelogram.



Given: —$\triangle ABC$ and ∥^{m} BCDE stand on the same base BC and have the same altitude.

Required :—To prove that $\triangle ABC = \frac{1}{2}$ ∥^{m} BCDE.

Construction :—Join BD.

Proof :—Now $\triangle BCD = \frac{1}{2}$ ∥^{m} BCDE.

(Diagonals of a ∥^{m} bisect it.)

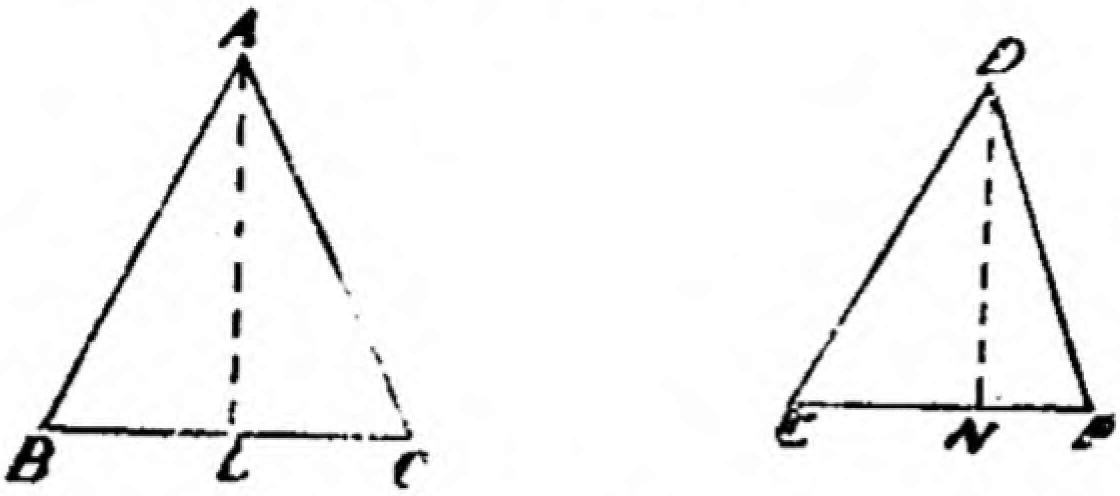
But $\triangle BCD = \triangle ABC$

(They stand on the same base BC and have the same altitude).

Hence $\triangle ABC = \frac{1}{2}$ ∥^{m} BCDE.

$= \frac{1}{2}$ base × height.

Cor. 3. *Triangles on equal bases, and of the same or equal altitudes are equal in area.*



Given :—Two triangles on equal bases BC and EF and having the same or equal altitudes AL and DM.

Required :—To prove that $\triangle$s ABC and DEF are equal in area.

Proof :—Area of $\triangle ABC = \frac{1}{2} BC \times AL$.

" $DEF = \frac{1}{2} EF \times DM$.

But $BC = EF$ (given)

and $AL = DM$ (given)

$\therefore \frac{1}{2} EF \times DM = \frac{1}{2} BC \times AL$

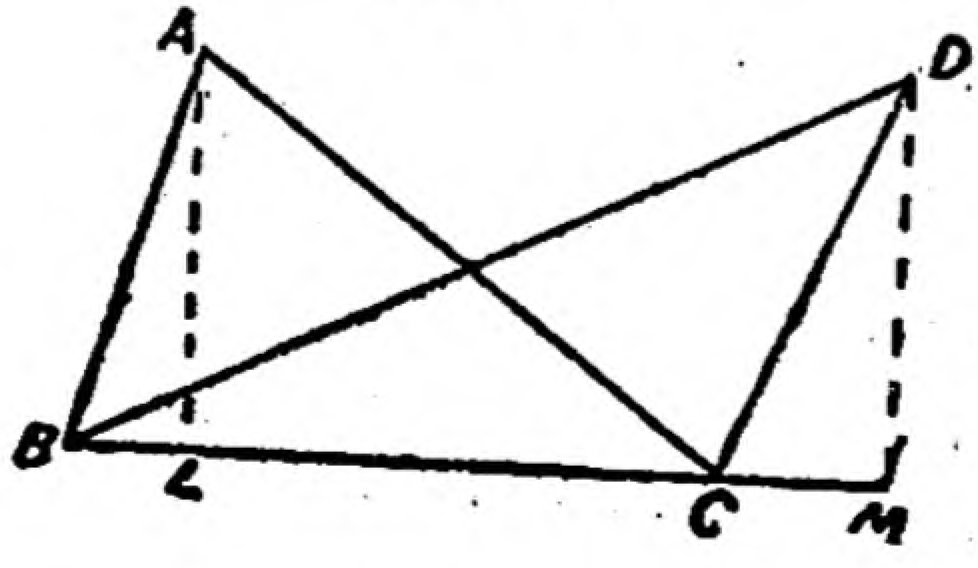
or $\triangle DEF = \triangle ABC$.

Q. E. D.

Exercises.

1. Equal triangles on the same or equal bases are of the same altitude.

CASE 1.



Given:—$\triangle$s ABC and DBC on the same base BC and of equal areas.

Required:—To prove that their alts. AL, DM are equal.

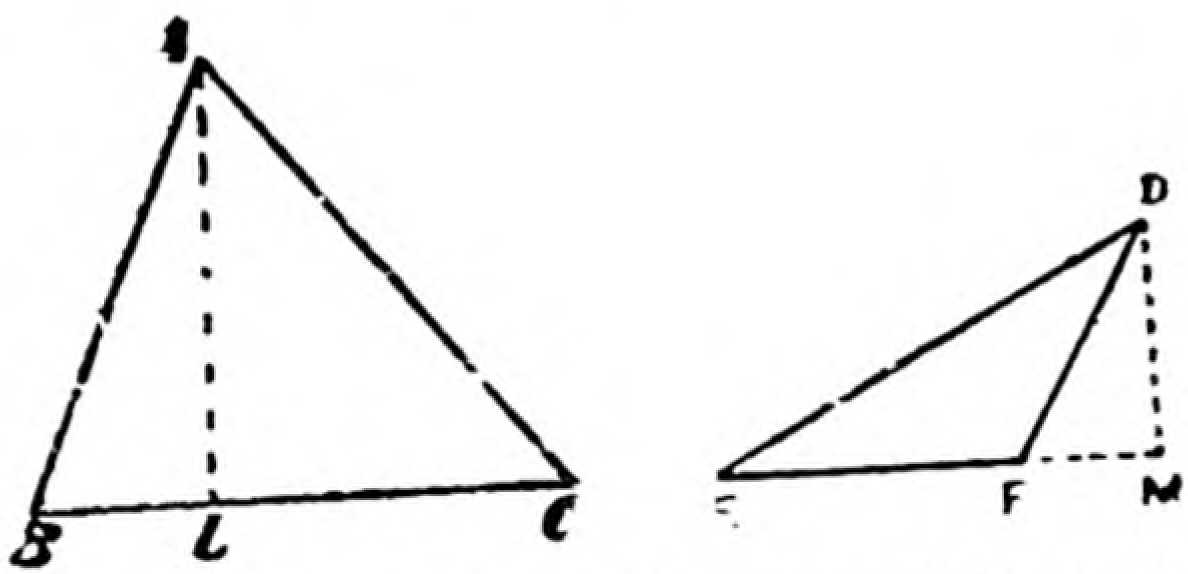
Proof:—Area of $\triangle ABC = \frac{1}{2} AL \times BC$.

Area of $\triangle DBC = \frac{1}{2} DM \times BC$.

But the areas are equal and BC is common.

$\therefore$ AL=DM. (Given).

CASE II.



Given:—$\triangle$s ABC and DEF on equal bases BC and EF and of equal areas.

Required:—To prove that their alts. AL and DM are equal.

Proof :—Area of $\triangle ABC = \frac{1}{2} AL \times BC$.

Area of $\triangle DEF = \frac{1}{2} DM \times EF$.

But the areas are equal and BC=EF (Given)

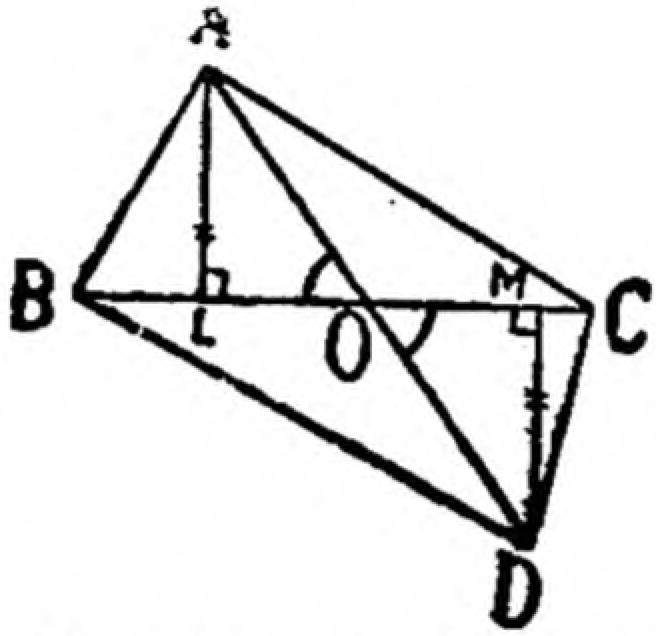
$\therefore$ AL=DM.

2. *Triangles of equals area, on equal bases in the same st. lines and on the same side of it, are between the same parallels.*

3. If a quadrilateral is bisected by each of its diagonals it is a parallelogram.

4. *Two triangles of equal area stand on the same base but on opposite sides of it ; show that the straight line joining their vertices is bisected by the base or the base produced.*

(Punjab, 1926 and Cambridge).

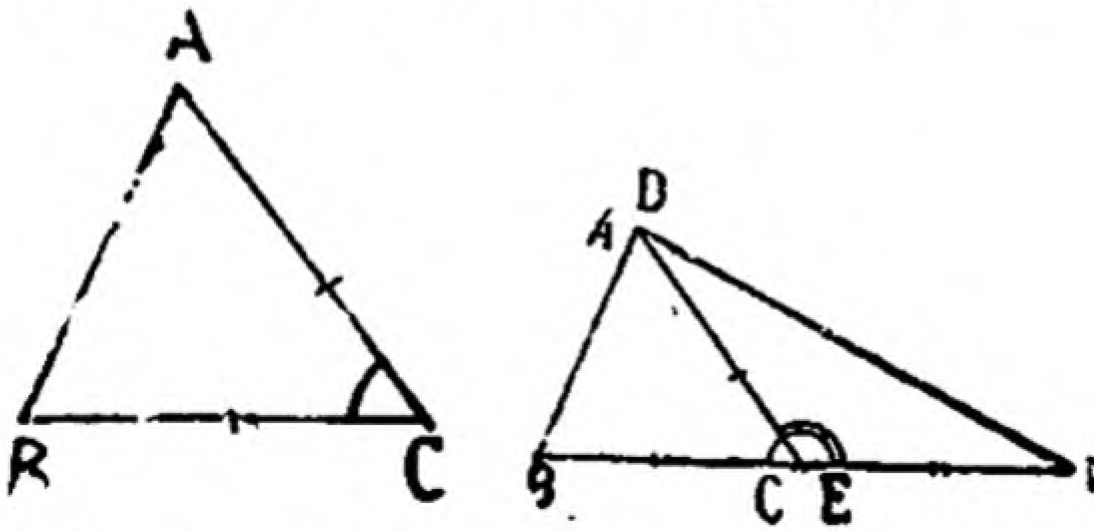


5. *Every median of a triangle bisects the triangle.*

6. The diagonals of a ||m divide it into four equivalent triangles. (Cambridge).

7. Two triangles have two sides of the one equal to two sides of the other, each to each, and the contained angles are supplementary. (Punjab, 1914)

Prove that their areas are equal.



8. If from any point in the base of an isosceles △ perpendiculars be drawn to the other sides their sum will be equal to the perpendicular from either extremity of the base upon the opposite side.

9. *The sum of the perpendiculars drawn from a point within an equilateral triangle on the sides is equal to the altitude of the triangle.* (Calcutta).

10. If one diagonal of a quadrilateral bisects the other, it also bisects the quadrilateral.

11. AD, BE and CF are the medians of the triangle ABC intersecting at G ; show that $\triangle ABG = \triangle GBC = \triangle GAC = \triangle \frac{1}{3}$ ABC.

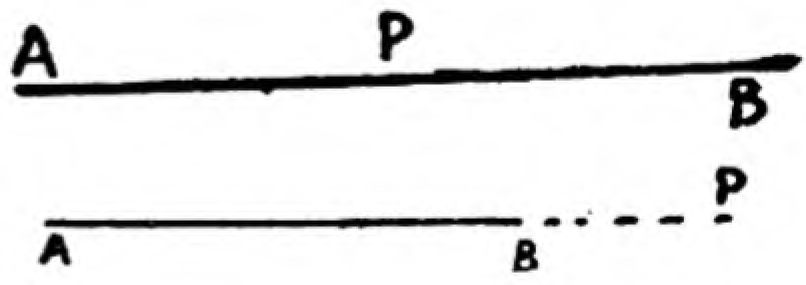
12. Triangles having the same altitude are in the ratio of their bases.

13. Given a triangle ABC, construct another triangle ABD, so that the area of the latter is half as much again as the area of the former triangle.

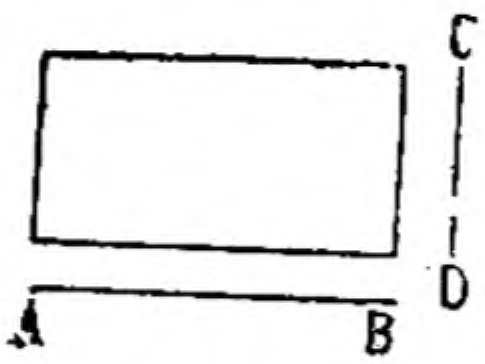
14. EF is any st. line parallel to the base BC of a triangle ABC. BG and CH are drawn parallel to AC and AB meeting EF in G and H respectively. Show that the $\triangle$s ABG and ACH are equal in area.

Def. If a point P be taken in a line AB, then AB is said to be divided *internally* at P and the segments are AP, PB.

If a point P be taken in AB produced, then AB is said to be divided *externally* at P and the segments are AP, PB.



Def. If AB, CD be two st. lines, any rectangle, which has one side equal to AB and the adj. side equal to CD, is called the rectangle contained by AB, CD.

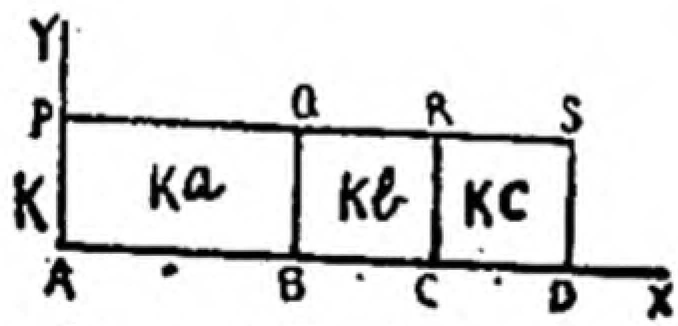


The rect. contained by AB, CD is denoted by AB, CD where AB stands for the number of units of length in AB and CD for the number of units of length in CD.

Similarly the sq. drawn on the side AB is denoted by the sq. on AB or AB^2.

Proposition 42. (*Theorem*)

Illustration and explanation of the geometrical theorem corresponding to the algebraic identity, $K\ (a+b+c)=Ka+Kb+Kc$.



Draw two st. lines AX, AY at right $\angle$s to each other. Along AX set off AB, BC, CD equal to a, b, c units of length respectively so that AB$=(a+b+c)$ units of length. Along AY set off AP$=$K units of length. Complete the rects. as shown in the figure.

Proof :—All the figures are rectangles.

Area of rect. AS$=$K$(a+b+c)$, units of area.

Area of rect. AQ$=$Ka. ,,

Area of rect. BR$=$Kb. ,,

Area of rect. CS$=$Kc. ,,

But rect. AS$=$Rect. AQ$+$Rect. BR$+$Rect. CS.

Hence K$(a+b+c)=$K$a+$K$b+$Kc.

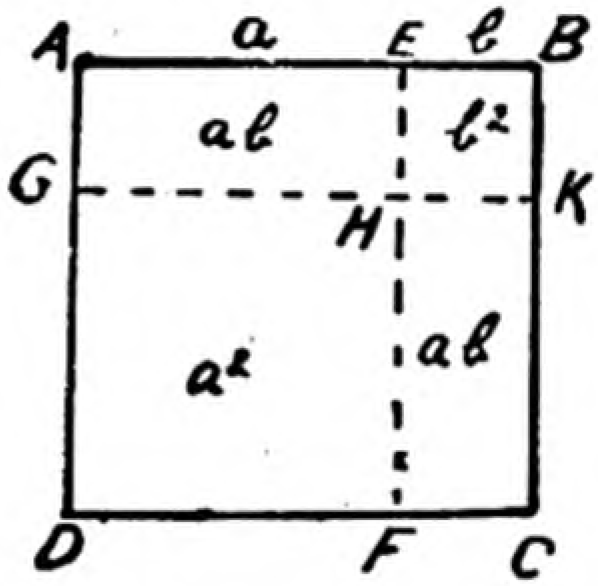
This gives the following geometrical theorem :--

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line, and the several parts of the divided line.

Proposition 43. (*Theorem.*)

Illustration and explanation of the geometrical theorem corresponding to the algebric identity.

$$(a+b)^2=a^2+b^2+2ab.$$



Take AB divided at E, so that AE$=a$ units of length and EB$=b$ Units of length.

On AB construct the square ABCD.

Through E draw EF $\parallel$ AD.

Cut off AG$=$b, and through G draw GHK $\parallel$ AB, cutting EF at H and BC at K.

Proof :—All the angles formed are right angles, therefore all the figures are rectangles or squares.

Sq. AC$=(a+b)^2$ units of area.

Sq. GF$=a^2$,, ,,

Sq. EK$=b^2$,, ,,

Rect. AH$=ab$,, ,,

Rect. HC$-ab$,, ,,

But the figure AC consists of sqs. GF, EK and rects AH, HC.

$\therefore\ (a+b)^2=a^2+b^2+2ab.$

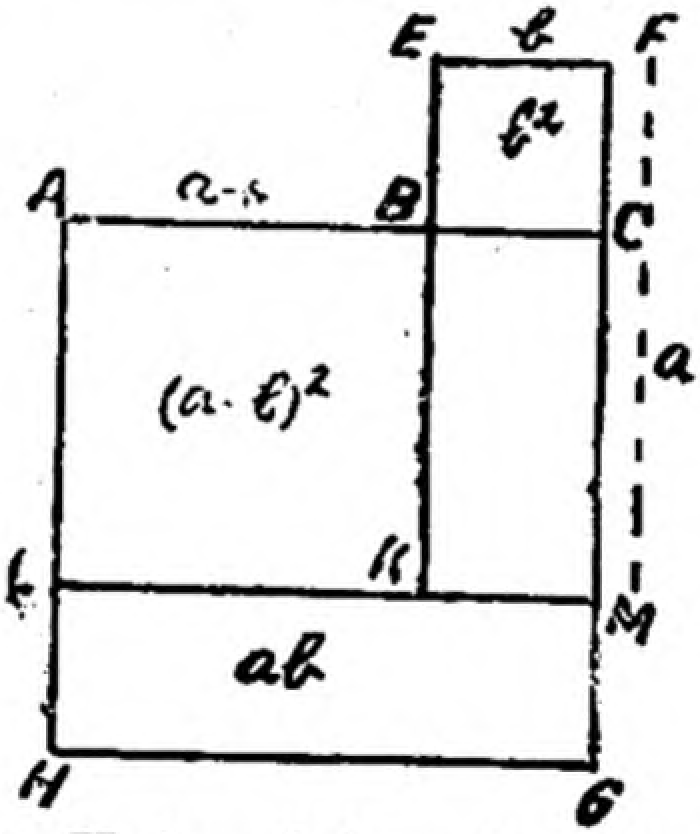
This gives the following geometrical theorem :—

If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts, together with twice the rectangle contained by the parts.

Proposition 44. (*Theorem*)

Illustration and explanation of the geometrical theorem corresponding to the algebraic identity.

$$(a-b)^2=a^2+b^2-2ab.$$



Take AC$=a$ Units of length and its part BC$=$b Units of length so that AB$=a-b$ Units of length.

On AC describe the sq. ACGH.

On BC describe the sq. BCFE.

On AB describe the sq. ABKL.

Produce LK to meet CG in M.

Proof :—All the figures are either sqs. or rects.

Now the whole figure$=$sq. AG$+$sq. EC$=a^2+b^2$ units of area.

Again the whole fig. is made up of three parts sq. AK, rects. LG and EM, which contain respectively $(a-b)^2$, ab and ab units of area.

$$\therefore\ (a-b)^2+ab+ab=a^2+b^2.$$

or $$(a-b)^2+2ab=a^2+b^2$$

or $$(a-b)^2=a^2+b^2-2ab.$$ [By transposition].

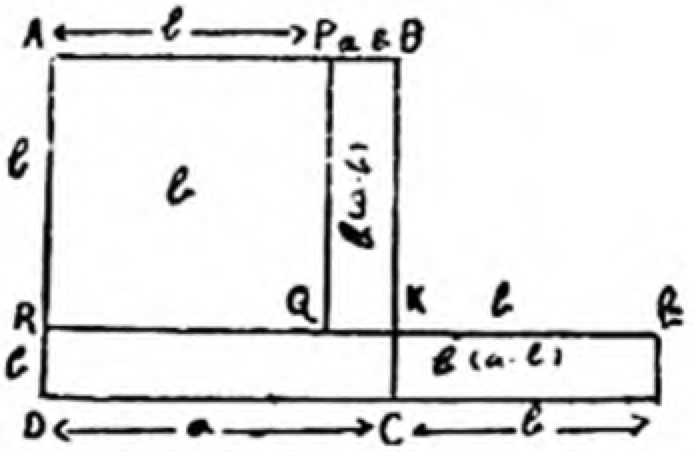
This gives the following geometrical theorem :—

If a straight line be divided externally into any two segments, the square on the straight line is equal to the sum of the squares on the segments minus twice the rectangle contained by the segments.

Proposition 45. (*Theorem*)

Illustration and explanation of the geometrical theorem corresponding to the algebraic identity :

$$a^2-b^2=(a+b)(a-b)$$



Take a st. line AB$=a$ units of length. From it cut off AP$=b$ units of length, so that PB$=a-b$ units of length.

On OB draw the sq. AC. On AP draw the sq. AQ. Produce RQ to L, so that KL$=b$ units of length. Complete the rect. KM.

Proof :—Rect. BQ$=$PQ.PB$=b(a-b)$ units of area.

Rect. KM$=$KL.LM$=b(a-b)$,,

$\therefore$ Rect. BQ$=$Rect. KM ,,

And Rect. RM$=$DM.LM$=(a+b)(a-b)$,,

Now Sq. AC$-$sq. AQ$=$Rect. BQ$+$Rect. RC.

$=$Rect. KM$+$Rect. RC.

[Rect. BQ$=$Rect. KM Proved]

$=$Rect. RM.

Hence $a^2-b^2=(a+b)(a-b)$.

This gives the following geometrical theorem :—

The difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.

Exercises.

1. Give geometrical representation of
$a(a+b)=a^2+ab$.

2. Prove geometrically that
$(a+b)^2=a(a+b)+b(a+b)$.

3. Prove geometrically that
$(a+b)(c+d)=ac+ad+bc+bd$.

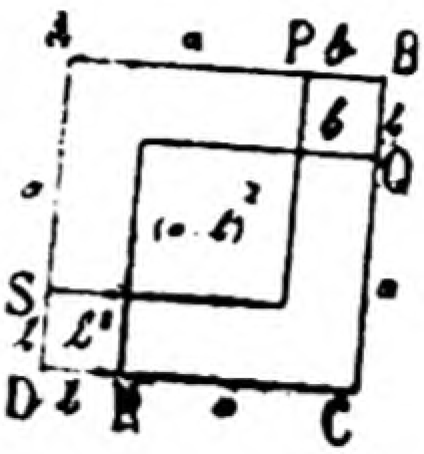
4. Prove taht the square on any st. line is equal to four times the square on half the line.

5. Draw a diagram to illustrate the identity :
$(p+q+r)^2=p^2+q^2+r^2+2qp+2pr+2qr$.

6. Illustrate and prove geometrically the identity ;

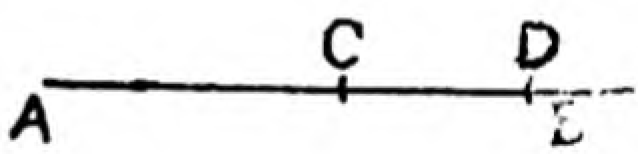
$(a+b)^2-(a-b)^2=4ab$

7. **Illustrate and prove geometrically the identity ;** $(a+b)^2+(a-b)^2=2(a^2+b^2)$.



8. If a st. line is divided equally and also unequally, the rect. contained by the unequal parts is equal to the difference on the squares on half the line and on the line between the pts. of division.

Thus, if AB is divided equally at C and unequally at D, then, AD. DB$=$AC$^2-$CD^2.

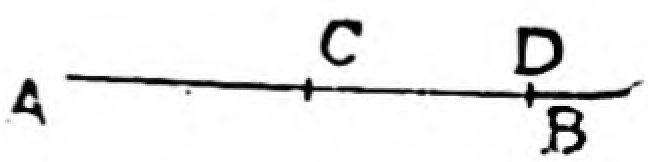


Hint.—AD, DB$=$(AC$+$CD)(BC$-$CD), etc.

9. If a rectangle and a square have the same perimeter, the square will have the larger area.

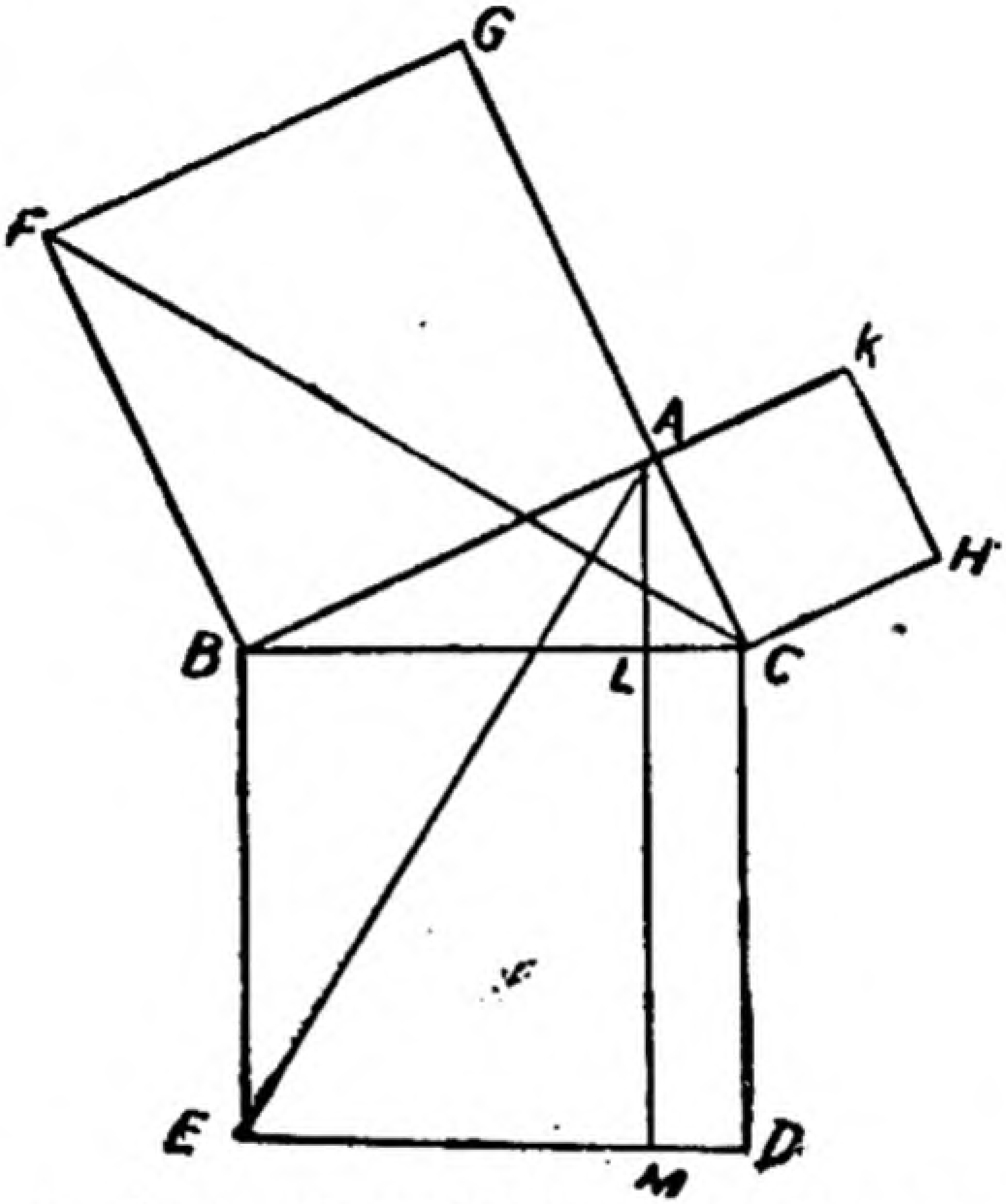
Hint.—Follows immediately from the identity $(a+b)^2=4ab+(a-b)^2$.

Let AB$=$semi-perimeter ; C is the mid-point of AB and D any other point. Now $\because$ AD.BD$+$CD$^2=$AC^2 $\therefore$ AD.BD is greatest when CD$=0$, that is, when C coincides with D or when AC$=\frac{1}{2}$ AB or AC is the side of a square.



Proposition 46. (*Theorem*).

In a right angled triangle, the square on the hypotenuse is equal to the sum of the squares on the sides containing the right angle.



Given :—A rt. ∠d △ABC having the rt. ∠ at A.

Required :—To prove $BC^2 = AB^2 + AC^2$.

Construction :—(1) On the sides describe squares as in the figure.

(2) Draw ALM ∥ BE.

(3) Join CF and AB.

Proof :—In △s ABE, FBC,

∠ABE = ∠FBC. (each $= 90° + \angle ABC$).

BE = BC (sides of a square).

BA = BF.

∴ △ABE ≡ △FBC.

Now CAG is a st. line ($\angle CAB + \angle BAG = 180°$).

$\therefore$ Sq. ABFG$=2\triangle$ FBC (same base FB and same $\parallel$s FB, GC).

Also rect. BEML$=2\triangle$ ABE (same base BE and same $\parallel$s BE, AM).

$\therefore$ Rect. BEML$=$sq. ABFG.

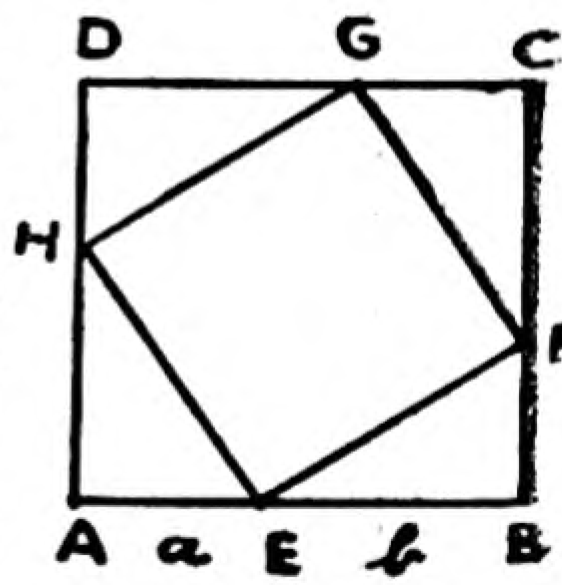
Similarly rect. CDML$=$sq. ACHK.

$\therefore$ By addition sq. BCDE$=$sq. ABFG$+$sq. ACHK.

Or, $BC^2=AB^2+AC^2$.

Cor. 1. In a rt. $\angle$d $\triangle$ABC. if LA is the perp. from the rt. $\angle$ upon the hyp. BC, then $AB^2=BC.\ BL$ and also $AC^2=BC.\ CL$.

Note.—This proposition is known as the Theorem of Pythagoras, as it was discovered by a Greek mathematician named Pythagoras about 550 B. C. The proposition is very useful and plays a very important part in the domain of mathematics Apparently it attracted the fancy of great mathematicians to an enormous extent with the result that over one hundred proofs of the proposition have already been advanced. Some of these proofs are given below :—



9. ABCD is a square on a straight line AB$=(a+b)$ units of length. From AB, BC, CD and DA lengths AE, BF, CG and DH each $=a$ have been cut off so that the remaining lengths EB, FC, GD and HA are each equal to b units of length. EF, FG, GH and HE have been joined. Prove the following results :—

(*a*) EFGH is a square

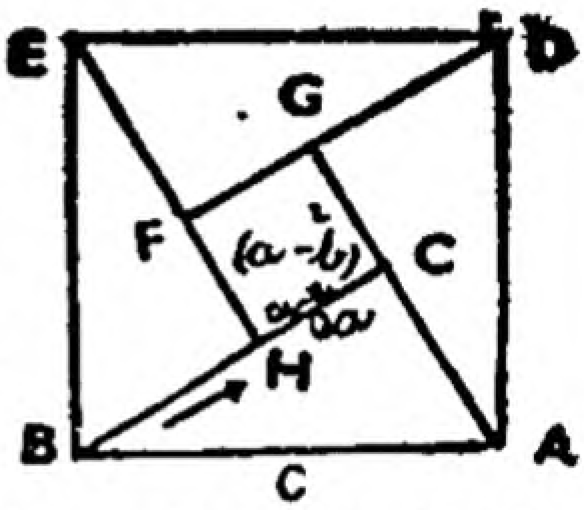
(*b*) Area of sq. ABCD$=AB^2=(a+b)^2=a^2+b^2+2ab$.

(*c*) Area of each of the right angled triangles HAE, FBE, FCG, GDH$=\frac{1}{2}\ a.\ b$.

(*d*) Area of sq. ABCD=sq. EFGH+2 ab.

(*e*) Compare results (*b*) and (*d*) and state the new results thus got in words.

[Result may be thus stated :—In a rt. ∠d. △ the square on the hypotenuse is equal to the sum of the squares on the remaining two sides.]



10. ABC is a right angled triangle with sides a, b and c units in length. On AB has been described the square ABED. From E and D are drawn EH and DF respectively perpendiculars to BC and EH. AC has been produced to meet DF at G. Prove the following results :—

(*a*) Triangles ABC, EHB, DFE and AGD are all right angled, each with an area$=\frac{1}{2}ab$.

(*b*) FGCH is a square with one side$=a-b$ units.

(*c*) Square on AB =sq. FC+ 4 △ ABC.

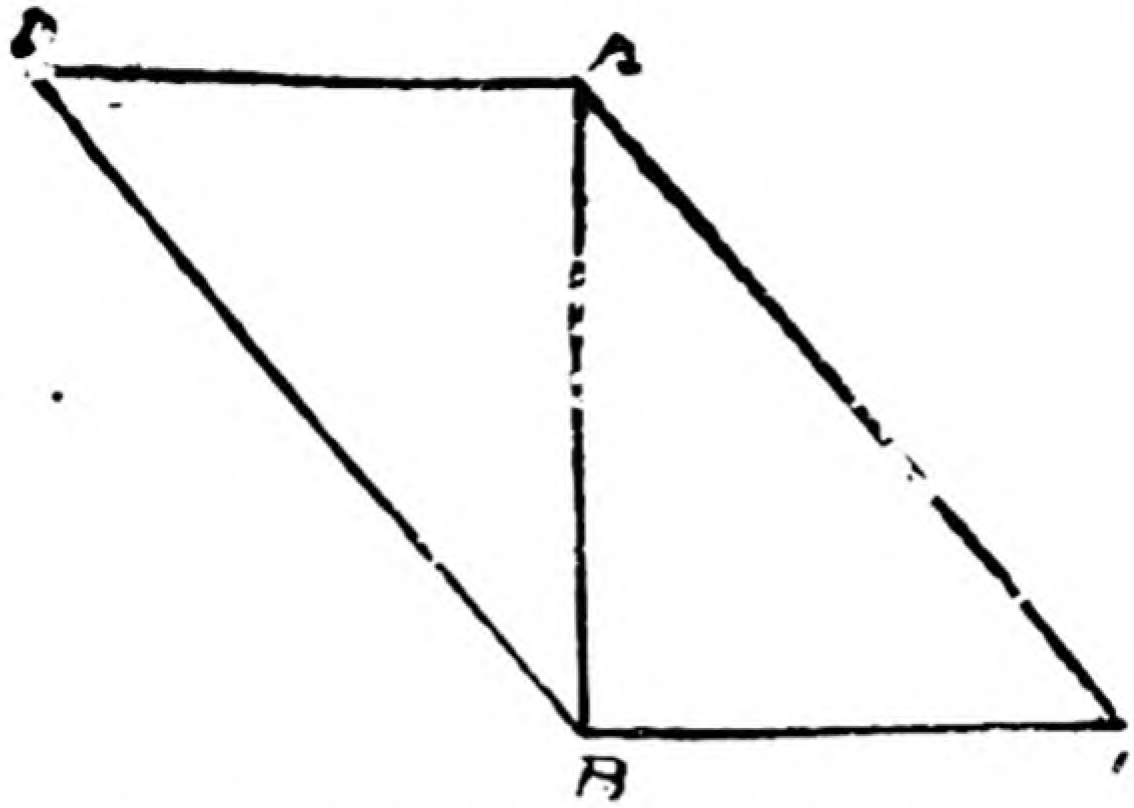
(*d*) From (C) infer that $c^2=a^2+b^2$.

(*e*) State the last result in words.

———

Proposition 47. (*Theorem*)

If the square on one side of a triangle be equal to the sum of the squares on the other two sides, the angle contained by these sides is a right angle.



Given :—In $\triangle$ ABC, $AC^2=AB^2+BC^2$.

Required :—To prove that $\angle$ABC is a right angle.

Construction :—Draw $AD \perp AB$ and make it equal to BC. Join BD.

Proof :—As angle BAD is a right angle,

$\therefore$ $BD^2=AD^2+AB^2$.

$=BC^2+AB^2$ (AD=BC)

$=AC^2$ (Given)

$\therefore$ BD=AC.

Now in $\triangle$s ABC and ABD.

AC=BD (Proved)

AB=AB

BC=AD (Const.)

$\therefore$ $\triangle$s are congruent.

Hence $\angle ABC=\angle BAD=$one right angle.

Q. E. D.

Note.—This proposition is the converse of Pythagoras Theorem.

Exercises.

1. *The square on the diagonal of a given square is double of the given square.*

2. PS is $\perp$ to the base of a $\triangle$ PQR ; prove that $PQ^2+RS^2=PR^2+QS^2$.

3. ABCD is a quadrilateral having AC $\perp$ to BD ; prove that $AB^2+CD^2=BC^2+DA^2$.

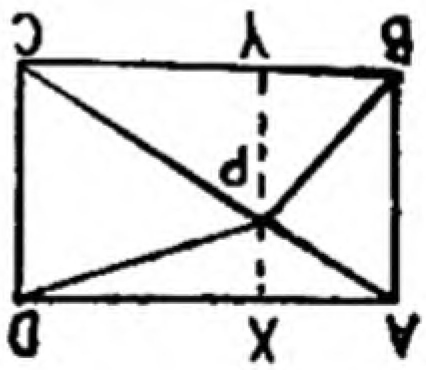
4. *The sum of the squares on the sides of a rhombus = the sum of the squares on the diagonals.*

(Calcutta, Matric.)

5. From the vertex of a triangle a perpendicular is drawn to the base. Prove that the difference of the squares on the two sides is equal to the difference of the squares on the two segments of the base.

6. *If any point P is joined to the angular points of a rectangle ABCD, prove that* $PA^2+PC^2=PB^2+PD^2$.

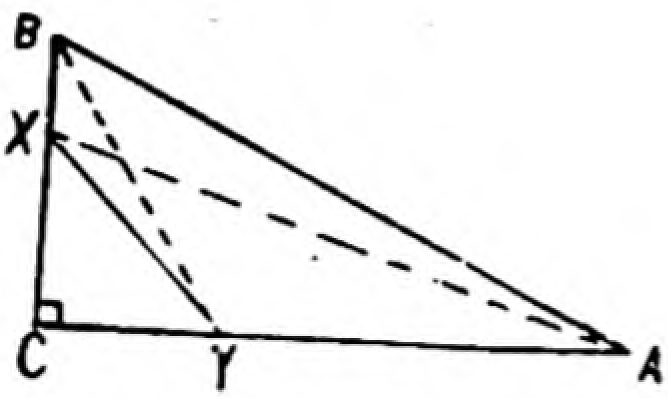
(Bombay Matric).



7. If D,E,F are feet of the $\perp$s from the angular points of a triangle ABC to the opposite sides, prove that $AE^2+BF^2+CD^2=AF^2+BD^2+CE^2$.

(Bombay)

8. ABC is a $\triangle$ right-angled at C and XY is a straight line intersecting CB at X and CA at Y. AX and BY are joined. Prove that $AB^2+XY^2=AX^2+BY^2$.



9. *ABC is a △ right-angled at A, BE, CF are the medians. Prove that* $4(BE^2+CF^2)=5BC^2$.

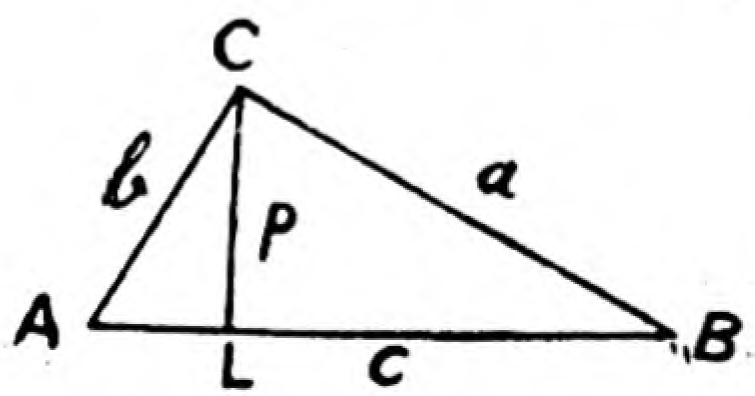
10. *In a right-angled triangle four times the sum of the squares on the medians is equal to six times the square on the hypotenuse.*

Hint.—ABC, a triangle having ∠B a rt ∠. AD, CE and BF the medians; $AD^2=AB^2+BD^2$ (Pythagoras) $CE^2=BC^2+BE^2$ $\therefore 4(AD^2+CE^2)=4AB^2+4BD^2+4BC^2+4BE^2=4AB^2+4BC^2+BC^2+AB^2=5AB^2+5BC^2=5AC^2$ $\therefore 4(AD^2+CE^2+BF^2)=5AC^2+AC^2=6AC^2(\because BF=\frac{1}{2}AC)$.

11. *In an equilateral triangle three times the square on a side is equal to four times the square on the ⊥ drawn from any vertex to the opposite side.*

12. ABC is a △ right-angled at C. If, a, b, c, denote the three sides opposite to ∠s. A, B, C respectively and p the perpendicular drawn from the right angle to the hypotenuse, prove that

(*i*) $pc=ab$; (*ii*) $\dfrac{1}{p^2}=\dfrac{1}{a^2}+\dfrac{1}{b^2}$



(*i*) $b^2=AL\,C$

$a^2=BL.C$

$\therefore a^2b^2=C^2.\ AL.\ BL.$

$=C^2P^2$ $[AL.BL=p^2]$

$\therefore ab=cp.$

(*ii*) $\dfrac{1}{a^2}+\dfrac{1}{b^2}=\dfrac{b^2+a^2}{a^2b^2}$

$$= \frac{c^2}{a^2b^2}$$

$$= \frac{c^2}{c^2p^2} \text{ [proved above]}$$

$$= \frac{1}{p^2}$$

13. *In a right-angled triangle the square on the* $\perp$ *from the rightangle on the hypotenuse is equal to the rectangle contained by the segments of the hypotenuse made by the perpendicular.*

14. If the difference of the squares on two sides of a triangle be equal to the square on the third side, the triangle is right-angled.

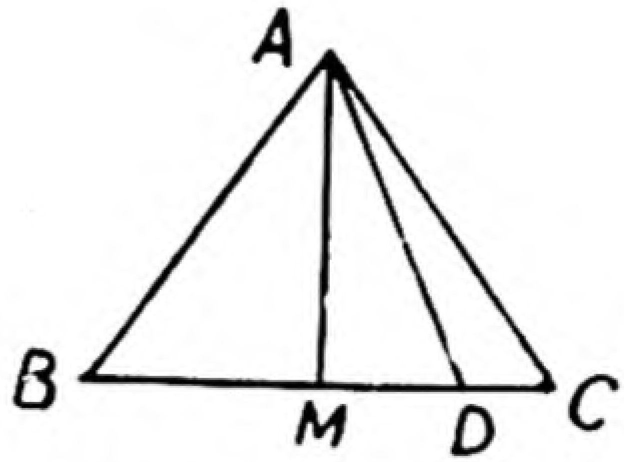
Hence or otherwise, show that the $\triangle$ whose sides are m^2+n^2, m^2-n^2, and $2mn$ is a right-angled $\triangle$.

15. Find the distance between the tops of two towers 116 ft. and 80ft. high and 48ft. apart.

16. The sides of a triangle are represented by the numbers 2, $\sqrt{3}$ and 1; show that it is a right-angled triangle having an acute angle of 60°.

[**Hint:** Join mid. pt. of longest side to the opp. $\angle$].

17. D is any pt. in the base BC of an isosceles triangle ABC. Prove that $AB^2-AD^2=BD.\,DC$.



Const:—Draw $AM \perp BC$.

Proof:—M is the mid. pt. of BC. ($\triangle$s. ABM and ACM).

$BD = BM + MD$

$DC = MC - MD$

$= BM - MD \qquad (\because BM = MC)$

$\therefore BD.\ DC = (BM + MD)(BM - MD)$

$= BM^2 - MD^2$

$= (BM^2 + AM^2) - (MD^2 + AM^2)$

[add and subtract AM^2]

$= AB^2 - AD^2$ **Q.E.D.**

PROJECTION.

Def. The Projection of a point on a given straight line is the foot of the perpendicular drawn from the point to the straight line.

When the point lies on the given line, *it is the projection of itself* on that line, as the point and the foot of the perpendicular coincide.

The projection of a *straight line* on another is the portion of the second line intercepted between the feet of the perpendiculars drawn from the ends of the first to the second.

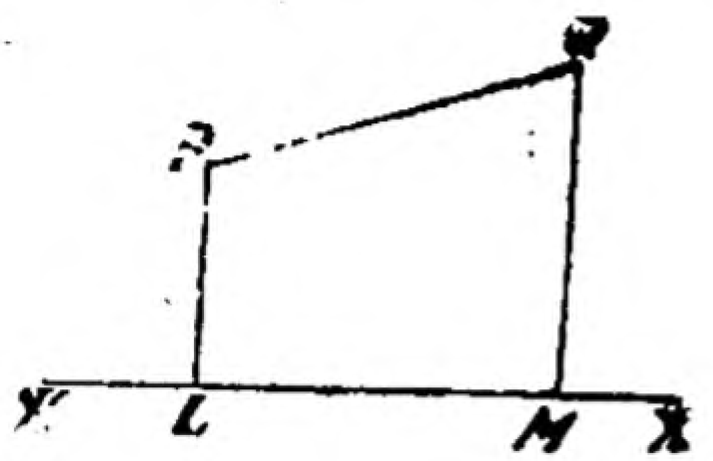


Fig. 1.

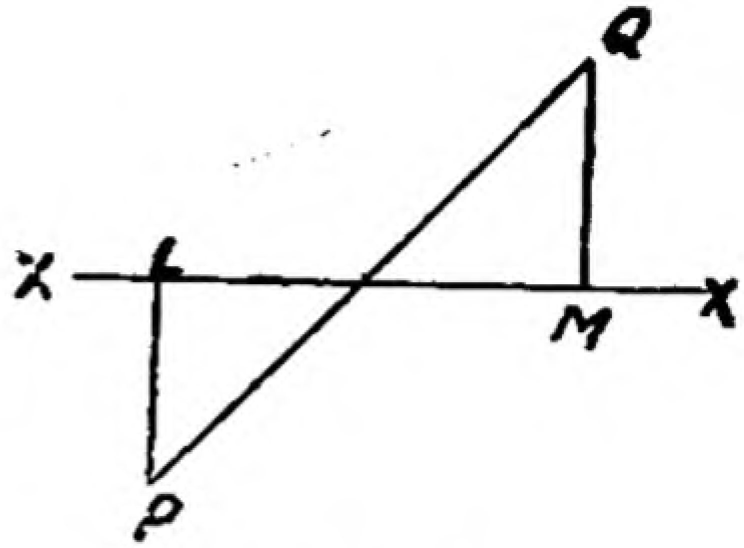


Fig. 2.

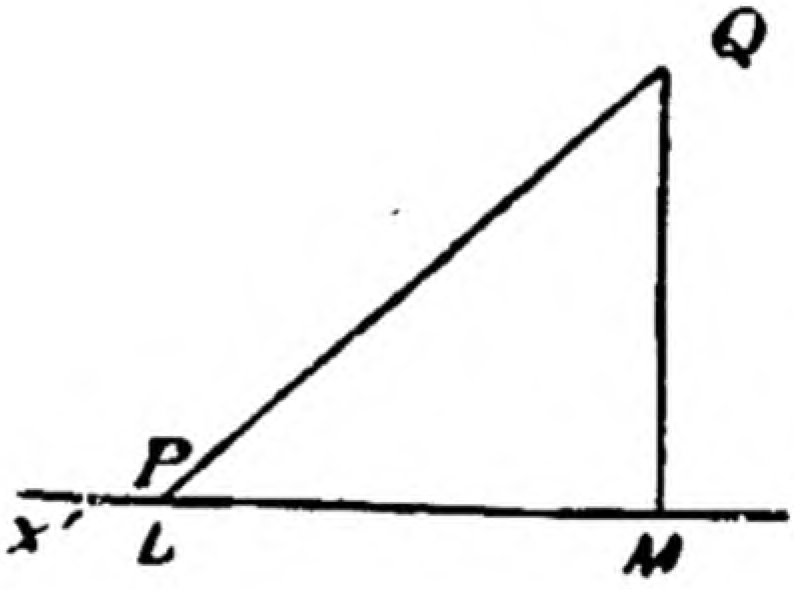


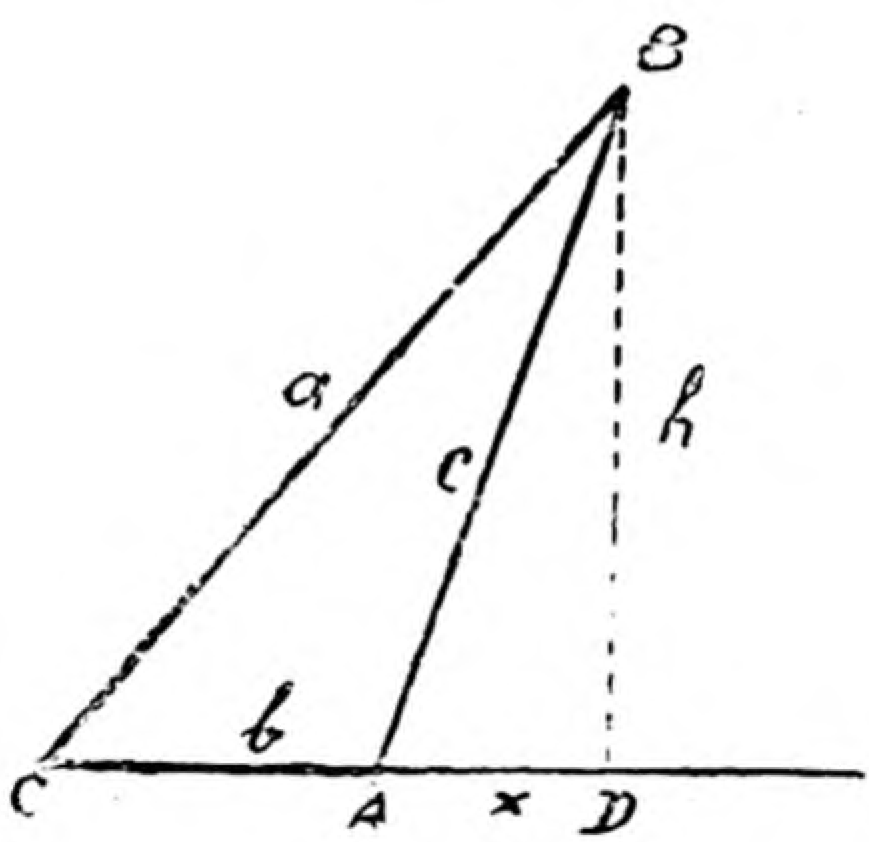
Fig. 3.

Thus in the above figures M is the projection of the point Q on the given line X′X and L is the projection of P ; but in Fig. 3 the point P lying on the line X′X coincides with its projection. Therefore, in each case, the line LM is the projection of the line PQ on the given line XX′.

Note.—Generally, in the triangle ABC, the sides opposite to angles A,B, and C are denoted by the corresponding small letters *a*, *b*, *c*.

Proposition 48 (*Theorem*)

In an obtuse angled triangle, the square on the side opposite to the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle together with twice the rectangle contained by one of the sides, and the projection on it of the other.



Given :—$\triangle$ABC having obtuse angle at A.

Let a, b, c be the usual names of the sides, of the $\triangle$ABC, and h, the perpendicular from B meeting CA produced in D, so that AD $(=x)$ in the projection of AB on CA.

Required :—To prove that $a^2=b^2+c^2+2bx$.

Proof :—$\because$ BDC is a rt. angled triangle (Const.)

$\therefore \quad a^2=(b+x)^2+h^2$

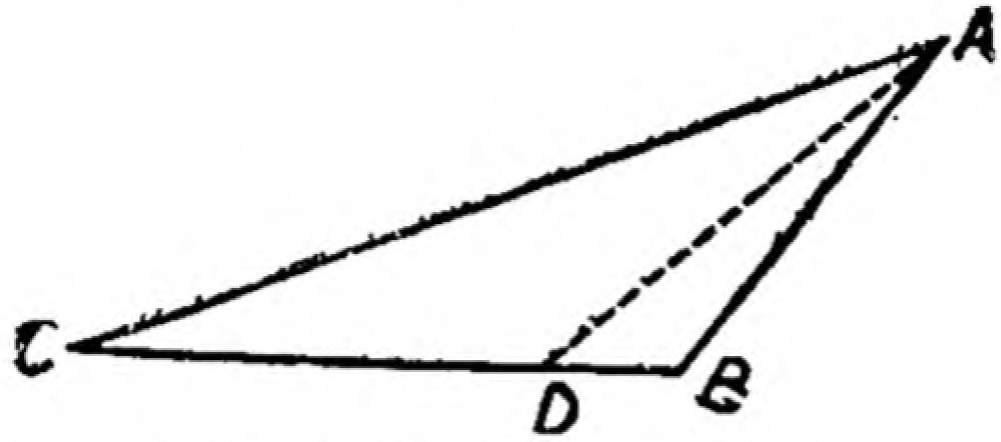
$\quad\quad =b^2+2bx+x^2+h^2$

But $x^2+h^2=c^2$. [ABD is rt. $\angle d$ $\triangle$ rt. $\triangle d$ at D].

$\therefore \quad a^2=b^2+2bx+c^2$. **Q. E. D.**

Exercises.

1. If the square on one side of a triangle is greater than the sum of the squares on the remaining two sides, then it is an obtuse angled triangle. (Converse of the theorem.)



Given :—$AC^2 > AB^2 + BC^2$

To prove :—$\angle ABC$ is obtuse.

Const :—If $\angle ABC$ is not obtuse, it is either right or acute.

Proof :If $\angle ABC$ is rt $\angle$, then $AC^2 = BC^2 + AB^2$.
But this is against what is given ($AC^2 > AB^2 + BC^2$.
$\therefore \angle ABC$ is not a rt. $\angle$.

If $\angle ABC$ is acute, perpendicular from A to BC must fall within the line BC. Let AD be such perpendicular.

Now $\angle ADC$ is a rt. $\angle$ (Sup.)
$\therefore AC^2 = CD^2 + AD^2$.

But $AD^2 < AB^2$ [$\because$ AD opp. $\angle B$ (less than a rt. $\angle$) is shorter than AB opp. to a rt. $\angle$ (Sup.).
and $CD^2 < BC^2$ [$\because$ CD is part of BC.]
$\therefore AC^2 < (AB^2 + BC^2)$.
This is also against what is given.

$\therefore$ Only one alternative remains i.e. $\angle ABC$ is obtuse.

Q. E. D.

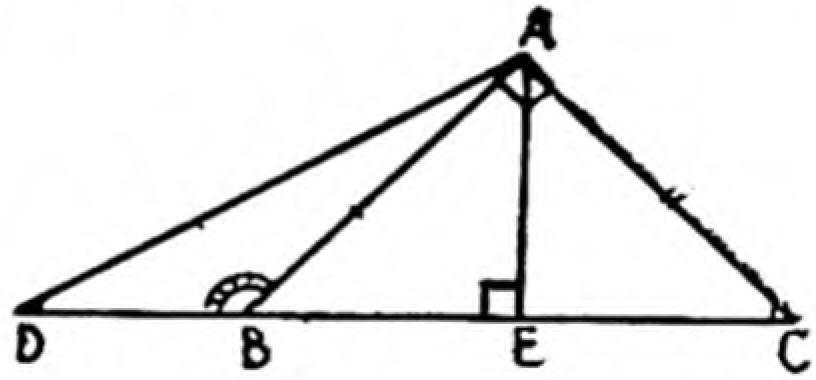
2. *The side BC of an equilateral triangle ABC is produced to D and AD is joined; show that $AD^2=BC^2+CD^2+BC.CD$.*

Hint.—Draw AE $\perp$ BC, EC$=\frac{1}{2}$ BC. $\angle ACD=120^\circ$ $=$obtuse angle $\therefore$ $AD^2=AC^2+CD^2+2EC.\ CD=BC^2+CD^2+BC.CD$.

3. In the triangle ABC, $\angle A=120^\circ$. Prove that $a^2=b^2+c^2+bc$.

4. *In an isosceles triangle ABC angle A is a right angle, D is a point in CB produced: if $AD^2=DB^2+3BA^2$, prove that $DB=BC$.* (*Madras. Matric.*

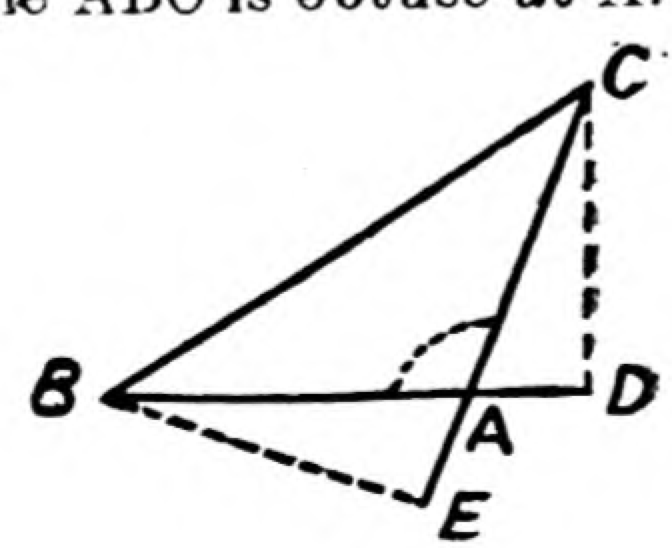
Hint.—Draw AE$\perp$BC; then $AD^2=DB^2+BA^2+2BD.\ BE=DB^2+3BA^2$. $\therefore$ $2DB.\ BE=2BA^2=BC^2$ $\therefore$ $DB=BC$. ($\therefore$ $2BE=BC$).



5. In a triangle ABC, C is an obtuse angle, AD $\perp$ BC produced and from AD produced DF is cut off$=$AB and DG$=$AC. Show that $FC^2=GB^2$, etc.

6. AB, the base of an isosceles triangle ABE is produced to L. Prove that $EL^2=BE^2+AL.\ BL$.

7. Triangle ABC is obtuse at A.



CD and BL are perpendiculars meeting the sides containing the obtuse angle at D and E.

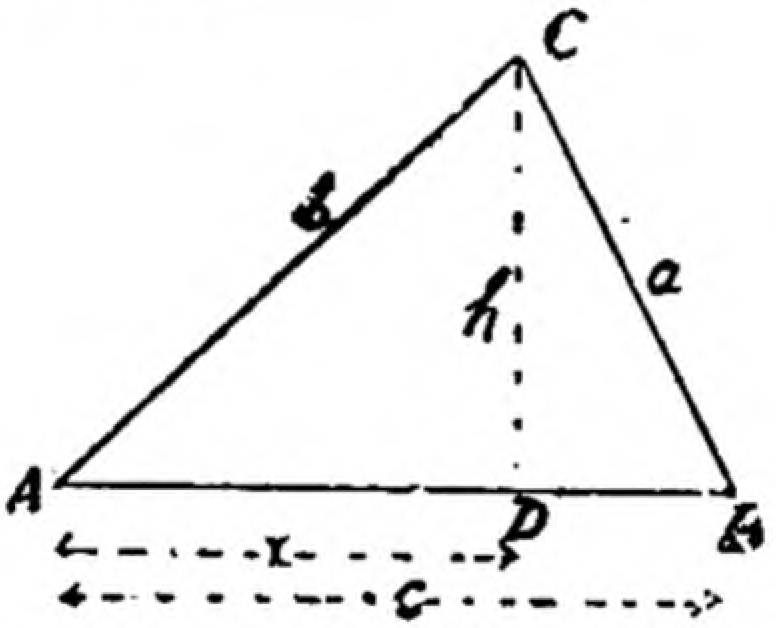
Show that BA.AD=CA.AE.

Hence prove that

$BC^2=AB.BD+AC.CE$.

Proposition 49 (*Theorem*).

In any triangle, the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing that acute angle diminished by twice the rectangles contained by one of those sides and the projection on it of the other.



Given :—$\triangle ABC$ having acute $\angle$ at A.

$BC=a$, $CA=b$, and $AB=c$, $CD \perp AB$, so that AD is the projection of AC on AB. Let $AD=x$ and $CD=h$.

Required :—To prove that $a^2=b^2+c^2-2cx$.

Proof :—$\because$ CDB is a rt. $\angle$ d $\triangle$. (Const.)

$\therefore a^2=h^2+DB^2$.

$=h^2+(c-x)^2$. $\because BD=(c-x)$

$=h^2+x^2+c^2-2cx$.

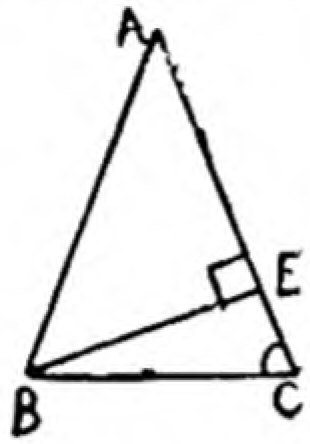
But $h^2+x^2=b^2$.

$\therefore a^2=b^2+c^2-cx$,

Q. E. D.

Exercises.

1. If from an extremity of the base of an isoceles $\triangle$ a perpendicular is drawn to the opposite side, twice the rectangle contained by that side and the segment adjacent to the base is equal to the square on the base.



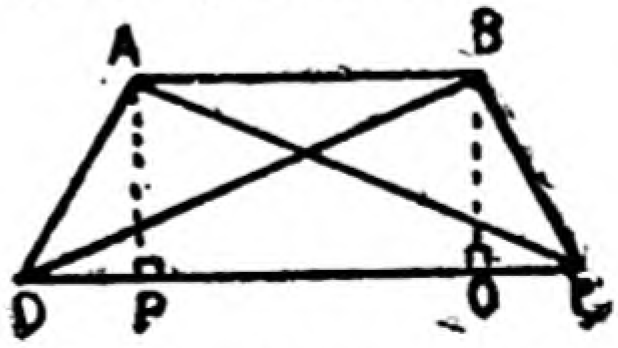
Hint.—$\angle C$ is acute and BE, $\perp$ AC in isosceles $\triangle ABC$ $\therefore AB^2=AC^2+BC^2-2AC.CE$: but $AB=AC$ $\therefore 2AC.CE=BC^2$.

2. In the triangle ABC, the angle A is 60°, prove that $a^2=b^2+c^2-bc$.

3. *In a triangle ABC, $\angle$s B and C are acute ; BD is $\perp$ to AC and CE $\perp$ AB. Show that $BC^2=AB.\ BE+AC.\ CD$.*

Hint.—$\angle B$ is acute $\therefore AC^2=BC^2+AB^2-2AB.\ BE$.
$\angle C$ is acute $\therefore AB^2=BC^2+AC^2-2AC.CD$.
$\therefore 2BC^2=2AB.BE+2AC.\ CD$, etc.

4. *In a trapezium ABCD, AB $\parallel$ CD. Show that $AC^2+BD^2=AD^2+BC^2+2AB.CD$.*



Hint.—$AC^2=AD^2+DC^2-2DC.DP$.
$DB^2=BC^2+DC^2-2DC.\ CQ$.
$\therefore AC^2+DB^2=AD^2+BC^2+2DC(DC-DP-CQ)$.
$=AD^2+BC^2+2DC.AB$.

5. The perpendicular AX on the base BC of a triangle cuts the base internally at X so that BX= =3CX. Show that $2AB^2=2AC^2+BC^2$.

Hint.—$\angle C$ is acute $\therefore AB^2=AC^2+BC^2-2CX.BC$.

$\therefore \quad 2AB^2=2AC^2+2BC^2-4CX.BC$.

$\qquad =2AC^2+2BC^2-BC^2$.

($\because$ $4CX=BC$). $\quad \therefore \quad 2AB^2=2AC^2+BC^2$.

6. In the figure of the proposition prove that

$$AD=\frac{c^2+b^2-a^2}{2c} \text{ and } BD=\frac{c^2+a^2-b^2}{2c}$$

7. Find the area of $\triangle ABC$ in terms of its sides.

Let CD be equal to p and AD the projection of AC on AB equal to x.

Area of the triangle ABC=$\frac{1}{2}$ base AB$\times$height CD $=\frac{1}{2}cp$.

Also $b^2=p^2+x^2$

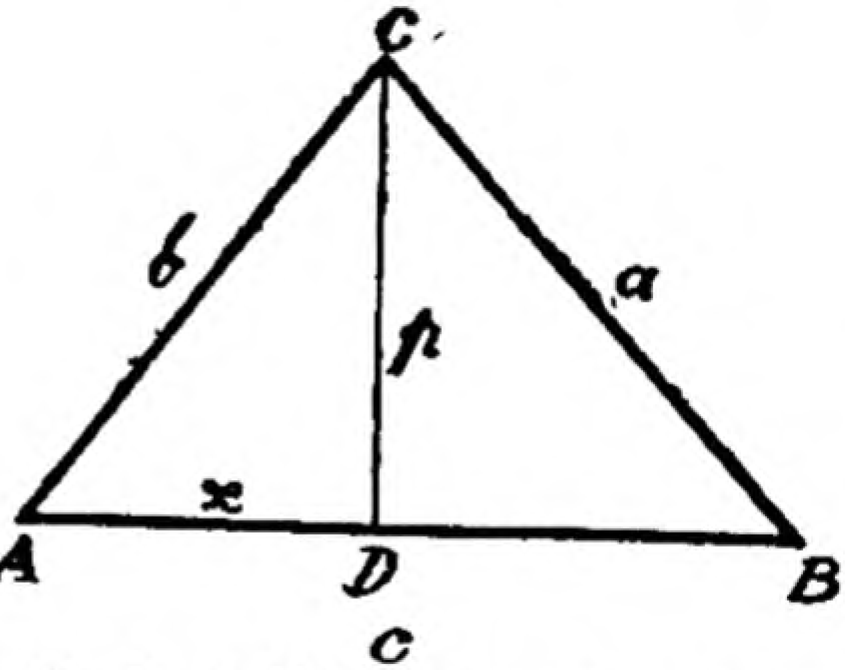
or $\quad p^2=b^2+x^2$

Angle A is acute

$\therefore \quad a^2=b^2+c^2-2cx$

or $2cx=b^2+c^2-a^2$

or $\qquad x=\dfrac{b^2+c^2-a^2}{2c}$



$$\therefore \quad p^2=b^2-\left(\frac{b^2+c^2-a^2}{2c}\right)^2=\frac{4c^2b^2-(b^2+c^2-a^2)^2}{4c^2}$$

$$=\frac{(2cb+b^2+c^2-a^2)(2cb-b^2-c^2+a^2)}{4c^2}$$

$$=\frac{[(b+c)^2-a^2]\,[a^2-(b-c)^2]}{4c^2}$$

$$=\frac{(b+c+a)(b+c-a)(a-b+c)(a+b-c)}{4c^2}$$

$$\therefore\ p=\frac{1}{2c}\sqrt{(a+b+c)(b+c-a)\ (a+c-b)(a+b-c)}$$

Put $a+b+c=2s\ \therefore \begin{cases} b+c-a=2\ (s-a) \\ a+c-b=2\ (s-b) \\ a+b-c=2\ (s-c) \end{cases}$

$$\therefore\ p=\frac{2}{c\sqrt{s(s-a)\ (s-b)(s-c)}}$$

$\therefore$ Area of the triangle $=\frac{1}{2}cp\sqrt{s(s-a)(s-b)(s-c)}$

This formula for the area of a triangle whose three sides are given was first found by Hero and is known after him as **Hero's formula**.

Area of a triangle is generally denoted by $\triangle$, a Greek letter and the formula then takes the form.

$\triangle=\sqrt{s(s-a)(s-b)(s-c)}$.

8. If p_1. p_2, p_3, are the lengths of the perpendiculars from the vertices of a triangle ABC upon the opposite sides, show that

$$p_1=\frac{2\triangle}{a},\ p_2=\frac{2\triangle}{b},\ p_3=\frac{2\triangle}{c}.$$

9. If the square on one side of a $\triangle$ is less than the sum of the squares on the remaining two sides, then the angle opposite to that side is acute.

Note.—Summary of the theorems.

In any triangle ABC

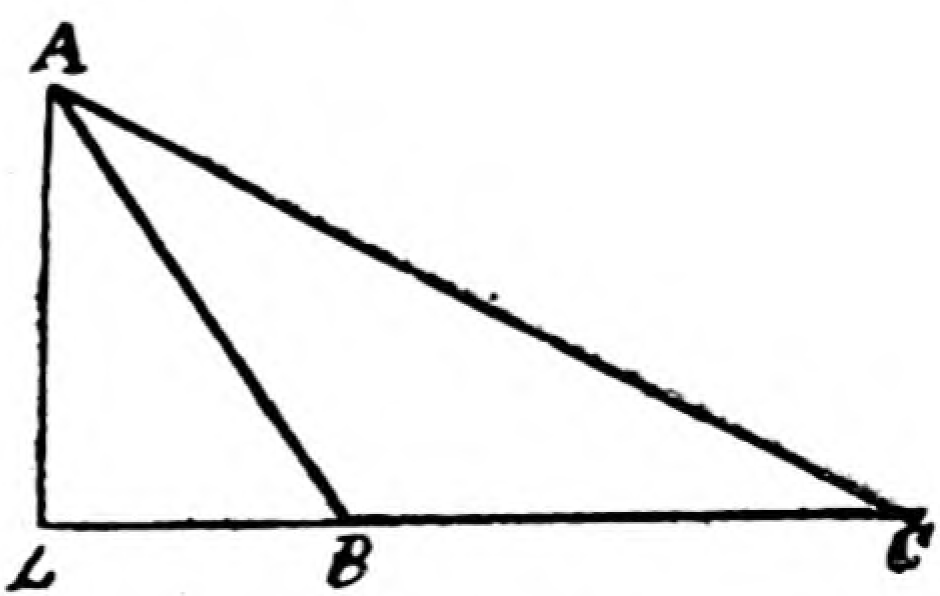


Fig. 1.

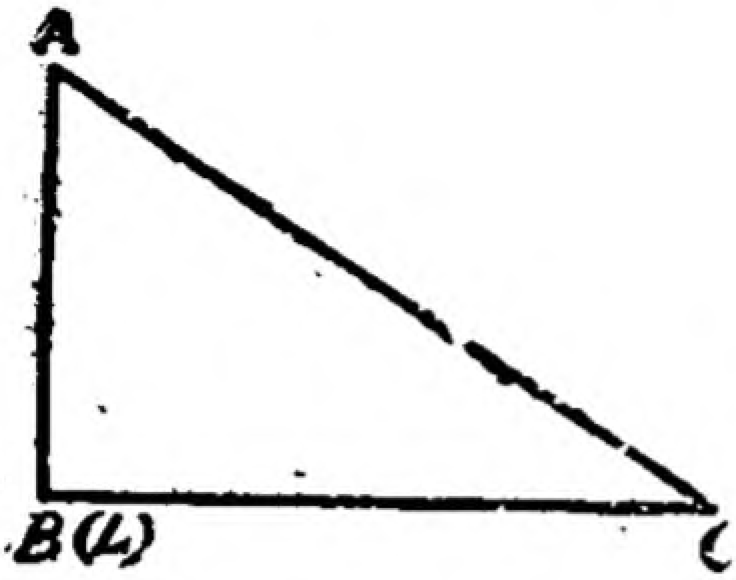


Fig. 2.

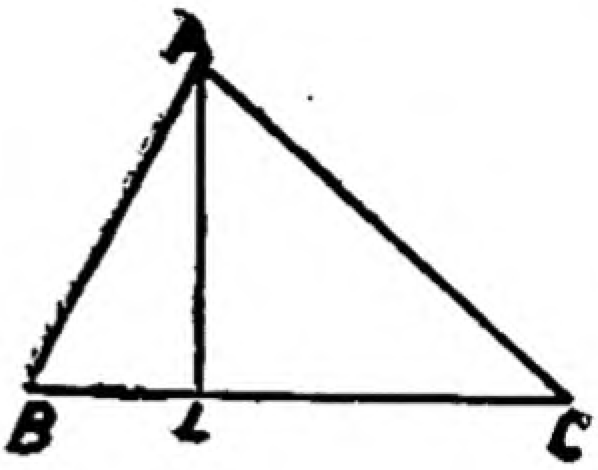


Fig. 3.

(*i*) $AC^2 = AB^2 + BC^2 + 2BC.BL$ if $\angle B$ is obtuse. (Fig. 1).

(*ii*) $AC^2 = AB^2 + BC^2$ if $\angle B$ is rt. $\angle$ (Fig. 2).

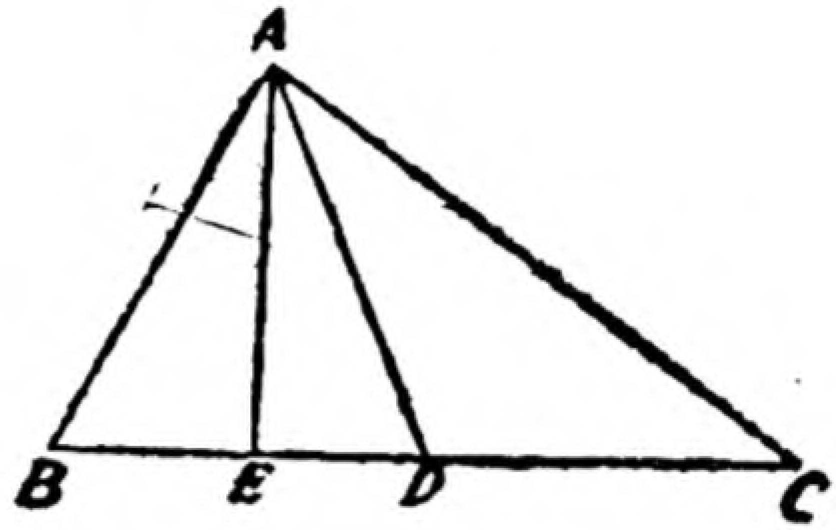
(*iii*) $AC^2 = AB^2 + BC^2 - 2BC.BL$ if $\angle B$ is acute (Fig. 3).

These propositions give the relation between the sides of a triangle and may, therefore, be included in one theorem which can be enunciated as under.

The square on a side of a $\triangle$ is $>$ than, $=$ to or $<$ than the sum of the squares on the other two sides according as the angle contained by these sides is obtuse, right or acute. The difference in the cases of inequality is twice the rectangle contained by one of the two sides and the projection on it of the other.

Proposition 50. (*Theorem*).

In any triangle the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.



Given :—A $\triangle ABC$, AD being the median which bisects BC.

Required :—To prove that $AB^2+AC^2=2BD^2+2AD^2$.

Construction :—Draw $AE \perp BC$.

Proof :—In the $\triangle ADB$

$AB^2=BD^2+AD^2-2BDED$...(1) $\because$ $\angle ADB$ is acute.

Again in the $\triangle ADC$

$AC^2=DC^3+AD^2+2DCDE$..(2) $\because$ $\angle ADC$ is obtuse.

But $BD=DC$.

$\therefore$ Adding (1) and (2) we get.

$AB^2+AC^2=2BD^2+2AD^2$

Q. E. D.

Note—This theorem is called **Appellonius Theorem**.

Cor. 1. In the triangle ABC if $AC>AB$ and AD and AE are the median and perpendicular respectively from the vertex to the base BC, prove that $AC^2-AB^2=2BC.ED$.

[Proof. Subtract (1) from (2) above.]

Cor. 2—If a straight line BC be bisected in D and E is any other point on BC or BC produced, prove that $CE^2+BE^2=2CD^2+2ED^2$.

Proof:—In the figure of the proposition let A

(Fig. 1)

(Fig. 2)

approach BC along the perpendicular AE. In the limiting position when A ultimately coincides with E, the result proved in the proposition reduces to $CE^2+BE^2=2CD^2+2ED^2$.

Cor. 2. May be enunciated thus :—

If a st. line is divided equally and also unequally (internally or externally) the sum of the squares on the two unequal parts is equal to twice the sum of the squares on half the line and on the line between the points of section.

Exercises.

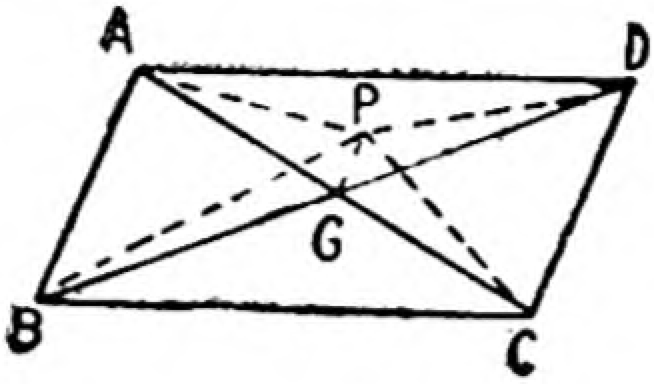
1. *Find the locus of a point such that the sum of the squares on its distances from two given points is equal to a given square.*

Hint.—In the fig. of Prop. $2BD^2+2AD^2=AC^2+AB^2$

$\therefore\ 2AD^2=AC^2+AB^2-2BD^2=$ a constant.

Hence AD is of constant length, hence the required locus is a circle whose centre is D (mid-point of BC) and radius AD.

2. P is any point within a $||^{m}$ ABCD, whose diagonals intersect at G; show that $PA^2+PB^2+PC^2+PD^2=AB^2+BC^2+4PG^2$. (Bombay)

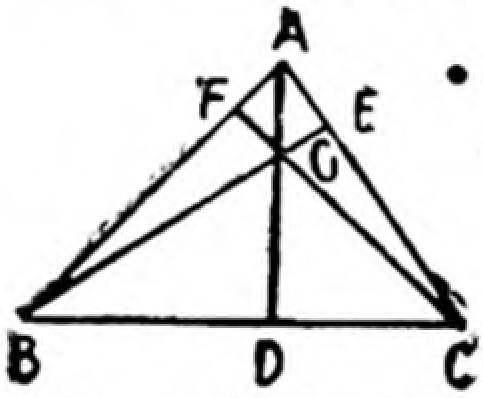


Hint.—

Join PG. $AP^2+CP^2=2PG^2+2AG^2$ and $BP^2+DP^2 =2PG^2+2BG^2$ $AP^2+CP^2+DP^2+BP^2=4PG^2+2AG^2+2BG^2=4PG^2+AB^2+BC^2$.

3. *Prove that three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians.* (Calcutta Matric.)

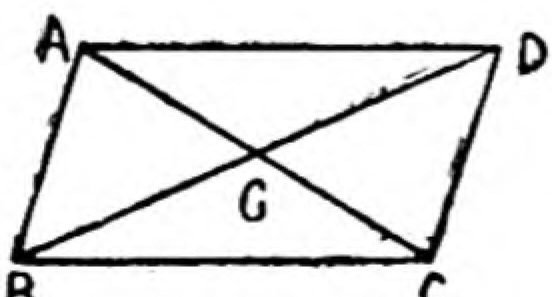
4. *G is the centroid of a $\triangle ABC$; prove that $3(GA^2+GB^2+GC^2)=AB^2+BC^2+CA^2$.*



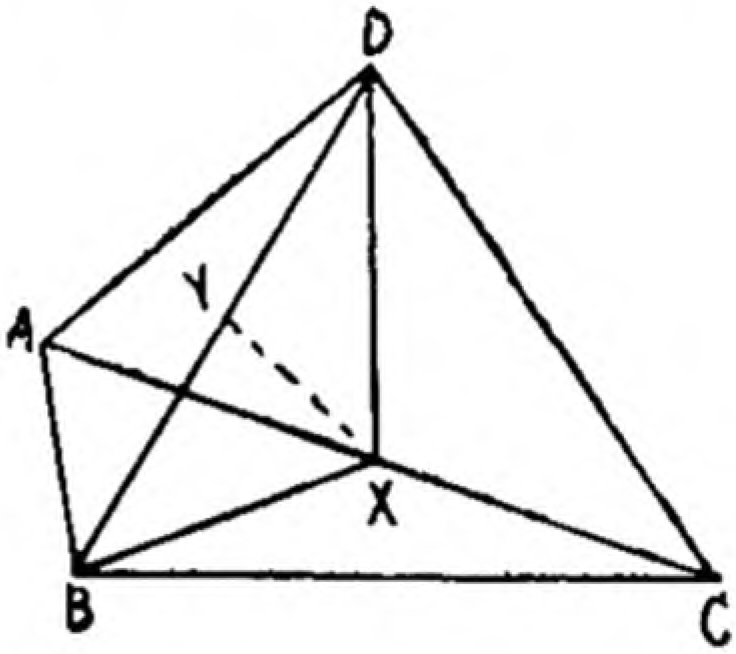
Hint.—$GB^2+GC^2=2BD^2+2GD^2$ (AD, BE, CF being medians) $GC^2+GA^2=2CE^2+2GE^2$; $GA^2+GB^2 =2AF^2+2GF^2$ $\therefore 2(GA^2+GB^2+GC^2)=2(BD^2+CE^2+AF)+2(GD^2+GE^2+GF^2$ $\therefore$ $4(GA^2+GB^2+GC^2)=4(BD^2+CE^2+AF^2)+4(GD^2+GE^2+GF^2)$ $=$ $BC^2+AC^2+AB^2+(AG^2+GB^2+GC^2)$.

$\therefore$ $3(AG^2+GB^2+GC^2)=BC^2+AC^2+AB^2$.

5. *The sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its four sides.*



6. In any quadrilateral the sum of the squares on the sides is greater than the sum of the squares on the diagonals by four times the square in the straight line joining the mid-points of the diagonals. Deduce Exercise 5 from this.



Hint.—ABCD is a quad.; X, Y mid-points of AC, BD.

In $\triangle ABC$, BX is a median $\therefore AB^2+BC^2=2AX^2+2BX^2$.

In $\triangle ADC$, DX is a median $\therefore CD^2+DA^2=2AX^2+2DX^2$.

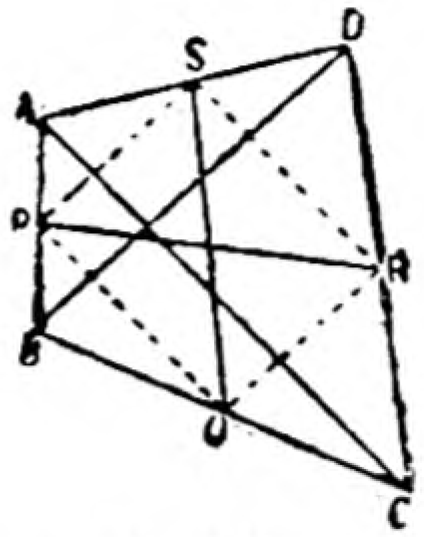
$\therefore$ (Adding) $AB^2+BC^2+CD^2+DA^2=4AX^2+2BX^2+2DX^2$.

But in $\triangle BDX$, XY is a median $\therefore BX^2+DX^2=2BY^2+2XY^2$.

$$\therefore AB^2+BC^2+CD^2+DA^2+4AX^2+4BY^2+4XY^2$$
$$=AC^2+BD^2+4XY^2.$$

If ABCD is $\|^m$ XY vanishes and sum of the squares on the sides=sum of squares on the diagonals.

7. The sum of the squares on the diagonals of a quadrilateral is equal to twice the sum of the squares on the lines joining the mid-points of the opposite



Hint.—ABCD a quad. PR and SQ join the mid-points of AB, CD and AD, BC. PQRS is a $\|^m$.

$\therefore\ PR^2+QS^2=PQ^2+QR^2+RS^2+SP^2$ (Ex 5).

$=2PQ^2+2QR^2$.

$\therefore\ 2PR^2+2QS^2=4PQ^2+4QR^2=AC^2+BD^2$.

8. If the squares on the sides of a quadrilateral are together equal to the squares on its diagonals the quadrilateral is a $\|^m$.

Hint.— Sum of the squares on the sides of a quad.= sum of the squares on the diagonals+4 times the square on the join of the mid-points of diagonals.

$\therefore$ In this case four times the square on the join of mid points of diagonals vanishes (=0) hence mid-points of diagonals coincide $\therefore$ the quad is a $\|^m$.

9. A railway station is mid-way between two villages A and B and is 64 miles distant from each. The distance of my village from A is 104 and from G 84 miles. Find the distance of my village from the station.

10. ABCD is a $\|^m$ of which AC, BD are the diagonals ; P is a point such that $PA^2+PC^2=PB^2+PD^2$; prove that ABCD is a rectangle. (Bombay, Matric).

Hint.—AC, BD cut in E. Join PE. Then $2AE^2+2PE^2=2BE^2+2PE^2$. Hence AE=BE $\therefore$ AC=BD.

$\therefore$ ABCD is a rect.

11. *If m_1, m_2, m_3 be the medians which bisect a, b, c, the sides of the triangle ABC, prove that*

$$m_1^2 = \frac{2b^2+2c^2-a^2}{4} \qquad m_2^2 = \frac{2c^2+2a^2-b^2}{4}$$

$$m_3^2 = \frac{2a^2+2b^2-c^2}{4}$$

Hence show that the greatest median of a $\triangle$ bisects the smallest side.

12. Say whether the following triangles are obtuse angled, right angled or acute angled. Why ?

(*a*) sides 13 cm., 10 cm., 6 cm.

(*b*) sides 5 cm., 12 cm., 13 cm.

(*c*) sides 9 cm., 13 cm., 5 cm.

(*d*) sides 9 cm., 10 cm., 13 cm.

13. In the $\triangle$ABC, calculate AC

(*a*) AB=4″, BC=5″, ∠ABC=30°.

(*b*) AB=4″. BC=5″, ∠ABC=150°.

(*c*) AB=4″, BC=5″, ∠ABC=45°.

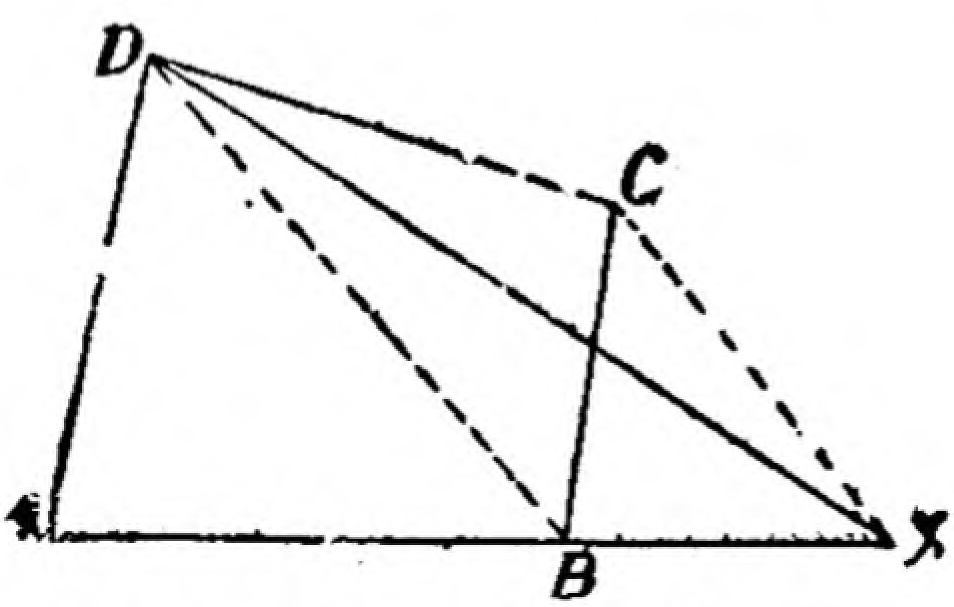
(*d*) AB=4″, BC=5″, ∠ABC=120°.

(*e*) AB=4″, BC=5″, ∠ABC=60°.

(*f*) AB=4″, BC=5″, ∠ABC=90°.

Proposition 51. (*Problem*)

To construct a $\triangle$ equal in area to a given quadrilateral.



Given :—A quadrilateral ABCD.

Required :—To construct a $\triangle$ equal to the quad. ABCD.

Construction :—1. Take three consecutive vertices B, C, D. Join BD (two alternate vertices).

2. Through C the middle vertex draw CX $\parallel$ DB, meeting AB produced in X.

3. Join DX. Then DAX is the reqd. $\triangle$ equal to the given quad. ABCD.

Proof :—$\because$ $\triangle$s DXB, CDB are on the same base DB and between the same $\parallel$s DB and CX.

$\therefore$ $\triangle$DXB$=$$\triangle$CDB.

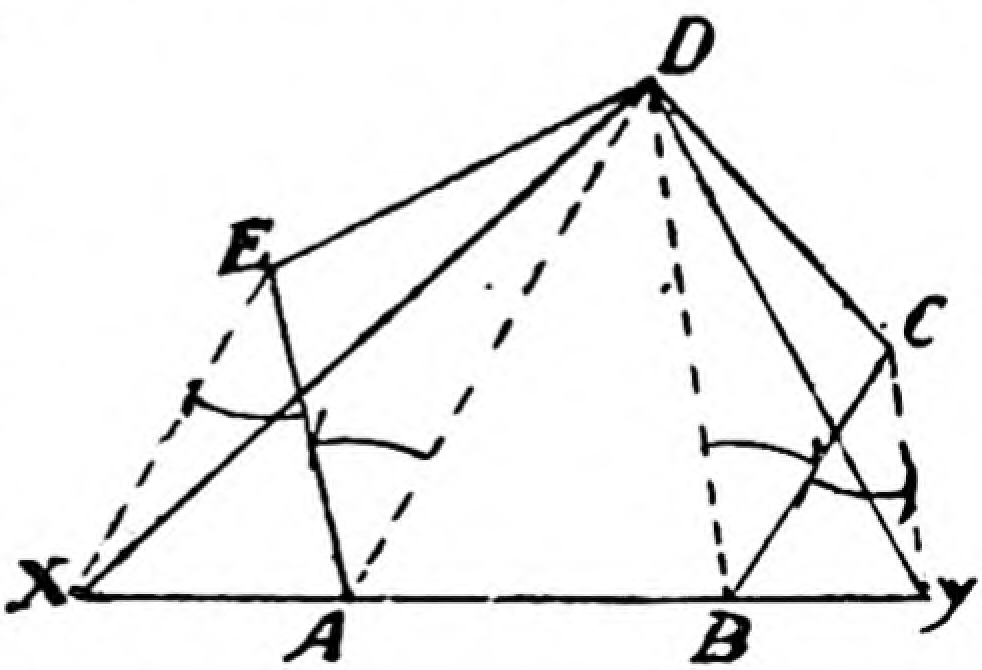
To each of these equals add $\triangle$DAB.

Then $\triangle$DAX$=$quad. ABCD.

Q. E. F.

Note.—By applying the above method once, we can reduce a pentagon to an equivalent quad., a hexagon to an equivalent pentagon and so on; and thus by the repeated application of the same process a polygon may be reduced to a $\triangle$ of equal area.

For example, in the adjoining diagram, the five-sided figure ABCDE has been reduced first by a single application of the above problem to an equivalent quad. AYDE. Then by applying the above problem again quad. AYDE is reduced to an equivalent $\triangle$DXY.



Thus the given pentagon ABCDE has been reduced to the equivalent $\triangle$DXY.

Note.—When an equivalent △ is to be constructed on a *particular side* of the quadrilateral that side **is not to be produced** but one of the two adjacent sides is produced in the same direction as the particular side in which the quadrilateral lies.

Exercises.

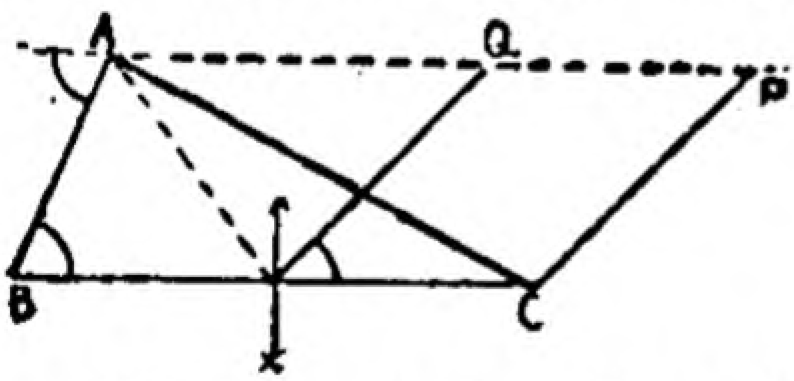
1. Draw a quad. ABCD having given AB=2·8″, BC=3·1″, CD=3·3″, DA=3·5″, and BD=3″. Construct a △ equal to this quad.

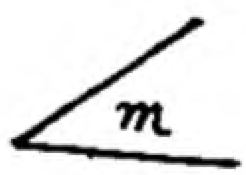
2. *Draw a quadrilateral ABCD when AB=60 mm., BD=45 mm., CD=28 mm., DA=55 mm., and ∠A=60°. On side AD construct a triangle equal to the quadrilateral. Measure the greatest side.*

3. Construct a △ equal in area to a quadrilateral ABCD, having its vertex at a given point P in PC and its base in the same st. line as AB.

Proposition 52. (*Problem*)

To construct a ∥m equal in area to a given △ and having one of its ∠s equal to a given ∠.





Given:—A △ ABC and an ∠m.

Required—To const. a ∥m equal to the △ABC, and having an ∠ equal to ∠m.

Construction:—1. Bisect BC at M.

2. Make ∠CMQ=∠m.

3. Through A, draw AP ∥ BC.

4. Through C, draw CP ∥ MQ.

Then MCPQ is the required ∥m.

Proof :—Join AM.

$\because$ $\parallel^m$ MP and $\triangle$ AMC stand on the same base MC and are between the same $\parallel$s MC, AP $\therefore$ $\parallel$s MP$=2\triangle$ AMC......(*i*).

Also $\because$ $\triangle$s AMC, AMB are on equal bases and of the same altitude $\therefore$ $\triangle$AMC$=$AMB.

$\therefore$ $\triangle$ABC$=2\triangle$AMC.........................(*ii*)

Hence $\parallel^m$ MP$=\triangle$ABC.

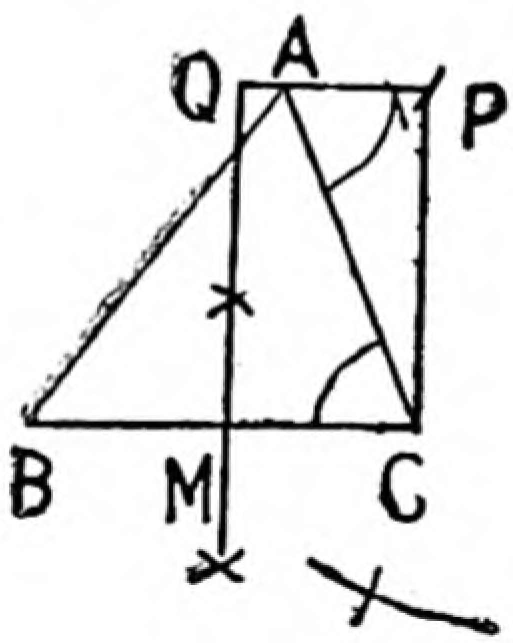
and its $\angle$CMQ$=\angle^m$.

Q. E. F.

Exercises.

1. Draw a $\triangle$ with sides 4 cm., 5cm. and 6cm. and construct a $\parallel^m$ of equal area and having (*i*) an angle$=60°$. (*ii*) an angle$=120°$ (*iii*) an angle $=90°$.

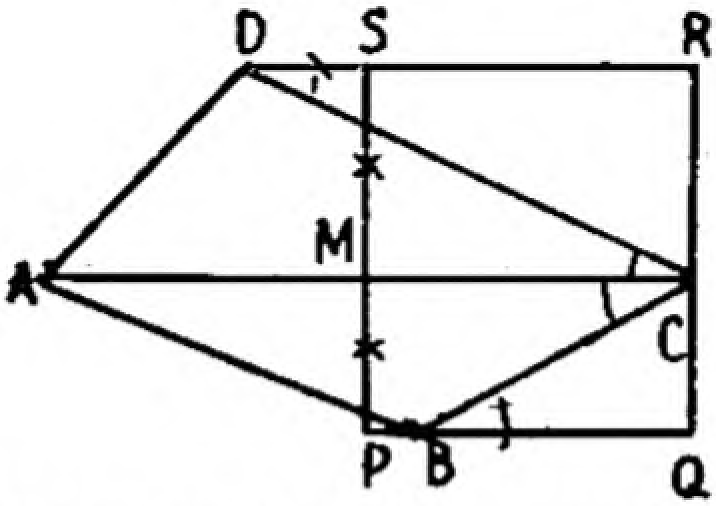
2. Construct a rectangle equal to a given $\triangle$.



3. Construct a rectangle equal to a given quad.

Hint.—First method : Reduce the given puad. to an equivalent $\triangle$ and then reduce this $\triangle$ to an equivalent rect.

Second method :—Draw DR, BQ each $\parallel$ to the diag. AC. Draw SP the $\perp$ bisector of AC. Draw RCQ $\perp$ AC. Then PQRS is the required rect.



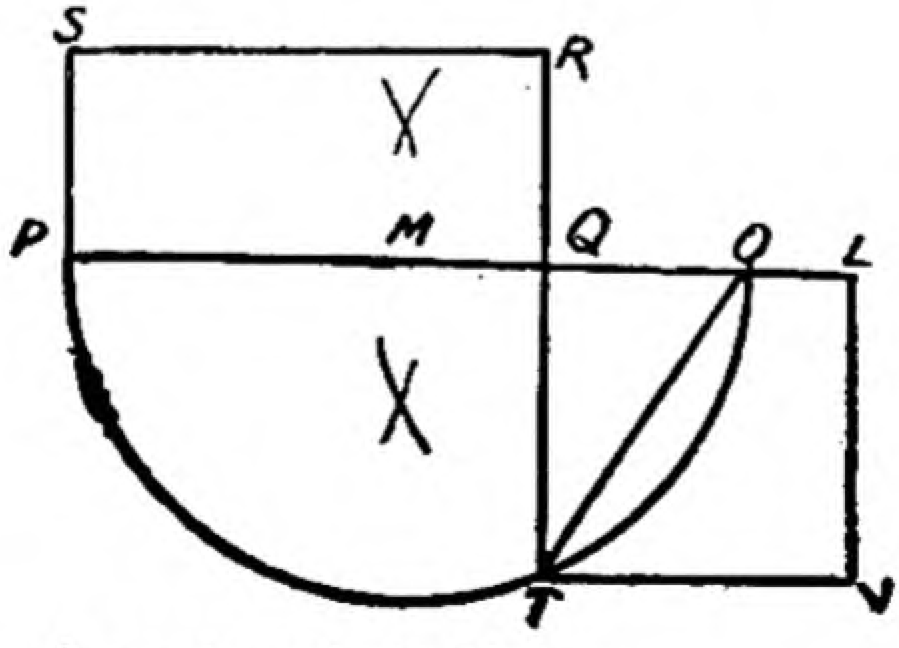
4. Construct a rectangle equal to a given polygon.

Hints.—Reduce the given polygon to an equivalent quad. and then proceed as in Ex. 3 given above.

5. *Construct a rectangle with sides 4·6 cm., and 3·2 cm., and transform it into a* $\parallel^{m}$ *having (i) one side 5·8 cm., (ii) one angle*$=60^{\circ}$, *(iii) one diagonal*$=8·7$ *cm.*

Proposition 53. (*Problem*).

To construct a square equal in area to a given rectangle.



Given —A rectangle PQRS.

Require :—To construct a square equal in area to it.

Construction :—Produce PQ to O making QO

equal to QR. Bisect PO in M and with M as centre and MO as radius draw a semi-circle. Produce RQ to meet the semi-circle in T. On QT construct the square QTVL.

Then square QTVL is the required square.

Proof :— Join PT, OT. ∵ △PTO is a right-angled triangle and TQ is perpendicular from the right angle to the hypotenuse. { ∠PTO = 1rt. ∠ being in the semicircle.

$\therefore TQ^2 = PQ.\ QO$ (QO = QR. Constr.)
$= PQ.\ QR.$

Therefore area of the square QLVT = area of the rect. PQRS.

Hence QLVT is the required square.

Q E. F.

Exercises.

1. Construct a rect. 2·1″ by 1·6″ and draw a square of equal area.

2. *Construct a square equal to ▱ᵐ whose adj. sides are 2.1″ and 1·6″ long and the included ∠ is of 60°.*

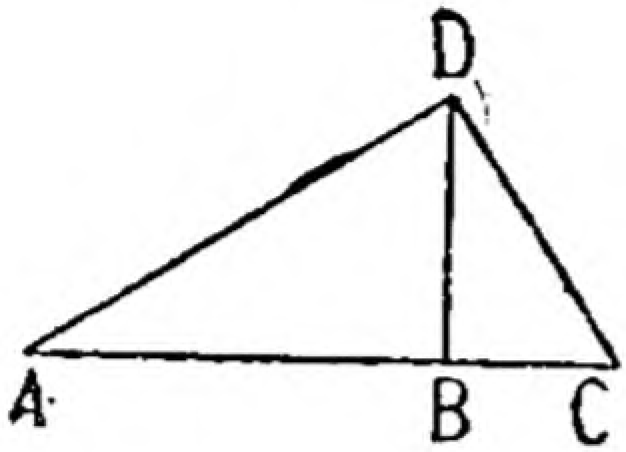
3. *Construct a square equal to a rhombus whose semidiagonals are 1·2″, 1·5″.*

4. *Construct the side of a square equal to a △ whose sides are 1·7″, 2·2″, 2″.*

5. *Construct the quad. ABCD in which AB = 2″, BC = 2″, CD = 2″, DA = 1 5″. BD = 2″. Reduce it to an equivalent rect. Draw a sq. equal to this rect.*

6. *Find the side of a sq. whose area is 4 sq. inches.*

7. On a given st. line as base draw a rect. equal to a given square.

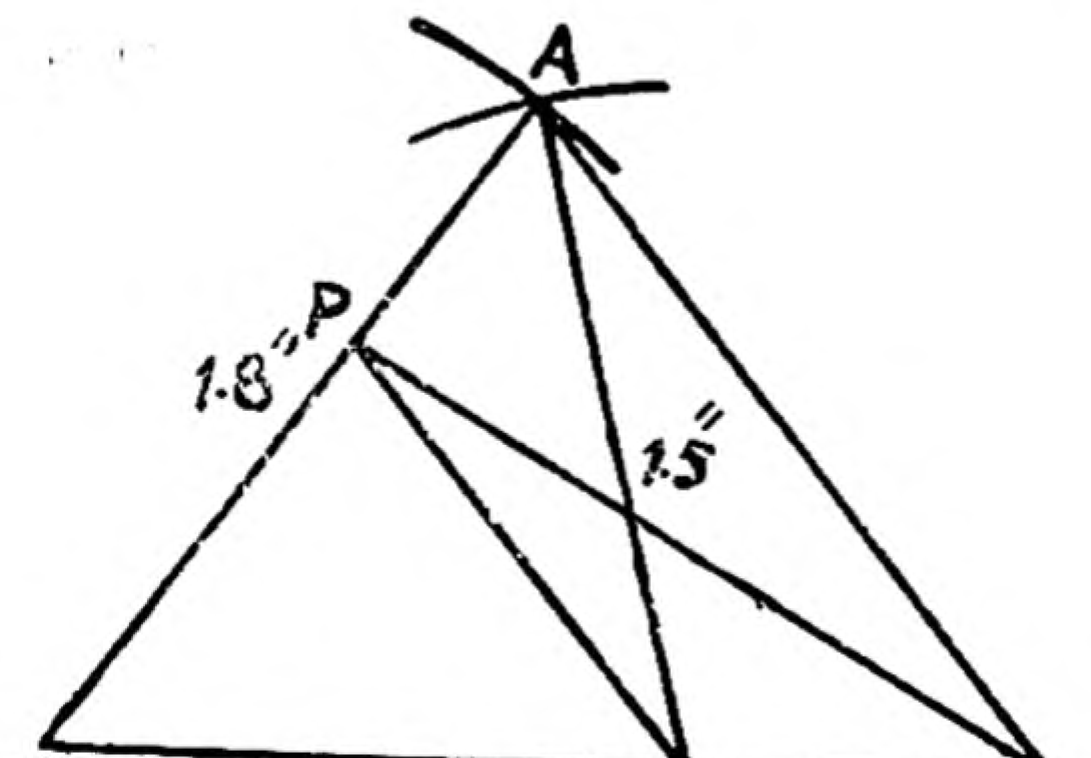


Hints.—Let AB be the given side of the rect. Draw BD ⊥ AB. Cut off BD = the side of the given sq. Draw DC ⊥ AD meeting AB produced in C. Then BC is the other side of the reqd. rect.

8. Find the length of the side of a rect. whose area is 36 sq. cm. and one side is 7 cm.

Exercises.

1. *Draw a △ whose sides are 1·4″, 1·5″ and 1·8″ and construct a △ of equivalent area on a base of 2·2″ long.*

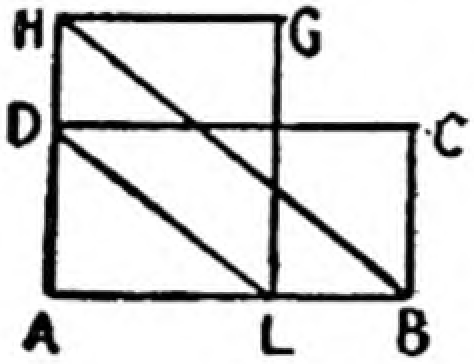


Hint.—Construct △ABC with the given data.

In BC produced take a point Q such that BQ = 2·2″.

Join AQ and draw CP ∥ to it meeting AB in P. Join PQ. $\triangle$BPQ is the required triangle.

2. *Draw a rectangle ABCD, AB=6 cm. and AD =3·5 cm. Construct a rectangle of equal area having one side equal to 5 cm.*



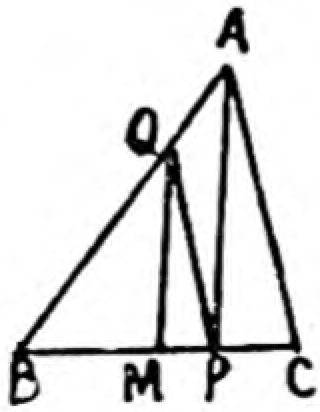
Hint.—From AB cut off AL=5 cm. Join LD and draw BH ∥ LD cutting AD produced at H. Complete the rectangle AHGL and prove that this rectangle is equal to the given one.

3. *Draw a $\triangle$ whose sides are 4 cm., 5 cm. and 7 cm. and construct a triangle equal to it having an angle of 60° and a side=6 cm.*

Hint.—Proceed as in Ex. 1 and then change the $\angle$.

4. *Construct a triangle equal in area to a given $\triangle$ and having a given altitude*

5. *Bisect a triangle ABC by a straight line through P, a point in BC.*

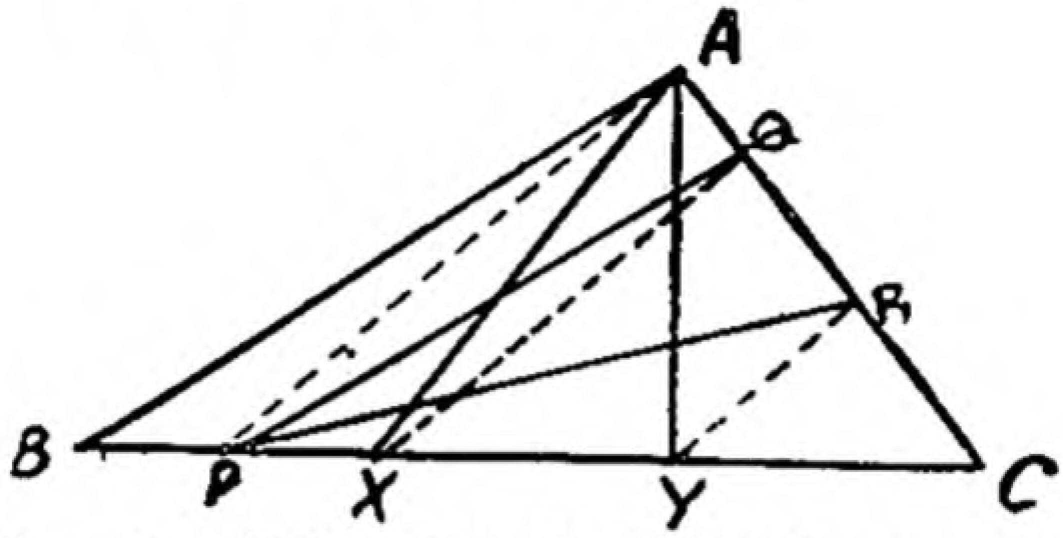


Hint.—Bisect BC at M. Join AP. Through M, the middle point of BC, draw MQ ∥ AP. Join QP.

[For proof join AM]

6. *Trisect a $\triangle$ ABC by lines drawn through P, a point in BC.*

Hint.—Trisect BC at X and Y. Join AP and draw XQ and YR $\parallel$ AP to meet AB and AC in Q and R. (State the general rule for such divisions.)



Proof :—$\triangle$ APQ=$\triangle$APX [same base and same $\parallel$s]

Add $\triangle$ ABP to both sides

$\therefore$ $\triangle$ ABX=Quad. ABPQ

But $\triangle$ ABX=$\frac{1}{3}$ $\triangle$ ABC

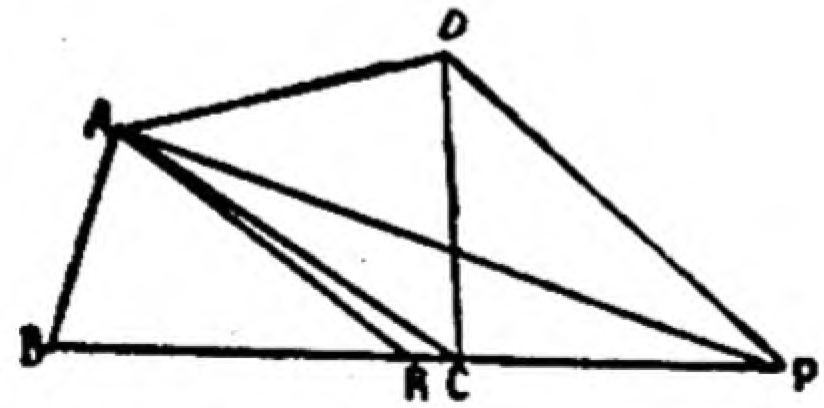
[$\because$ BX=$\frac{1}{3}$ BC and same altitude]

Similarly $\triangle$ PRY=$\triangle$AYC=$\frac{1}{3}$$\triangle$ABC

$\therefore$ $\triangle$PRY=Quad ABPQ=$\triangle$PQR

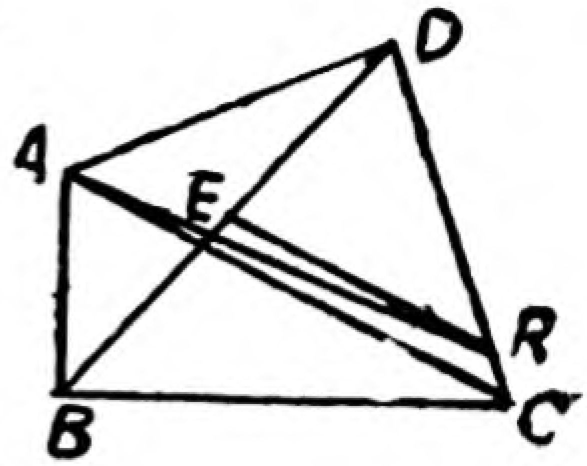
[the remaining $\frac{1}{3}$ $\triangle$ABC.]

7. *Bisect a quadrilateral by a st. line drawn through an angular point.*



Hint.—Reduce the quadrilateral ABCD to an equivalent triangle ABP with vertex at the given angular point A. Join this vertex A with the middle point R of the base BP of this triangle. AR is the required line that bisects the quadrilateral.

Alternative Const. join diagonal BD and bisect it in E. Join AC. Draw ER ∥ AC. Join AR. AR bisects the quad. through A.

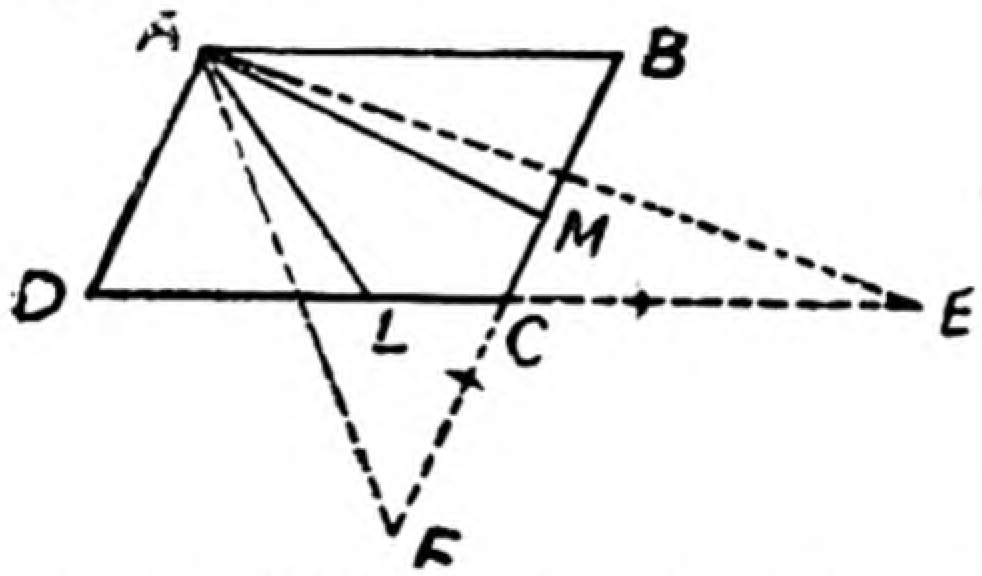


Note.—Join CE and supply proof.

8. *Bisect a quadrilateral by a straight line through a point in one of its sides.*

Hint.—Reduce the quadrilateral to a triangle and apply example 5.

9. Trisect a ∥^{m} by drawing two lines through one of its angular points.



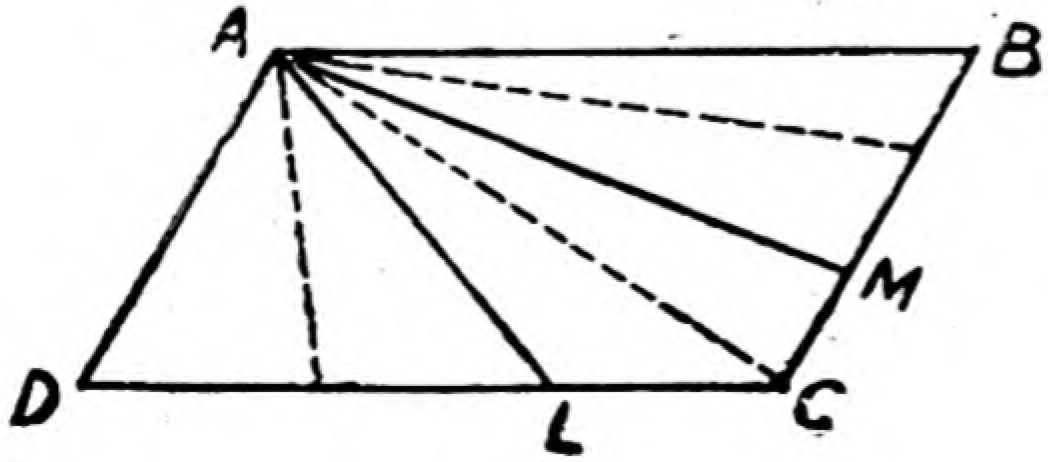
Hint.—Produce DC to E making CE=DC. Cut off DL=$\frac{1}{3}$ DE.

Then $\triangle$ ADL=$\frac{1}{3}$ $\triangle$ADE=$\frac{1}{3}$ ∥^{m} ABCD

Similarly produce BC to F making CF=BC. Cut off BM=$\frac{1}{3}$ BF.

Then $\triangle$ ABM=$\frac{1}{3}$ $\triangle$ ABF=$\frac{1}{3}$ ∥^{m} ABCD.

Or, simply



Trisect DC and BC and join the points of trisection to A. The $\|^m$ is split up into six equal $\triangle$s. Mark off three groups of two $\triangle$s each.

Thus AL, AM divide the $\|^m$ into 3 equal parts.

Miscellaneous Exercises on Areas.

1. ABC is a $\triangle$ whose base BC is bisected at X. If Y is any point in the median AX, show that $\triangle$ABY $=\triangle$ACY in area.

2. ABC is a $\triangle$ and XY is drawn $\|$ base BC, cutting the other sides at X and Y. Join BY and CX; and show that

(*i*) $\triangle XBC = \triangle YBC$.
(*ii*) $\triangle BXY = \triangle CXY$
(*iii*) $\triangle ABY = \triangle ACX$.
If BY and CX cut at P. show that
(*iv*) $\triangle BPX = \triangle CPY$.

3. **ABCD** is a $\|^m$ and BP, DQ are $\perp$s from B and D on the diagonal AC, show that $BP = DQ$.

Hence if E is any point in AC or AC produced, prove (*i*) $\triangle ADE = \triangle ABE$, (*ii*) $\triangle CDE = \triangle CBE$.

4. ABCD is a $\|^m$ and E, F, are the mid-points of AD, BC; if G is any point in EF or EF produced, show that $\triangle AGB = \frac{1}{4} \|^m$ ABCD.

5. ABCD is a $\|^m$ and X and Y any points in DC and AD respectively; show that the $\triangle$s AXB, BYC are equal in area.

6 *Show how to draw on the base of a given $\triangle$ an isosceles $\triangle$ of equal area.*

7. The $\|^m$ formed by joining the mid-points of the sides of a quad. is half the quadrilateral.

8. ABC is a $\triangle$ and E, F the mid-points of the sides AB, AC; show that if BF and CE intersect in G, the $\triangle$ BGC=quad. AFGE.

9. M, N are the mid-points of sides AB, AC of $\triangle$ABC; MNPQ is a $\|^m$ P, Q being on side BC. Prove that MNPQ is equivalent to $\triangle$ABN.

10. ABCD is a $\|^m$; F is on side AB; DF, CB cut in E. Prove that $\triangle$ADE=$\triangle$DFC.

11. P is any point on the diagonal AC of a $\|^m$ ABCD. Prove that $\triangle$ABP=$\triangle$ADP.

12. ABCD is a trapezium; AB $\|$ CD; a line $\|$ AB cuts sides BC, AD in P, Q respectively. Prove $\triangle$BQC=$\triangle$APQ.

13. The diagonals of a $\|^m$ ABCD cut in E; F is a point on side AD. Prove that the quad. BFCE is equivalent to $\triangle$ BCE.

14. In the trapezium ABCD, AB $\|$ DC, and X is the mid-point of BC. Through X draw PQ $\|$ AD to meet AB and DC produced in P and Q. Then prove that

(*i*) trapezium ABCD=$\|^m$APQD.

(*ii*) trapezium ABCD=twice the $\triangle$AXD.

15. Draw a square equal to the sum of two given squares.

16. Draw a square whose area is twice that of a given square.

17. Draw a square equal to the difference of two given squares.

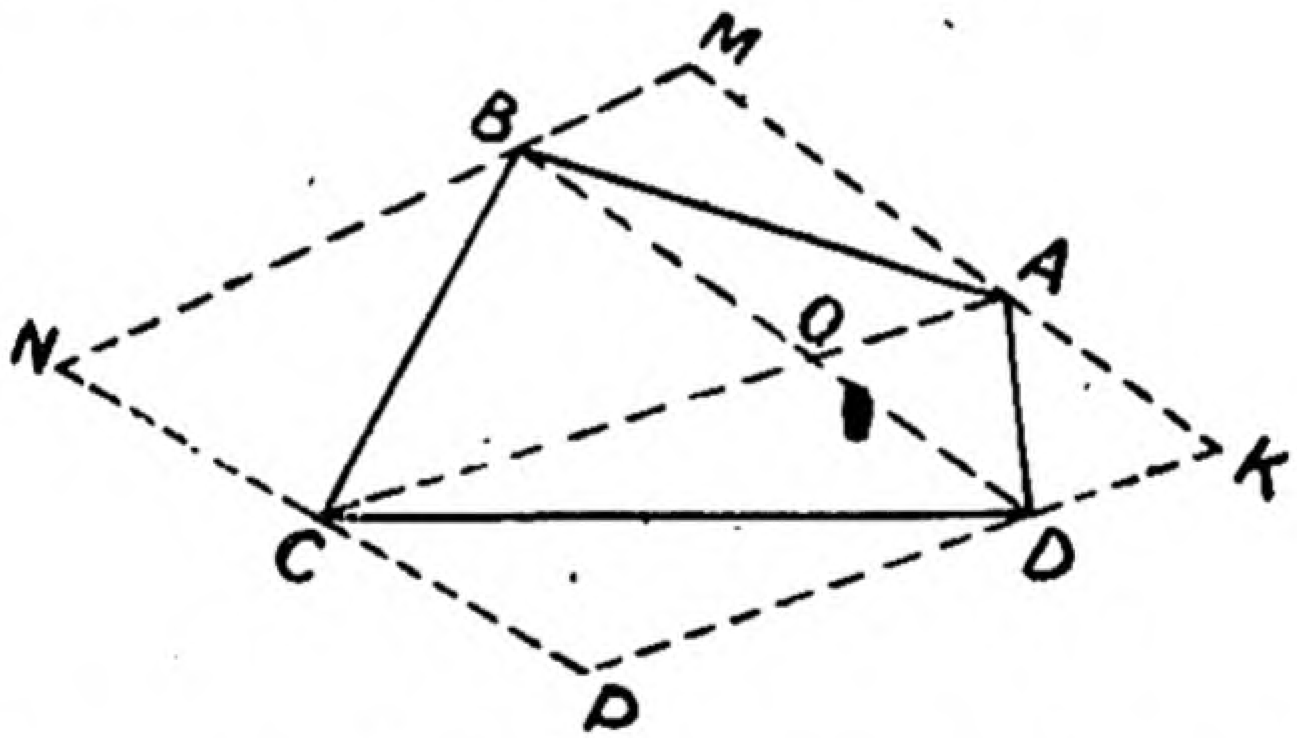
18. Draw a square whose area is half that of a given square.

19 Draw a $\|^m$ of equal perimeter and area to a given $\triangle$.

20. Construct a $\|^m$ on a base of 3″ with diagonals 5″ and 3·2″. Make a $\triangle$ of the same area and having an angle of 45°.

21. The sum of the perpendiculars drawn from any point within a regular polygon to its sides is the same wherever the point is taken.

22. (*a*) ABC is a quad. Through A and C draw parallels to BD and through B and D draw parallels to AC ; show that the area of ABCD is equal to half the area of resulting parallelogram.



Hint.—$\triangle ABC = \frac{1}{2} \|^m KB$ (same base, same $\|$s).
$\triangle BCD = \frac{1}{2} \|^m DN$ (,,)

(*b*) The area of a quad. is equal to that of a triangle, two-sides of which are equal and parallel to the diagonals.

23. If a st. line AB is divided into two parts at O, show that (*i*) $OA^2 + OB^2$ is least ; (*ii*) rectangle OA.OB is greatest when the st. line is bisected at O.

24. If L is a point in the side AB of a triangle

ABC such that $AL^2 - BL^2 = AC^2 - BC^2$; then AL is $\perp$ AB.

25. ABCD is a $\|^m$ and O is any point without the angle BAD and its opposite vertical angle; show that the triangle OAC is equal to the sum of the triangles OAD and OAB.

Hint.—Through O draw OE $\|$ BC or AD cutting AB in E. Now $\triangle ADE = \triangle ADO, = \triangle ADE = \triangle AEC$ $\therefore$ $\triangle ADO = \triangle AEC$. $\triangle OEC = \triangle OEB$, $\triangle AOE = \triangle AOE$, adding $\triangle AEC + \triangle AEC + \triangle AOE = \triangle ADO + \triangle OEB + \triangle AOE$ $\therefore$ $\triangle OAC = \triangle ADO + \triangle OAB$. Or *otherwise*.

Draw BL, CM, DN, $\perp$ s to OA meeting OA or OA produced in L, M, N. Draw DE $\perp$ CM. $\triangle$s ABL and DEC are congruent. $\therefore$ BL = CE. In rec. NDEM, DN = EM

$\therefore$ BL + DN = CE + EM = CM. $\triangle OAB = \frac{1}{2} OA.BL$, $\triangle OAD = \frac{1}{2} OA.DN$, $\triangle OAC = \frac{1}{2} OA.CM$

$\therefore$ $\triangle OAC = \triangle OAD + \triangle OAB$ $\because$ BL + DN = CM and OA (common base).

26. Prove that the triangle whose sides are $\frac{1}{2}[m+n)$, $\frac{1}{2}(m-n)$, and $\sqrt{mn}$ is right-angled.

27. The projections of two equal and parallel st. lines on any other st. line are equal.

28. If the square on one side of a triangle be *greater* than the sum of the squares on the other two sides, the angle contained by these sides is obtuse.

[Ex. 1. Theorem 48.)

29. If the square on one side of a triangle be *less* than the sums of the squares on the other two sides, the angle between them is acute.

30. Prove that twice the square on the line joining any point on the hypotenuse of an isosceles right angled triangle to the vertex is equal to the sum of the squares on the segments of the hypotenuse.

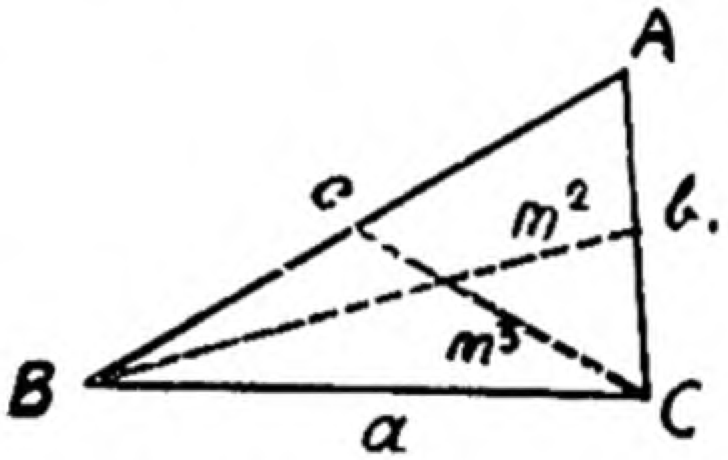
Hint.—ABC is an isosceles $\triangle$ having hypotenuse BC. D is any point in BC. Draw $AO \perp BC$, then $2AD^2 = BD^2 + DC^2$. $OA = OB = OC$, $AD^2 = OA^2 + OD^2 = BO^2 + OD^2$.

$\therefore$ $2AD^2 = 2BO^2 + 2OD^2$ and $BD^2 + CD^2 = (BO - DO)^2 + (CO + DO)^2 = 2BO^2 + 2OD^2$ $\therefore$ $2AD^2 = BD^2 + CD^2$.

31. Construct a trapezium and reduce it to an equivalent $\|^m$.

Hint.—Through the middle point of one of its non-parallel sides draw a line $\|$ to the opposite side.

32. *Of the two medians of a $\triangle$ that which is shorter bisects the longer side.*



Hint.—From Ex. 11 Theorem 50 we have

$$m_2^2 = \frac{2a^2 + 2c^2 - b^2}{4} \text{ and } m_3^2 = \frac{2a^2 + 2b^2 - c^2}{4}$$

$\left\{\begin{array}{l}\text{Given : } AB > AC\\ \text{To prove : } m_2 > m_3.\end{array}\right.$

$$\therefore \quad m_2^2 - m_3^2 = \frac{2a^2 + 2c^2 - b^2}{4} - \frac{2a^2 + 2b^2 - c^2}{4}$$

$$= \frac{2a^2 + 2c^2 - b^2 - 2a^2 - 2b^2 + c^2}{4}$$

$$= \frac{3c^2 - 3b^2}{4} = \tfrac{3}{4}(c^2 - b^2)$$

But $c > b$ $\therefore$ $c^2 > b^2$ $\therefore$ $c^2 - b^2$ is positive $\therefore$ $m_2^2 - m_3^2$ is positive. $\therefore$ $m_2 > m_3$

Q. E. D.

33. *In any $\triangle$ the shortest median bisects the longest side.*

(Punjab. 1924).

Hint.—Follows immediately from Ex. 32.

34. In a right-angled triangle the equilateral △ on the hypotenuse is equal to the sum of the equilateral △s on the two sides.

35. If the middle points of the sides of a quad. be joind, the figure formed is a parallelogram whose area is half that of the quad.

36. *Construct a △ equal in area to the sum of two given △s.*

37. Trisect a ||^{m} by lines drawn from a given point in one of its sides.

Hint.—ABCD is a ||^{m}, P a point in AB. Trisect AB in F and E (E near to A). Through F and E draw ||s to BC or AD meeting CD in G and H and let P be in EF. Bisect FG and EH in M and O. Join PM and PO and produce them to meet CD in N and L respectively, then PN and PL trisect the ||^{m}. If P is in BF, join PM and produce it to meet CD in K, then PBCK=$\frac{1}{3}$||^{m} ABCD; Now bisect quad. ADKP by drawing a line through P.

38. The square on any st. line drawn from the vertex of an isosceles △ to the base is less than the square on one of the equal sides by the rectangle contained by the segments of the base.

39. ABC is a triangle right angled at C and from any point D in AC a perpendicular DE is drawn to the hypotenuse; show that the rectangle AB.AE=the rectangle AC.AD.

40. AB is a diameter of a circle and DC is a chord parallel to AB; if any point P in AB is joined to the extremities of chord CD, show that $PC^2+PD^2=PA^2+PB^2$.

Hint.—Let O be the centre and E the mid-point of CD. Then $AP^2+BP^2=2OA^2+2OP^2=2OC^2+2OP^2=2CE^2+2OE^2+2OP^2=2CE^2+2EP^2=CP^2+DP^2$ ($\angle$s CEO and POE are rt. $\angle$s)

41. The st. line joining the mid-points of the parallel sides of a trapezium divides it into two equivalent quads.

42. From any point O within a triangle ABC, perpendiculars OD, OE, OF are drawn to the sides BC, CA, AB, respectively. Prove that $AF^2+BD^2+CE^2=CD^2+BF^2+AE^2$.

Hint.—$AF^2=AO^2-OF^2$, $BD^2=BO^2-OD^2$, $CE^2=CO^2-OE^2$ $\therefore$ $AF^2+BD^2+CE^2=CO^2-OD^2+BO^2-OF^2+AO^2-OE^2=CD^2+BF^2+AE^2$.

43. ABC is any $\triangle$ and BE, CF $\perp$s on CA, AB; prove that AB.AF$=$AC. AE.

44. ABC is a triangle having the $\angle A=120^\circ$; prove that $BC^2=AB^2+AC^2+AB.AC$.

45. O is the orthocentre of the $\triangle$ABC. Prove that AO. BC$=$BO. CA$+$CO. AB.$=4\triangle$ABC.

Hint:—$\triangle ABC=\frac{1}{2}AD.DC$, $\triangle ABC=\frac{1}{2}BE.AC$

$\triangle ABC=\frac{1}{2}CA.AB$ $\therefore$ $3ABC=\frac{1}{2}(AD.BC+BE.AC+CF.AB)$ $\therefore 6\triangle ABC=AD.BC+BE.AC+CF.AB$

also $2\triangle ABC=2(\triangle OBC+\triangle OCA+\triangle OAB)$ **(ii)**

$=2(\frac{1}{2}OD.BC+\frac{1}{2}OE.AC+\frac{1}{2}OF.AB$

Subtracting (*i*) and (*ii*).

$4\triangle ABC=(AD-OD)BC+(BE-OE)AC+(CF-OF)AB$

$=AO.BC+BO.AC+CO.AB.$

46. ABC is an acute-angled $\triangle$; from A,B,C $\perp$s AD,BE CF are drawn to the opposite sides. Prove that $BC^2+CA^2+AB^2=2(AB\,AF+BC.BD+CA.CE)$.

Hint:—$\because \angle C$ is acute $\therefore$ $AB^2=BC^2+AC^2-2AC.CE$
similarly $BC^2=AB^2+AC^2-2AB.AF$ by addition
and $AC^2=AB^2+BC^2-2BC.BD$.
$AB^2+BC^2+AC^2=2(AB^2+BC^2+AC^2)-2(AB.AF+BC.BD+AC.CE)$.

$\therefore 2(AB.FA+BC.BD+AC.CE)=AB^2+BC^2+AC^2$.

47. The perimeter of an isosceles $\triangle$ is $<$ than that of any other $\triangle$ of equal area standing on the same base.

48. ABCD is the diameter of two circles having the same centre and P, Q are any two points on the outer and inner circles respectively. Prove that $BP^2+CP^2=AQ^2+DQ^2$

Hint: – Let O be the common centre; join OP, OQ. AQ,DQ and apply Prop. 50.

49. From a point within an equilateral $\triangle$, perpendiculars are drawn to the three sides and are 8, 10 and 12 ft: ; find the sides and area of the triangle.

50. Through the vertices A,B,C of $\triangle ABC$ are drawn three parallel lines which cut the sides or the sides produced, of the triangle, in D,E,F respectively. Prove $\triangle DEF=2\triangle ABC$.

Hint:—AD, BF, CE parallels meet BC, CA produced and BA produced in D F and E respectively. $\triangle CEF=\triangle CEB$; take away $\triangle ACE$ from each $\therefore$ $\triangle AEF=\triangle ABC$ (*i*) $\triangle BAD=\triangle ADF$; $\triangle CAD=\triangle EAD$ $\therefore$ $\triangle ABC=\triangle ADF+\triangle EAD$ (*ii*).

Adding (*i*) and (*ii*) $\triangle AEF+\triangle ADF+\triangle EAD=2\triangle ABC$ or $\triangle DEF=\triangle ABC$.

51. M, N are the mid-points of sides AC, AB of $\triangle ABC$, CN and BM cut in G. Prove that area ANGM $=\triangle BGC$.

52. If a quad. be divided into four equivalent $\triangle$s by its diagonals, it is a parallelogram.

53. X,Y,Z,W are points on the sides AB, BC, CD, DA of the rectangle ABCD.; XY= ZW; prove that $AZ^2+CW^2=AY^2+CX^2$

Hint:—Join AC and apply Prop. 47 Ex. 8.

54. ABC is a $\triangle$ right-angled at B ; from M, the mid-point of AB a perpendicular is drawn to AC. If P is the foot of this perpendicular, prove that $CP^2=AP^2+BC^2$.

Hint:—$CP^2+PM^2=CM^2=BM^2+BC^2=AM^2+BC^2$
$=PM^2+AP^2+BC^2$.

Take away PM^2 $\therefore CP^2=AP^2+BC^2$.

55. Given C and D points on a perpendicular to the line AB; prove that $AC^2-BC^2=AD^2-BD^2$.

56. ABC is an acute-angled $\triangle$; AD, BE are $\perp$s to BC and CA intersecting at O ; prove that AD.OD =BD.CD.

57. *Find graphically* $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$, $\sqrt{5}$.

58. The diagonal BD of a $\|^m$ ABCD is at right angles to the sides AB and CD ; prove that the difference between the squares on the diagonals is four times the square on AB.

Hint.—Draw CE $\|$ BD, meeting AB produced in E and apply Cor. 1 Prop. 50.

59. In a quadrilateral ABCD, the opposite angles at B and D are right angles ; AD, BC are produced to meet at E ; prove that rect. AE, DE=rect. BE.CE.

Hint.—Join AC. $\because \angle E$ is acute in $\triangle AEC$ and CD $\perp AE$ $\therefore AC^2=EC^2+EA^2-2AE.DE$. (i)

Again in $\triangle$ECA, $\angle$E is acute and AB $\perp$ EC produced $\therefore$ $AC^2=EC^2+EA^2-2EC.BE$ (*ii*) ; from (*i*) and (*ii*) AE. DE = EC.BE.

60. The angles B and C of a $\triangle$ABC are acute. BE, CF are perpendiculars to AC and AB ; prove that $BC^2=AB.BF+AC.CE$.

Hint.—See Prop. 49, Ex. 3.

61. ABC is a $\triangle$ rt. $\angle$d at C ; from a point D in AC, DE is drawn $\perp$ AB; show that AB.AE = AC.AD.

Hint.—Proof similar to Ex. 59.

62. ABC is a $\triangle$; on the side of BC promote from A, describe a square BDEC ; show that $AD^2-AE^2=AB^2-AC^2$

Hint.—$AD^2\sim AE^2=AB^2\sim AC^2$.

From A draw AL and AM $\perp$ s to DB and EC produced $\therefore$ $\angle$ABD is obtuse.

$\therefore$ $AD^2=AB^2+BD^2+BD.BL$.

$\therefore$ $\angle$ACE is obtuse $\therefore$ $AE^2=AC^2+CE^2+2CE.CM$.

But BD = CE and BL = CM.

$\therefore$ $AD^2\sim AE^2=AB^2\sim AC^2$

$\therefore$ The difference between the squares on the diagonals of a $\|^m$ is four times rectangle contained by one of the sides and the projection upon it of the adjacent side.

Hint.—Draw DE $\|$ AC meeting BC produced in E and drop DF $\perp$ BE.

Now $DB^2-DE^2=4BC.CF$.

64. *A point moves so that the sum of the squares on its distances from two fixed points is constant ; prove that the locus of a point is a circle.*

Hint.—See Prop. 50, Ex. 1.

65. In a $\triangle ABC$, X is point of trisection of the base BC adjacent to the angle B ; prove that $2AB^2+AC^2=6BX^2+3AX^2$.

Hint.—Let Y be the second point of trisection.

From $\triangle ABY$, $AB^2+AY^2=2BX+2AX^2$. *(i)*

$\therefore 2AB^2+2AY^2=4BX^2+4AX^2$ *(ii)*

From $\triangle AXC$, $AX^2+AC^2=4AY^2+2XY^2$ *(iii)*

Adding *(ii)* and *(iii)*

$2AB^2+2AY^2+AX^2+AC^2=4BX^2+4AX^2+2AY^2+2XY^2$

Or $2AB^2+AC^2=6BX^2+3AX^2$ $\therefore$ $BX=XY$.

66. Prove that the difference of the squares on the sides of a $\triangle$ is equal to three times the difference of the squares on the joins of the vertex to the joins of trisection of the base.

Hint :—(Solution of Ex. 65) subtract *(iii)* from *(i)*

$AB^2-AC^2+AY^2-AX^2=2AX^2-2AY^2+2BX^2-2XY^2$

Or $AB^2-AC^2=3AX^2-3AY^2=3(AX^2-AY^2)$

$\therefore$ $BX=XY$.

67. If in a $\triangle ABC$, AD is drawn $\perp$ BC and $AD^2=BD.CD$ prove that $\angle CAB$ is a rt. angle.

68. *A and B are two given points ; show that the locus of P which moves so that PA^2-PB^2 is constant, is a st. line $\perp AB$.*

Hint :—(By Prop. 50. Cor. 1)

If PM is $\perp$ AB, $PA^2-PB^2=2AB.\ DM$. But AB is constant. $\therefore$ DM is constant, that s, M is a fixed point. Hence the projection of P on AB is a fixed point and thereforethe required locus is a st. line perpendicular to AB.

69. ABC is a $\triangle$; BM, CN are perpendiculars to the *external* bisector of the $\angle A$; prove that the square on the sum cf AB, AC is *greater* than the square on BC by four times the rect. BM.CN.

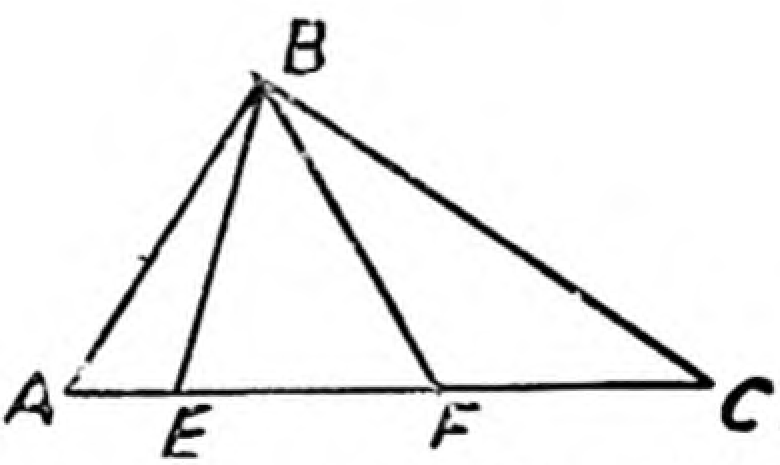
Hint:—Produce BA,CN to meet in D. Then AD$=$AC and CN$=$DN; also BD$=$AB$+$AC. Draw BP$\perp$CD. Then BPNM is a rect. and BM$=$PN. Thus BN is a median of $\triangle$BCD and NP its projection on CD. Hence $BD^2-BC^2=$ 2CD. PN$=$4CN.BM.

70. ABC is a $\triangle$: BM, CB are perpendiculars to the *internal* bisector of the $\angle$A ; prove that the square on the difference of AB.AC is *less* than the square on BC by four times the rect. BM.CN.

Hint: – Let AB>AC. Produce CN to cut AB in D. Then AD$=$AC and CN$=$DN; also BD$=$AB$-$AC. Draw BP$\perp$CD Then BPNM is a rectangle ; so that BM$=$PN. Now BN is a median of $\triangle$BCD, and NP its projection on CD. Thus $BC^2-BD^2=2$ CD.PN$=4$ CN.BM.

71. The perimeter of an an isosceles $\triangle$ is greater than that of an equal rectangle of the same altitude.

72. From the hypotenuse of a right-angled $\triangle$ portions are cut off equal to the adjacent sides; prove that the square on the middle segment is equal to twice the rectangle contained by the extreme segments.



Hint: –ABC a $\triangle$ rt. $\angle$d at B. From AC, AF is cut$=$AB and from CA, CE is cut$=$BC. Then $EF^2=2$AE. CF $\because \angle$B is a rt. $\angle$. $\therefore AC^2-BC^2=AB^2$

$\therefore AC^2-CE^2=AF^2$ ($\because$ BC$=$CE and AB$=$AF)

$\therefore (AE^2+CE^2+2AE.CE)-CE^2=AE^2+EF^2+2AE.EF.$ $[AC^2=(AE+CE)^2]$

$\therefore 2\ AE.CE=EF^2+2\ AE.EF.$ $\therefore 2\ AE(EF+CF)=EF^2+2AE.EF.$

$\therefore 2\ AE.EF+2\ AE.CF=EF^2+2AE.EF.$

$\therefore 2\ AE.CF.=EF^2.$

SECTION III

RATIO AND PROPORTION

Def.—The *ratio* between any two quantities or magnitudes of the same kind is the relation that indicates how often one quantity or magnitude is contained in the other.

For instance, Rs. 10 are contained in Rs. 20 two times and this fact is expressed as $\frac{\text{Rs. } 20}{\text{Rs. } 10} = \frac{2}{1}$. $\frac{2}{1}$ in this case indicates the relation between two quantities *of the same kind, i. e.*, Rs. 20 and Rs. 10, and is called a *ratio* between them and is purely a fractional number.

A ratio is, therefore, always expressible in the form of a fraction such as $\frac{a}{b}$. It may also be written $a \div b$ or $a : b$ and is read as a is to b.

Both the numerator and the denominator of a fraction representing a ratio are called its **terms**, the numerator being known as the **antecedent** and the denominator as the **consequent** of the given ratio.

Thus in $\frac{a}{b}$, a is called the antecedent and b the consequent of the ratio.

The fraction representing a ratio may be exact or *inexact*, *i. e.*, the quotient obtained by dividing the numerator by the denominator may be a terminating decimal or a non-terminating one. If the fraction representing a ratio is exact, the two quantities compared are called **commensurable** because in this case a common measure can be determined which is

contained an exact number of times in both. But if the fraction is inexact the quantities are known as **incommensurable quantities.**

The ratio of the side of a square to its diagonal is $\frac{1}{\sqrt{2}}$, which is not an exact fraction, so the side of a square and its diagonal are incommensurable quantities.

Def. Proportion is a statement of equality of two or more ratios.

When two ratios are equal, the fact is always expressed by putting them in the form of an equation. If two ratios $\frac{a}{b}$ and $\frac{c}{d}$ are equal, they are written as $\frac{a}{b}=\frac{c}{d}$ and the four magnitudes a, b, c, d, are said to be **in proportion or proportionals**. The proportion is then read as *a is to b as c is to d* and can also be expressed as $a : b = c : d$ or $a : b :: c : d$.

a, b, c and d are called **terms** of the proportion, the first and the fourth being known as the **extremes** and the second and the third as the **means**. In the above proportion a and d are the extremes and b and c the means.

In $\frac{a}{b}=\frac{c}{d}$ by multiplying across, it is obvious that $ad=bc$, or in other words **the product of the extremes is equal to the product of the means.**

The last term of a proportion is called the *fourth proportional* to the first three. Thus if $a : b :: c : d$; d is the fourth proportional to a, b and c.

a, b and c are said to be in **continued proportion** $a : b = b : c$. In this case b is called the **mean proportional** between a and c and is called a **third proportional** to a, and b. From $\frac{a}{b} = \frac{b}{c}$ we have $b^2 = ac$ or the **squares of the mean proportional = the product of the extremes.**

If $a : b = c^2 : d^2$, then $a : b$ is said to be in the duplicate ratio of c to d.

Students should refresh their memory of the ordinary properties of proportional magnitudes by making a reference to "Wilsingh Series" Arithmetic and Algebra. For their convenience we give below the important relations.

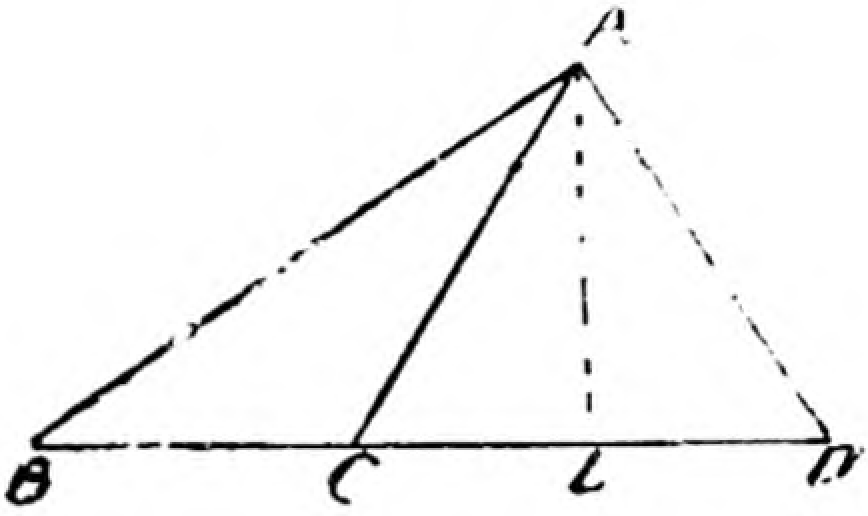
If $a : b = b : c$.
then $b^2 = ac$..............................(*i*)
If $a : b = c : d$,
then $ad = bc$..............................(*ii*)
$b : a = d : c$..............................(*iii*) (Invertendo).
$a : c = b : d$..............................(*iv*) (Alternendo).
$a+b : b = c+d : d$......................(*v*) (Componendo).
$a-b : c-d : d$..........................(*vi*) (Dividendo).
and $a+b : a-b = c+d : c-d$....(*vii*) (Componendo and Dividendo).

If $a : b = c : d = e : f = \ldots\ldots$

then each ratio $= \dfrac{a+c+e+\ldots\ldots}{b+d+f+\ldots\ldots}$ (*viii*)

Proposition 54. (*Theorem*)

Triangles which have the same altitudes are to one another as their bases.



Given :—$\triangle$s ABC and ACD having the same altitude AL.

Required :—To prove that

$$\frac{\triangle ABC}{\triangle ACD}=\frac{BC}{CD}$$

Proof :—$\triangle ABC=\frac{1}{2}AL.BC.$

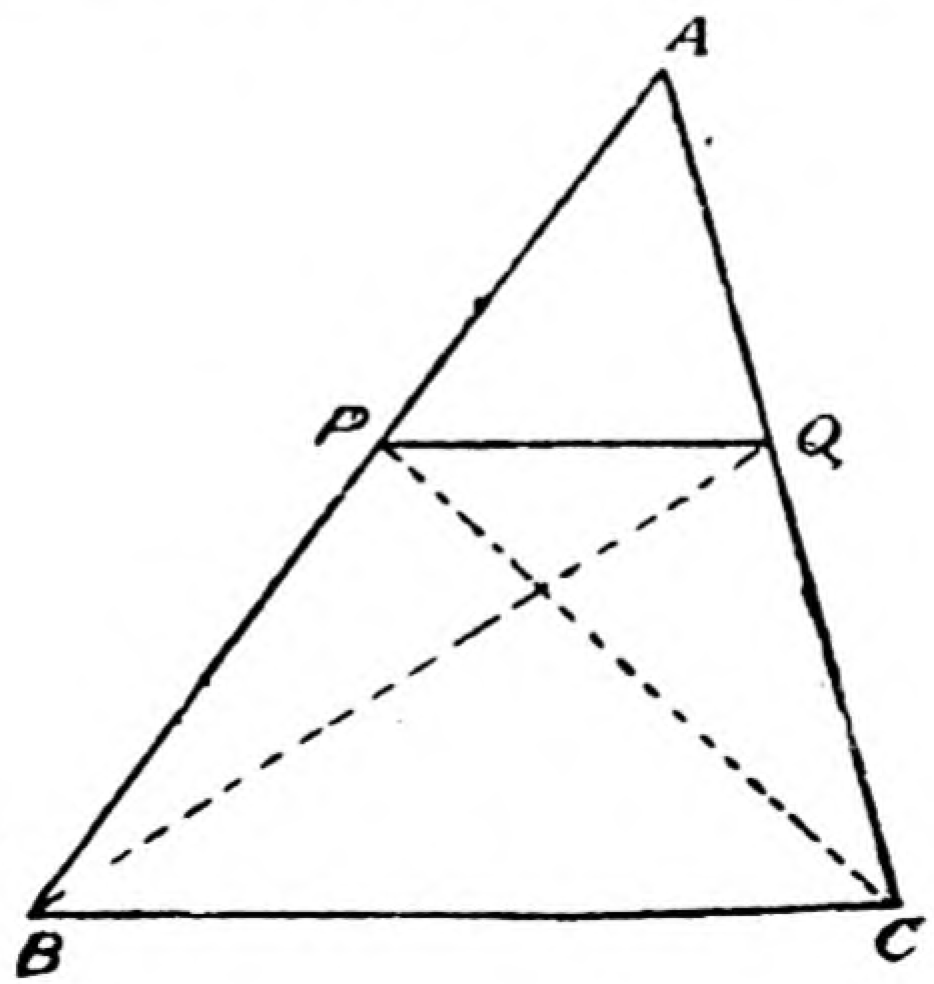
and $\triangle ACD=\frac{1}{2}AL.CD.$

$$\therefore \quad \frac{\triangle ABC}{\triangle ACD}=\frac{\frac{1}{2}AL.BC}{\frac{1}{2}AL.CD}=\frac{BC}{CD}.$$

Q. E. D.

Proposition 55. (*Theorem*)

If a st. line is drawn parallel to one side of a triangle, the other two sides are divided proportionally.



Given :—In the $\triangle ABC$, PQ is drawn $\parallel$ to BC.

Required :—To prove that $\frac{AP}{PB}=\frac{AQ}{QC}$.

Construction :—Join PC, QB.

Proof :—

$\frac{AP}{PB}=\frac{\triangle APQ}{\triangle PBQ}$ ($\triangle$s APQ, PBQ are of the same alt.)

So also $\frac{AQ}{QC}=\frac{\triangle AQP}{\triangle QCP}$ ($\triangle$s AQP, QCP are of the same alt.)

But $\triangle PBQ=\triangle QCP$ (They are on the same base PQ and are between the same $\parallel$s).

$\therefore \frac{AP}{PB}=\frac{AQ}{QC}$.

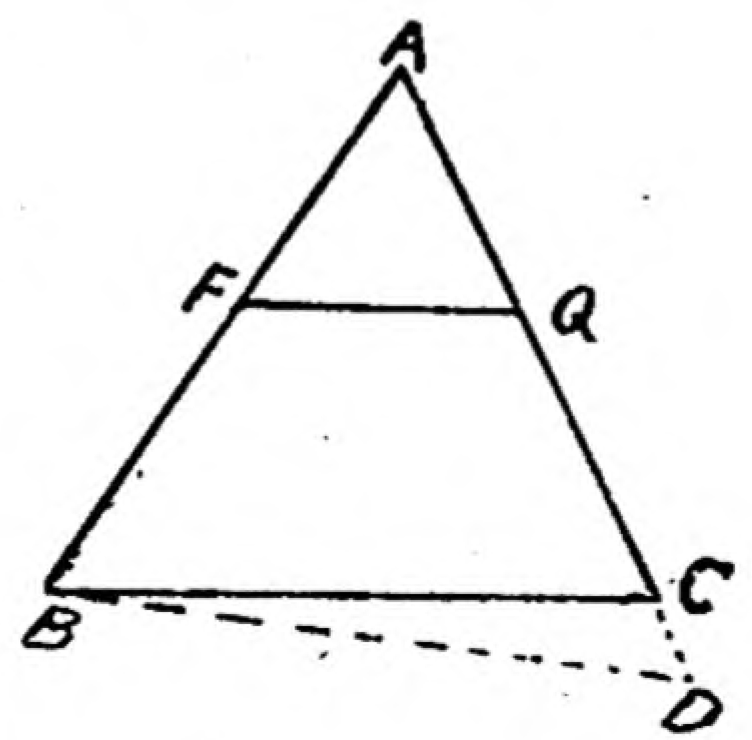
Q. E. D.

Cor :—In the $\triangle ABC$, if PQ is drawn $\parallel$ to BC,

$\frac{AP}{AB}=\frac{AQ}{AC}$.

Converse of the above Theorem.

A straight line which divides two sides of a triangle proportionally is parallel to the third side.



Given :—PQ is a st. line cutting AB, CD the sides of the $\triangle ABC$ so that

$$\frac{AP}{PB}=\frac{AQ}{QC}.$$

Required :—To prove that PQ ∥ BC.

Construction :—If BC is not ∥ to PQ, draw BD ∥ to PQ.

Proof :—As PQ ∥ BD.

$$\therefore \frac{AP}{PB}=\frac{AQ}{QD}.$$

$$\text{But } \frac{AP}{PB}=\frac{AQ}{QC} \qquad \text{(Given)}$$

$$\therefore \frac{AQ}{QD}=\frac{AQ}{QC}.$$

i.e., QD=QC which is impossible unless D coincides with C.

Hence PQ ∥ BC.

Q. E. D.

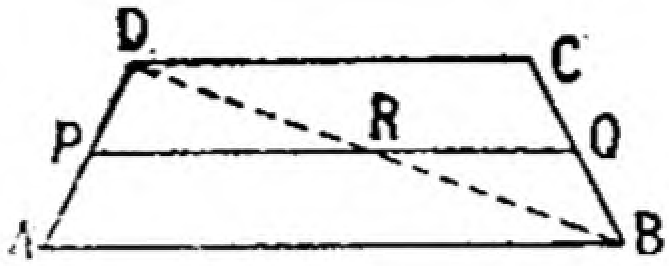
Cor.:—If PQ is a st. line cutting the sides AB, AC of a $\triangle ABC$, so that $\frac{AP}{AB}=\frac{AQ}{AC}$, then shall PQ be ∥ to BC.

Exercises.

1. The straight lide drawn through the middle point of a side of a triangle ∥ to the case bisects the other side.

2. The straight line joining the middle points of two side of a triangle is ∥ to the third side.

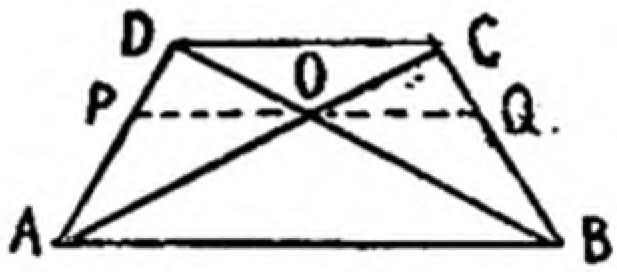
3. ABCD *is a quadrilateral having* AB *parallel to* CD *; prove that any st. line drawn* ∥ *to* AB *cuts* AD *and* BC *proportionally.*



4. The st. line which joins the middle points of the oblique sides of a trapezium is parallel to the parallel sides.

5. *The intercepts made by three or more parallel straight lines on any two st. lines are proportional.*

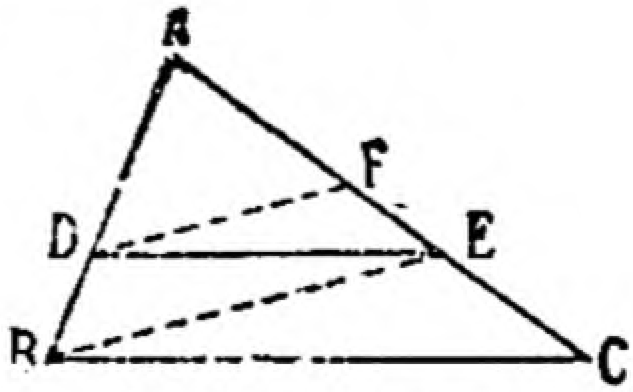
6. *The diagonals of a trapezium cut each othe proportionally.*



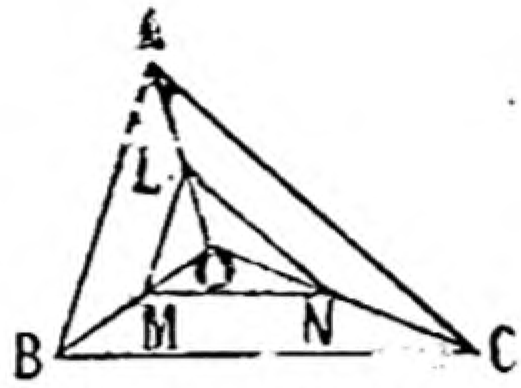
7. From a point E in the common base of two triangles ACB ADB, straight lines are drawn parallel to AC, AD meeting BC, BD at F, G; show that FG ∥ CD.

8. E is a point in the side AB of a quadrilateral ABCD; EF ∥ AD meets BD in F and EG ∥ AC meets BC in G. Prove that FG ∥ CD.

9. In this $\triangle$ABC, DE ∥ BC and DF ∥ BE. Prove that AF : AE=AE : AC.



10. In this $\triangle$ ABC, O is any point, LM ∥ AB, MN ∥ BC. Prove that NL ∥ AC.



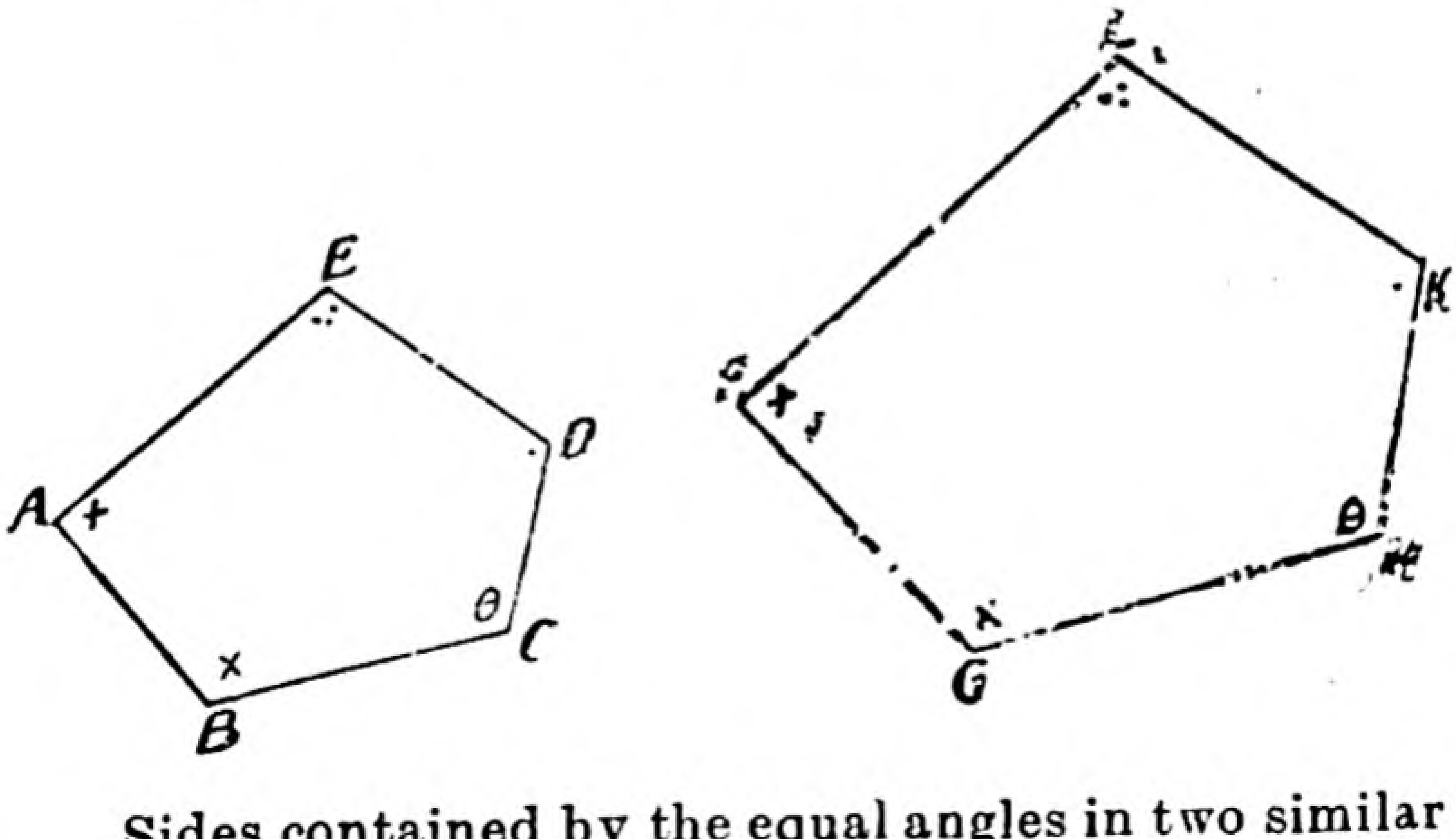
SIMILAR FIGURES.

Def.—Two rectilineal figures are said to be **equiangular,** if the angles of one figure, taken in order, are respectively equal to the angles of the other, taken in order.

Two rectilineal figures are said to be **similar,** if they are equiangular and have also their sides about equal angles proportional.

Thus ABCDE and FGHKL are similar figures, if $\angle A=\angle F$, $\angle B=\angle G$, $\angle C=\angle H$, $\angle D=\angle K$, $\angle E=\angle L$.

and also $\dfrac{AB}{FG}=\dfrac{BC}{GH}=\dfrac{CD}{HK}=\dfrac{DE}{KL}=\dfrac{EA}{LF}$



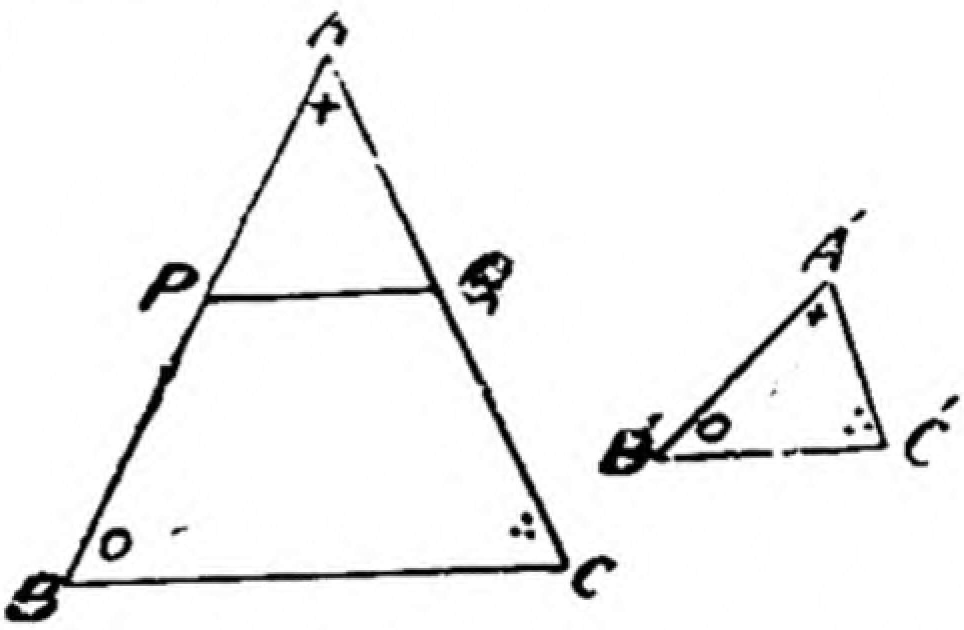
Sides contained by the equal angles in two similar figures are said to be **corresponding sides** and the

angles contained by sides which are proportional are called **corresponding angles.** Thus ∠A, ∠F; ∠B, ∠G; ∠C, ∠H ∠D, ∠K; ∠E, ∠L; are the corresponding angles and AB, FG; BC, GH; CD, HK; DE, KL; EA, LF; are the corresponding sides.

N.B.—In the case of a △ corresponding sides may also be taken as those opposite to equal angles.

Proposition 56. (*Theorem*)

If two triangles are equiangular, their corresponding sides are proportional.



Given : △s ABC, A′B′C′ having ∠A=∠A′, ∠B=∠B′ and ∠C=∠C′.

Required :—To prove that

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{AC}{A'C'}.$$

Construction :—Cut off AP=A′B′ and AQ=A′C′. Join PQ.

Proof :—In the △s APQ, A′B′C′.

$$\because \begin{cases} AP=A'B' & \text{(Constr.)} \\ AQ=A'C' & \text{(Constr.)} \\ \angle A=\angle A' & \text{(Given)} \end{cases}$$

∴ the △s are congruent,

and in particular ∠APQ= ∠B′=∠B.

∴ PQ ∥ BC.

Hence $\frac{AB}{AP}=\frac{AC}{AQ}$.

i.e., $\frac{AB}{A'B'}=\frac{AC}{A'C'}$ (for $AP=A'B'$ and $AQ=A'C'$)

Similarly it may be proved that $\frac{BC}{B'C'}=\frac{AC}{A'C'}$.

Hence $\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{AC}{A'C'}$. **Q. E. D.**

Cor. 1.—If two $\angle$s of one $\triangle$ are equal to two $\angle$s of another $\triangle$ each to each, the two $\triangle$s are **similar.**

Cor. 2. All parallels drawn to a side of a triangle cut off from it triangles which are similar to the whole triangle.

Exercises

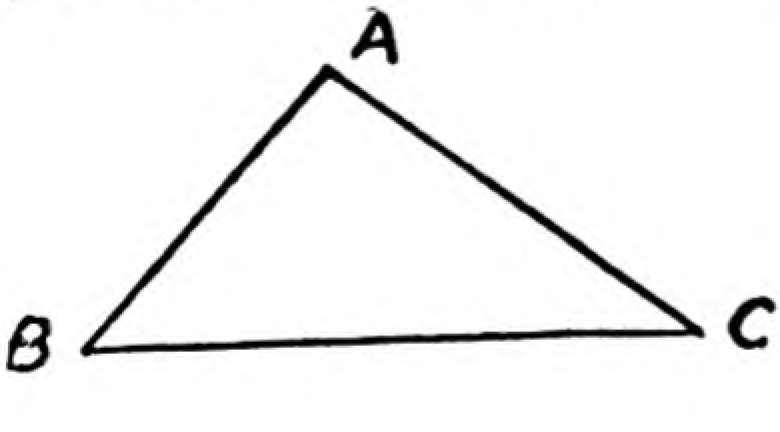
1. In equiangular triangles the perpendiculars drawn from the angles on corresponding sides are proportional to these sides.

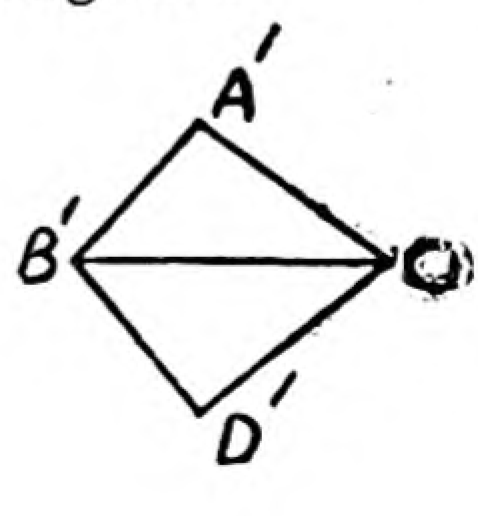
2. *All parallels drawn to the base of a triangle whic are terminated by the sides are bisected by the median to that base.*

3. The straight lines joining the middle points of the sides of a triangle divide the triangle into four triangles each similar to the whole triangle.

Proposition 57 (*Theorem*)

If the corresponding sides of two triangles are proportional, the triangles are equiangular.





Given :—Two $\triangle$s ABC, A′B′C′, such that

$$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CA}{C'A'}$$

Required :—To prove the $\angle A=\angle A'$,
$\angle B=\angle B'$ and $\angle C=\angle C'$

Const. :—Make $\angle C'B'D'=\angle ABC$ and
$\angle B'C'D'=\angle ACB$

Proof :—$\triangle D'B'C'$ is equiangular to $\triangle ABC$ (Const.)

$$\therefore \quad \frac{D'B'}{B'C'}=\frac{AB}{BC}$$

But $\dfrac{A'B'}{B'C'}=\dfrac{AB}{BC}$ (Given)

$\therefore \quad \dfrac{D'B'}{B'C'}=\dfrac{A'B'}{B'C'}$ or $D'B=A'B'$.

Similarly $D'C'=A'C'$.

Hence in the $\triangle$s A′B′C′, D′B′C′.

$$\therefore \begin{cases} A'B'=D'B' \\ A'C'=D'C' \\ B'C' \text{ is common.} \end{cases}$$

$\therefore$ $\triangle$s are congruent.

Hence $\angle A'B'C'=\angle D'B'C'=\angle B$

and $\angle A'C'B'=\angle D'C'B'=\angle C$.

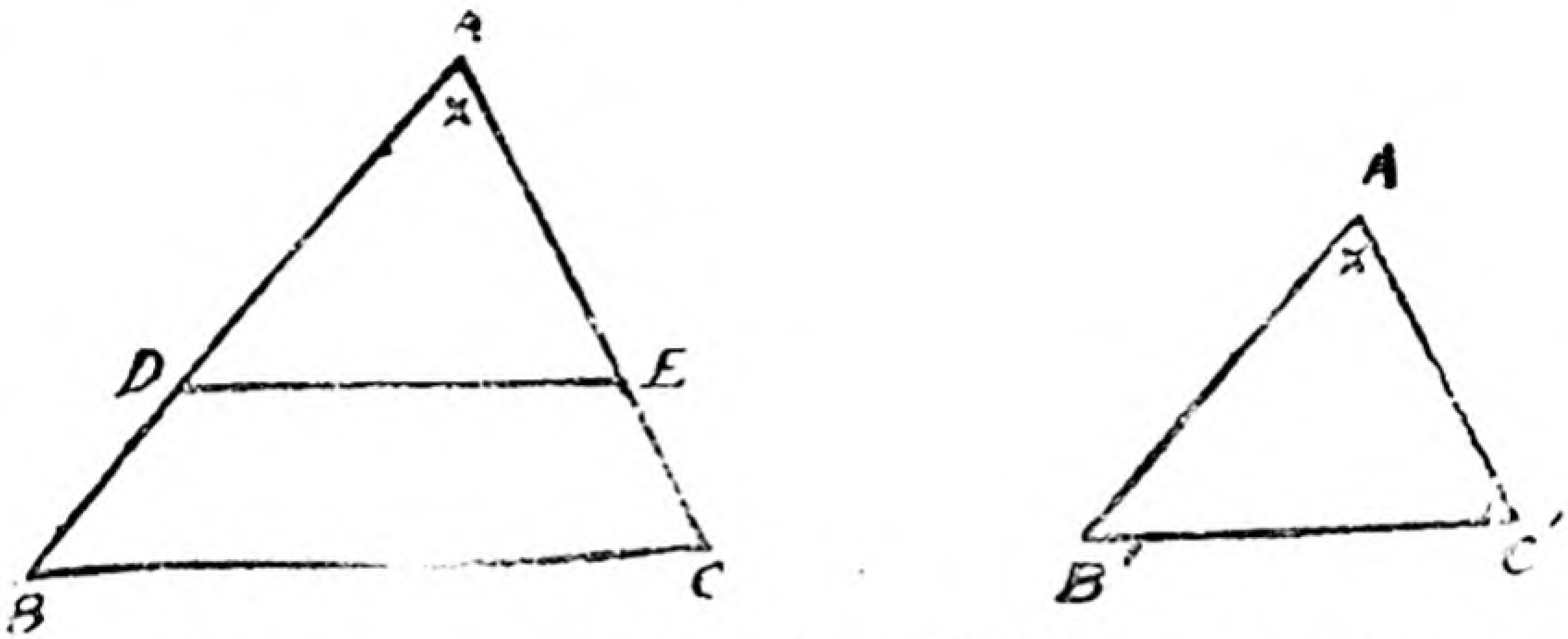
and $\angle A'=\angle D'=\angle A$.

Q. E. D.

Cor.—If the corresponding sides of two triangles are proportional, the triangles are similar.

Proposition 58. (*Theorem*).

If two triangles have one angle of the one equal to one angle of the other and the sides about these angles proportional, the triangles are similar.



Given :—$\triangle$s ABC and A'B'C', having $\angle A = \angle A'$ and $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$

Required :—To prove that the $\triangle$s are similar

Constr ;—From AB and AC, cut off AD=A'B' and AE=A'C' respectively. Join DE.

Proof :— $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$ (Given)

But AD=A'B' and AE=A'C' (Constr)

$\therefore \dfrac{AB}{AD} = \dfrac{AC}{AE}$

$\therefore$ DE $\parallel$ BC

$\therefore \angle D = \angle B$ and $\angle E = \angle C$.

But $\angle D = \angle B'$ and $\angle E = C'$.

$\therefore \angle B = \angle B'$ and $\angle C = \angle C'$.

$\therefore$ the $\triangle$s are equiangular and hence similar.

Q. E. D.

Exercises

1. In the isosceles triangle ABC the base AB is produced both ways to X and Y so that $AX.BY = AC^2$; prove that the triangles XAC and YBC are similar.

2. The diagonal of a four-sided figure cut one another in the same ratio. Prove that the figure has two sides parallel.

3. Two triangles are similar if two sides and the median to one of these two sides of the one are proportional to the corresponding parts of the other.

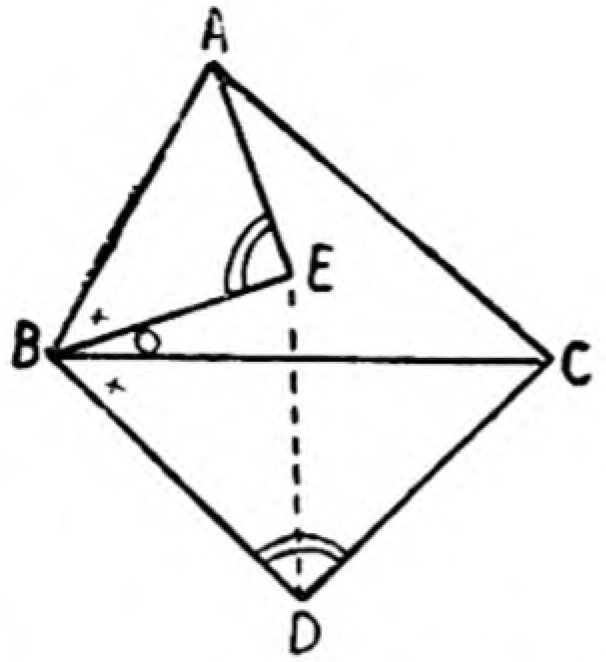
4. *The perpendicular drawn from the vertex of a* $\triangle$ *on the base is a mean proportional between the segments of the base. Prove that the triangle is right-angled.*

5. *From an external point O two st. lines OAB and OT are drawn meeting the circle at AB and T respectively, such that* $OT^2 = OA.OB$. *Prove that OT is a tangent to the circle.*

6. *The vertical angle A of the triangle ABC is bisected by a straight line which cuts the base at D and meets the circumference of the circumcircle of the triangle at E. Prove that the* $\triangle$*s BAD and EAC are similar.*

7. ABC is a triangle and E a point in it. On the base BC and outside it a triangle BCD is drawn such that $\angle CBD = \angle ABE$ and $\angle BDC = \angle BEA$; show that triangles BED and ABC are similar.

(Punjab, 191[illegible])



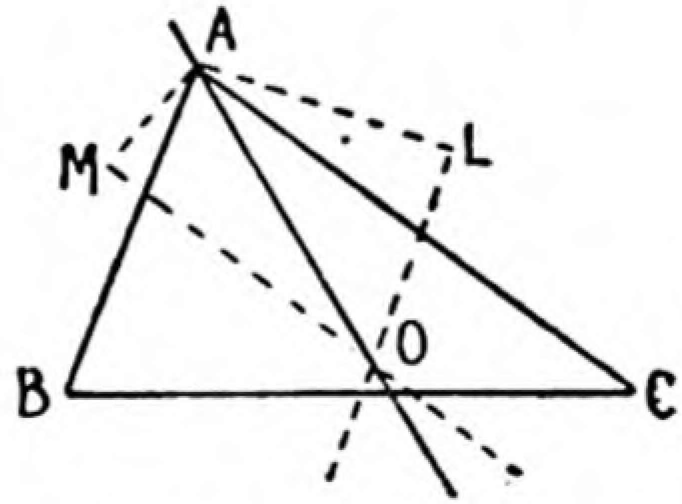
Hint :—Join DE $\therefore$ Δs BDC and ABE are equiangular

$\frac{BD}{BC}=\frac{BE}{AB}$ or $\frac{BD}{BE}=\frac{BC}{AB}$.

Again $\because$ $\angle CBD=\angle EBA$ to each of these equals add $\angle CBE$.

$\therefore$ $\angle EBD=\angle ABC$ $\therefore$ Δs DBE and ABC are similar.

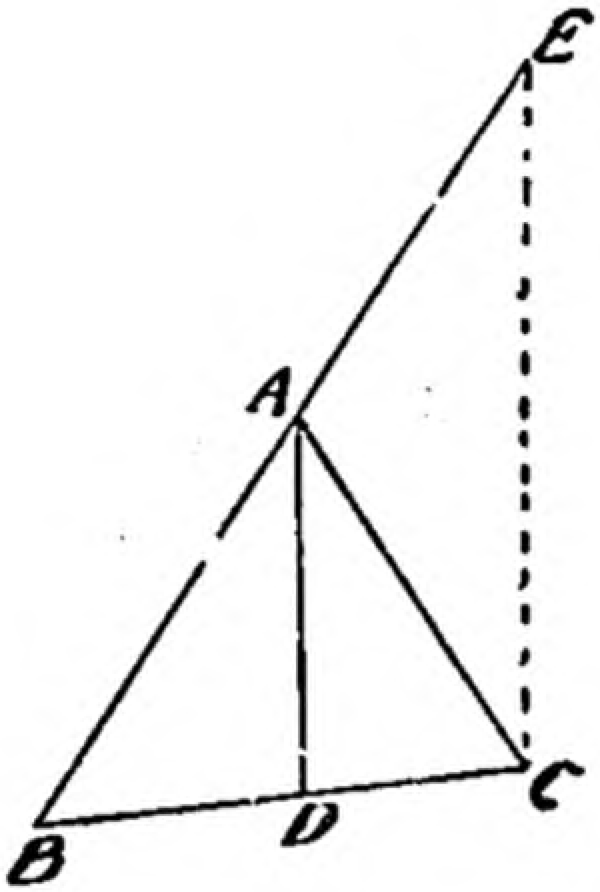
8. *Find the locus of a point which moves in such a way that of the perpendiculars drawn from it to the sides AB and AC of a triangle, the first is always double the other.*



Hint:—Draw AL $\perp$ AB towards the same side of AB as C, and of any length ; draw AM $\perp$ AC towards the same side of AC as B, and make it half as long as AL. Through L and M draw parallels to AB, AC meeting in O. Then AO produced both ways is the required locus.

Proposition 59 (*Theorem*)

The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.



Given :—AD the bisector of $\angle$A of the $\triangle$ABC.

Required :—To prove that BD : DC=BA : AC.

Construction :—Draw CE $\parallel$ to DA meeting BA produced at E.

Proof :—$\because$ AD $\parallel$ EC.

$\therefore$ $\angle$BAD=$\angle$AEC. (Corresp. $\angle$s)

and $\angle$CAD=$\angle$ACE (Alter. $\angle$s)

But $\angle$BAD=$\angle$CAD (Given)

$\therefore$ $\angle$AEC=$\angle$ACE

$\therefore$ AC=AE.

Now DA $\parallel$ CE. (Const).

$\therefore$ $\dfrac{BD}{DC}=\dfrac{BA}{AE}=\dfrac{BA}{AC}$ ($\because$ AE=AC.)

Q. E. D.

Converse of the above Theorem.

If a straight line drawn from an angle of a triangle divides the opposite sides internally in the ratio of the sides containing the angle, it is the internal bisector of that angle.

Given :—In $\triangle ABC$, AD has been drawn in such a way that $\frac{BD}{DC}=\frac{AB}{AC}$

Required: To prove that AD is the internal bisecor of $\angle BAC$.

Construction :—Draw CE parallel to DA cutting BA produced in E.

Proof :—$\because$ AD $\parallel$ CE.

$$\therefore \frac{BD}{DC}=\frac{BA}{AE}$$

But $\frac{BD}{DC}=\frac{AB}{AC}$ (Given)

$$\therefore \frac{AB}{AE}=\frac{AB}{AC}$$

i.e. $AE=AC$

$\therefore \angle ACE=\angle AEC$

But as AD $\parallel$ CE

$\angle ACE=\angle CAD$ (Alter. $\angle$s)

And $\angle AEC=\angle BAD$ (Corresp. $\angle$s)

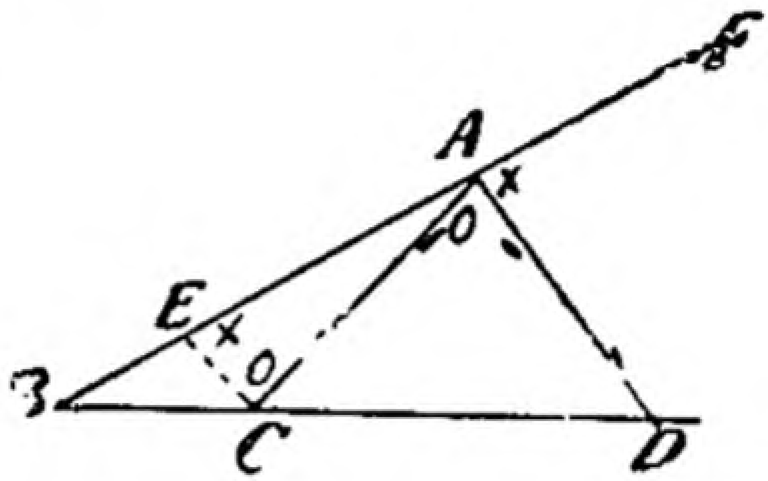
$\therefore \angle CAD=\angle BAD$

Or AD bisects the angle BAC.

Q. E. D.

Proposition 60. (*Theorem*).

The external bisector of an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle.



Given :—AD the bisector of $\angle$CAF, the external $\angle$ at A of the $\triangle$ABC, meeting BC produced in DB.

Required :—To prove that BD : CD=BA : AC.

Construction :—Draw CE $\parallel$ to DA meeting AB in E.

Proof :—$\because$ CE $\parallel$ DA.

$\therefore$ $\angle$CAD= $\angle$ACE. (Alter $\angle$s).

and $\angle$FAD=$\angle$AEC. (Corresp. $\angle$s)

But $\angle$CAD=$\angle$FAD. (Given).

$\therefore$ $\angle$ACE=$\angle$AEC.

$\therefore$ EA=AC.

Now EC $\parallel$ AD (Const.)

$$\therefore \frac{BD}{CD}=\frac{BA}{EA}$$

$$=\frac{AB}{AC} \qquad (\because EA=AC)$$

Q. E. D.

Converse of the above theorem

If a st. line drawn from an angle of a triangle divides the opposite side externally in the ratio of the sides containing the angle, it is the external bisector of that angle.

Given :—In the $\triangle ABC$, AD has been drawn in such a way that $\frac{BD}{CD}=\frac{AB}{AC}$.

Required :—To prove that AD is the external bisector of the $\angle BAC$.

Construction :—Draw CE parallel to AD cutting AB in E.

Proof :— $\because$ AD $\parallel$ CE.

$\therefore \frac{BD}{CD}=\frac{AB}{AE}$.

But $\frac{BD}{CD}=\frac{AB}{AC}$ (Given)

$\therefore \frac{AB}{AE}=\frac{AB}{AC}$.

i.e., $AE=AC$

$\therefore \angle ACE=\angle AEC$

But as AD $\parallel$ CE

$\angle ACE=\angle CAD$. (Alter. $\angle$s)

and $\angle AEC=\angle DAF$ (Corresp. $\angle$s).

$\therefore \angle CAD=\angle DAF$.

Or, AD bisects the $\angle BAC$ externally.

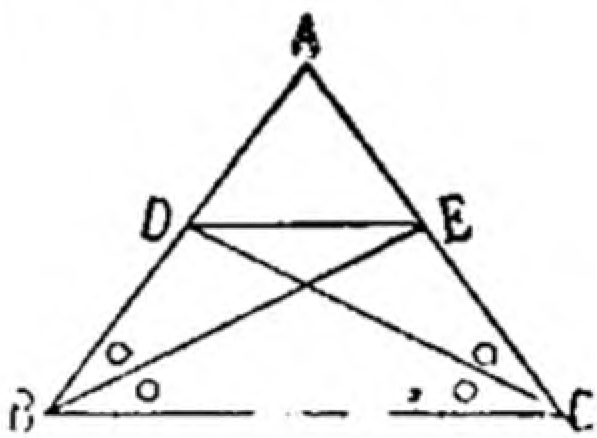
Q.E.D.

Note.—The theorem does not hold good when the $\triangle ABC$ is isosceles, because in that case AD becomes $\parallel$ to BC.

Exercises.

1. ABCD is a quadrilateral and the bisectors of angles B and D meet the diagonal AC at L; show that AB : BC = AD : CD. (Punjab, 1925)

2. The bisectors of the angles at the base BC of an isosceles triangle ABC meet the opposite sides at D and E; prove that DE ∥ BC. (Punjab)



3. The bisectors of the angles B and C of a $\triangle$ABC meet the opposite sides in E and D respectively. If DE is parallel to BC, prove that the triangle is isosceles.

Hint.—$\because$ BE bisects $\angle$B, $\therefore \frac{AB}{BC}=\frac{AE}{EC}$

Again $\because$ CD bisects $\angle$C. $\therefore \frac{AC}{BC}=\frac{AD}{BD}$

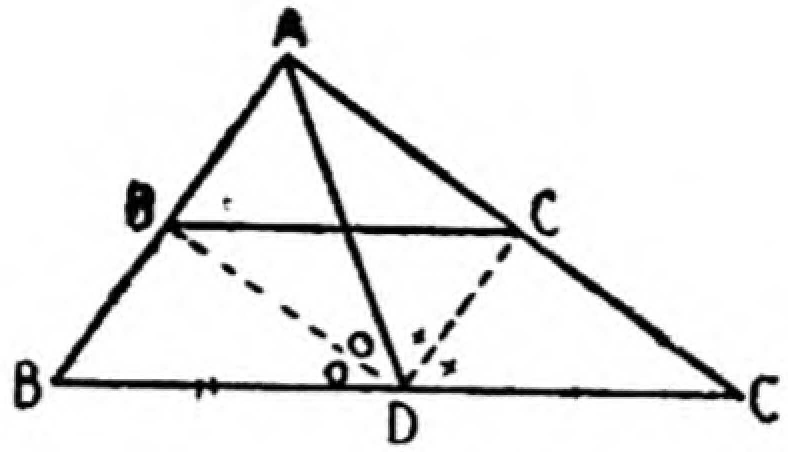
But $\because$ DE ∥ BC $\therefore \frac{AE}{EC}=\frac{AD}{BD}$

$\therefore \frac{AB}{BC}=\frac{AC}{BC}$ $\quad\therefore$ AB = AC

4. *The median AD bisects the base of the $\triangle$ABC in D, DB and DC are drawn bisecting the angles ADB and ADC respectively and meeting the sides AB, AC in B′ and C′. Prove that B′C′ ∥ BC.* (Punjab, 1914)

Hint. – $\because$ DB′ bisects $\angle$ADB $\therefore \frac{AD}{BD}=\frac{AB'}{BB'}$

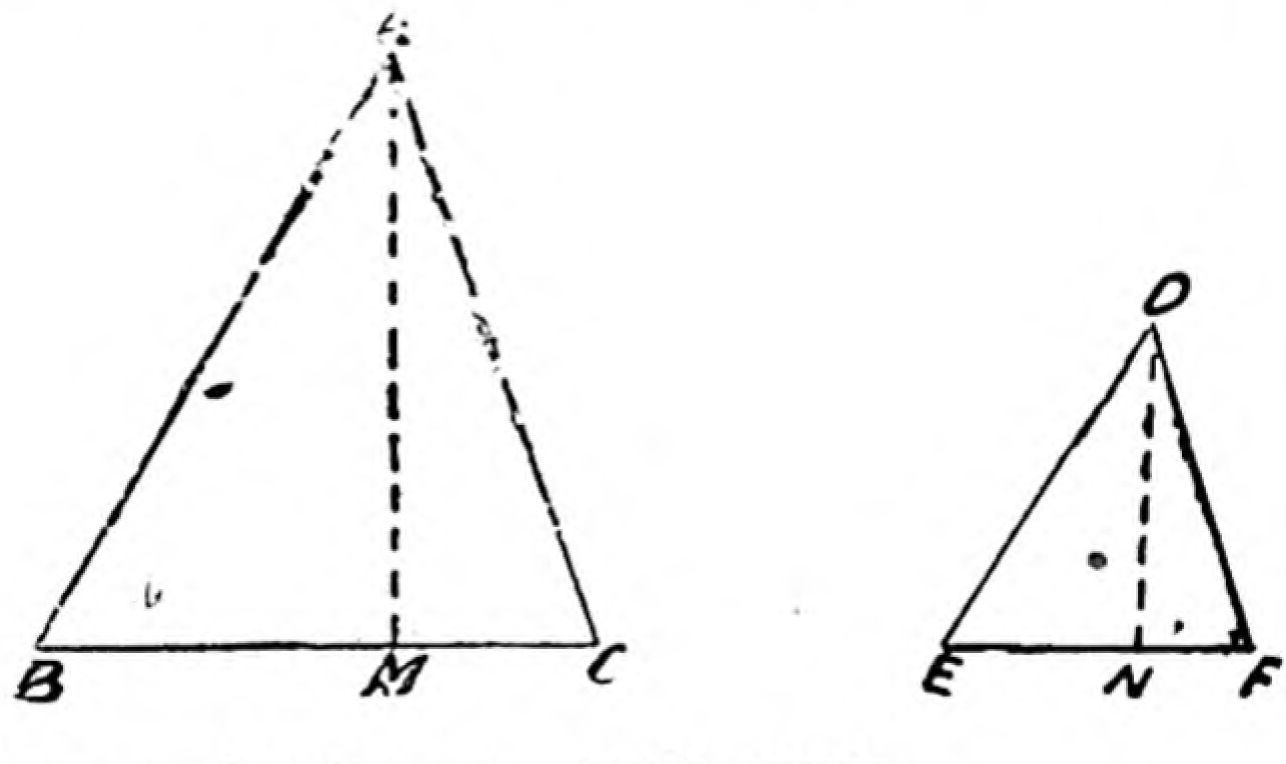
$\because$ DC′ bisects $\angle$ADC $\therefore \dfrac{AD}{CD}=\dfrac{AC'}{CC'}$



But $\dfrac{AD}{BD}=\dfrac{AD}{CD}$ $\because$ BD=CD $\therefore$ $\dfrac{AB'}{BB'}=\dfrac{AC'}{CC'}$ $\therefore$ BC′ $\parallel$ BC

Proposition 61 (*Theorem*)

The ratio of the areas of two similar triangles is equal to the ratio of the squares on the corresponding sides.



Given :—Similar $\triangle$s ABC, DEF

Required :—To prove

$$\frac{\triangle ABC}{\triangle DEF}=\frac{BC^2}{EF^2}=\frac{CA^2}{FD^2}=\frac{AB^2}{DE^2}.$$

Construction :—Draw AM and DN $\perp$s to BC and EF respectively.

Proof :— $\because$ $\triangle ABM$ is similar to $\triangle DEN$,

$$\left(\because \begin{array}{l}\angle B=\angle E \text{ (Given)}\\ \angle M=\angle N \text{ (Const.)}\end{array}\right)$$

$$\therefore \frac{AM}{DN}=\frac{AB}{DE}.$$

$$=\frac{BC}{EF} \qquad \text{(Given)}$$

Now $$\frac{\triangle ABC}{\triangle DEF}=\frac{\frac{1}{2}AM.BC}{\frac{1}{2}DN.EF}=\frac{AM.BC}{DN.EF}.$$

$$=\frac{BC.BC}{EF.EF} \quad \left(\because \frac{AM}{DN}=\frac{BC}{EF} \text{ proved}\right)$$

$$=\frac{BC^2}{EF^2}.$$

Similarly $$\frac{\triangle ABC}{\triangle DEF}=\frac{CA^2}{FD^2}=\frac{AB^2}{DE^2}.$$

Q. E. D.

Cor.—Areas of similar triangles are proportional to the squares on the corresponding altitudes.

Exercises

1. Equal similar triangles are equal in all respects.

2. AC and BD are the diagonals of a trapezium which intersect on O ; if the side AB is ∥ to and double of CD, find the ratio of triangles AOB and COD.

3. X, Y are the middle points of the sides AB, AC of a triangle ABC. Show that the triangle AXY is one-fourth of the triangle ABC.

4. AD and BE are the medians of the triangle ABC which meet in G. If DE be joined, compare the areas of the triangles ABG and DGE.

5. *The equilateral triangle described on the side of a square is half of the equilateral triangle described on the diagonal of the same square.*

Hint :—The equilateral $\triangle$s are similar and the square on the diagonal of a square is double the original square.

6. If the areas of two similar triangles be as **4 : 1** show that the sides of the one are double the corresponding sides of the other.

7. Two isosceles triangles have equal vertical angles, and their areas are in the ratio of 9 to 16. Compare their altitudes.

8. *Show that two similar triangles are to each other in the duplicate ratio of* :—

(*i*) *their corresponding medians* ;

(*ii*) *the bisectors of corresponding angles* ;

(*iii*) *the circum-radii.*

Hint :—(*i*) Similar $\triangle$s ABC, A′B′C′ and AD and A′D′ medians. $\therefore$ $\triangle$s are similar.

$$\therefore \quad \frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{2BD}{2B'D'}=\frac{BD}{B'D'} \text{ and } \angle B=\angle B'$$

$\therefore$ $\triangle$s ABD and $\triangle$A′B′D′ are similar

$$\therefore \quad \frac{AD}{A'D'}=\frac{AB}{A'B'}=\frac{BC}{B'C'}.$$

But $\dfrac{\triangle ABC}{\triangle A'B'C'}=\dfrac{BC^2}{B'C'^2}$, hence

$$\frac{\triangle ABC}{\triangle A'B'C'}=\frac{AD^2}{A'D'^2}$$

(*ii*) If AD and A′D′ are the bisectors of $\angle$s A and A′

$\therefore$ $\triangle$s. ABD and A′B′D′ are similar, hence $\dfrac{AD}{A'D'}=\dfrac{BC}{B'C'}$

and $\therefore \quad \frac{\Delta ABC}{\Delta A'B'C'} = \frac{BC^2}{B'C'^2} = \frac{AD^2}{A'D'^2}$

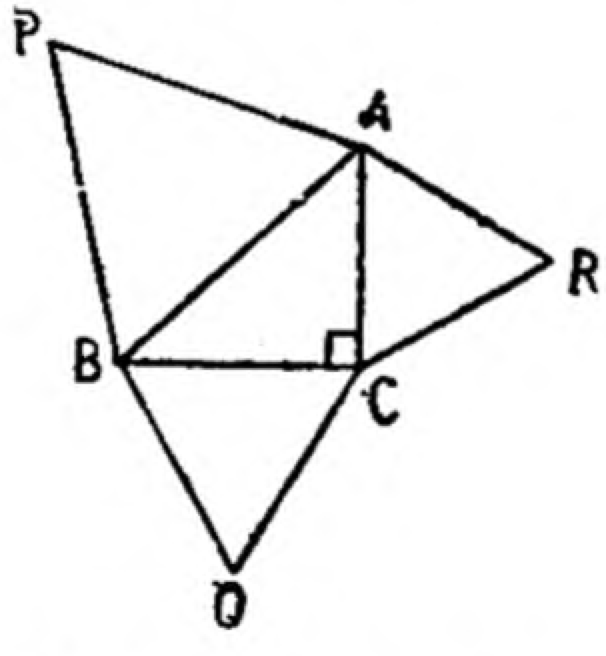
(*iii*) Use the fact that the circum-radii of similar Δs are as their corresponding sides.

9. *Prove that the equilateral triangle described on the hypotenuse of a right-angled triangle is equal to the sum of the equilateral triangles described on the sides containing the right angle.*

Hint :— $\frac{\Delta Q}{\Delta P} = \frac{BC^2}{AB^2}$

$$\frac{\Delta R}{\Delta P} = \frac{AC^2}{AB^2}$$

$$\therefore \quad \frac{\Delta Q + \Delta R}{\Delta P} = \frac{BC^2 + AC^2}{AB^2} = 1.$$



10. *The area of similar polygons are proportional to the squares on the corresponding sides.*

———

SECTION IV

THE CIRCLE

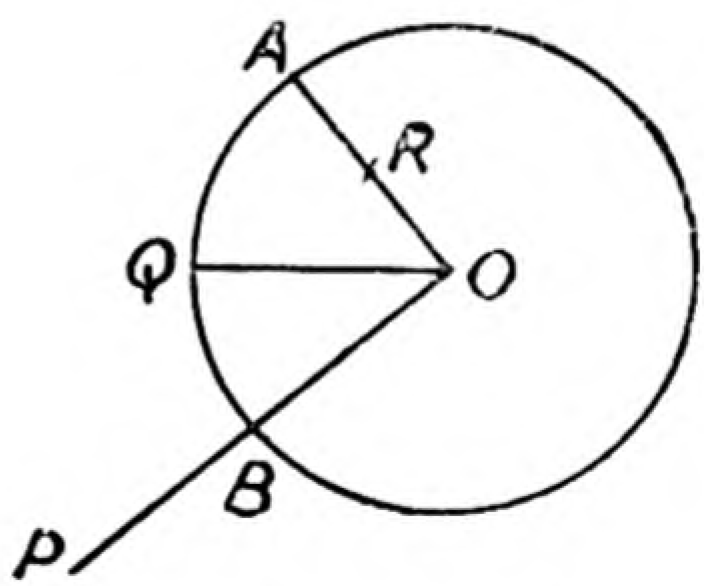
A circle is the locus of a point in a plane which moves in such a way that its distance from a fixed point is constant.

Two circles are said to be equal when their radii are equal.

For, if such circles are placed with their centres, one on the other, their circumferences must also coincide, all points on the circumferences of both the circles being equally distant from their centres.

A point lies without, upon, or within a circle according as its distance from the centre is greater than, equal to, or than the radius.

Three points P, Q, R, are taken respectively



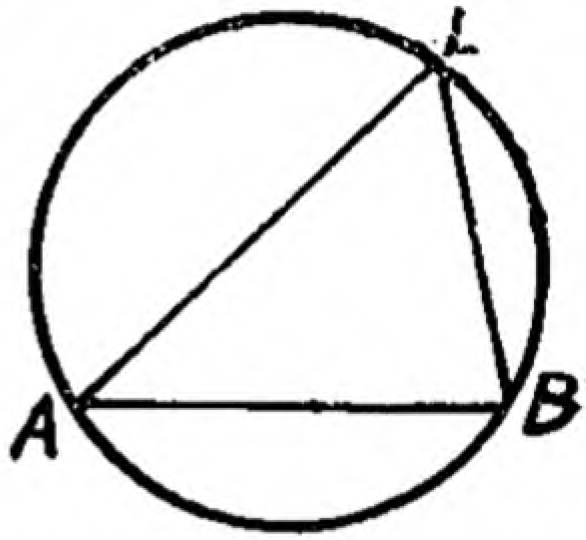
without, upon, and inside the circle, join the points to O, the centre of the circle.

OR when produced meets the circle in A. Therefore OR is evidently less than the radius OA.

Line OP cuts the circle in B. Evidently B is nearer to the centre O than P.

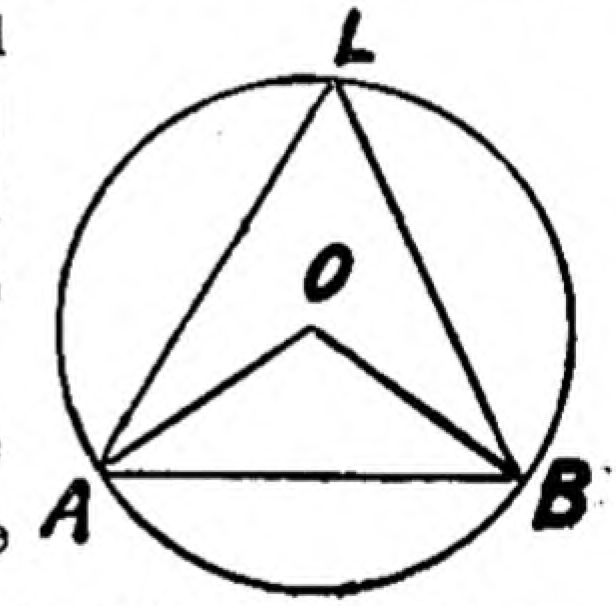
Therefore OP is greater than OB, the radius of the circle.

An angle in a segment of a circle is an angle contained by two straight lines drawn from any point in the arc of the segment to the extremities of its chord. $\angle$ALB is the angle in the segment ALB.



If two segments contain equal angles, they are said to be **Similar Segments**.

If any two points A and B are taken on the circumference of a circle and joined to the centre O, the angle formed *at the centre* by these lines is called an **angle at the centre** (as angle AOB).



If the same point A and B are joined to *any* other point L on the arc ALB, the angle ALB thus formed *at the circumference* is called the **angle at the circumference.**

If a polygon is so constructed as to have all its angular point on the circumference of a circle, it is said to be an **inscribed polygon ;** but if it is so made as to have all its sides touching the circle, it is a **circumscribed polygon.**

In the first figure we have an inscribed polygon ABCDE, but in the second figure the polygon cons-tructed is a circumscribed polygon.

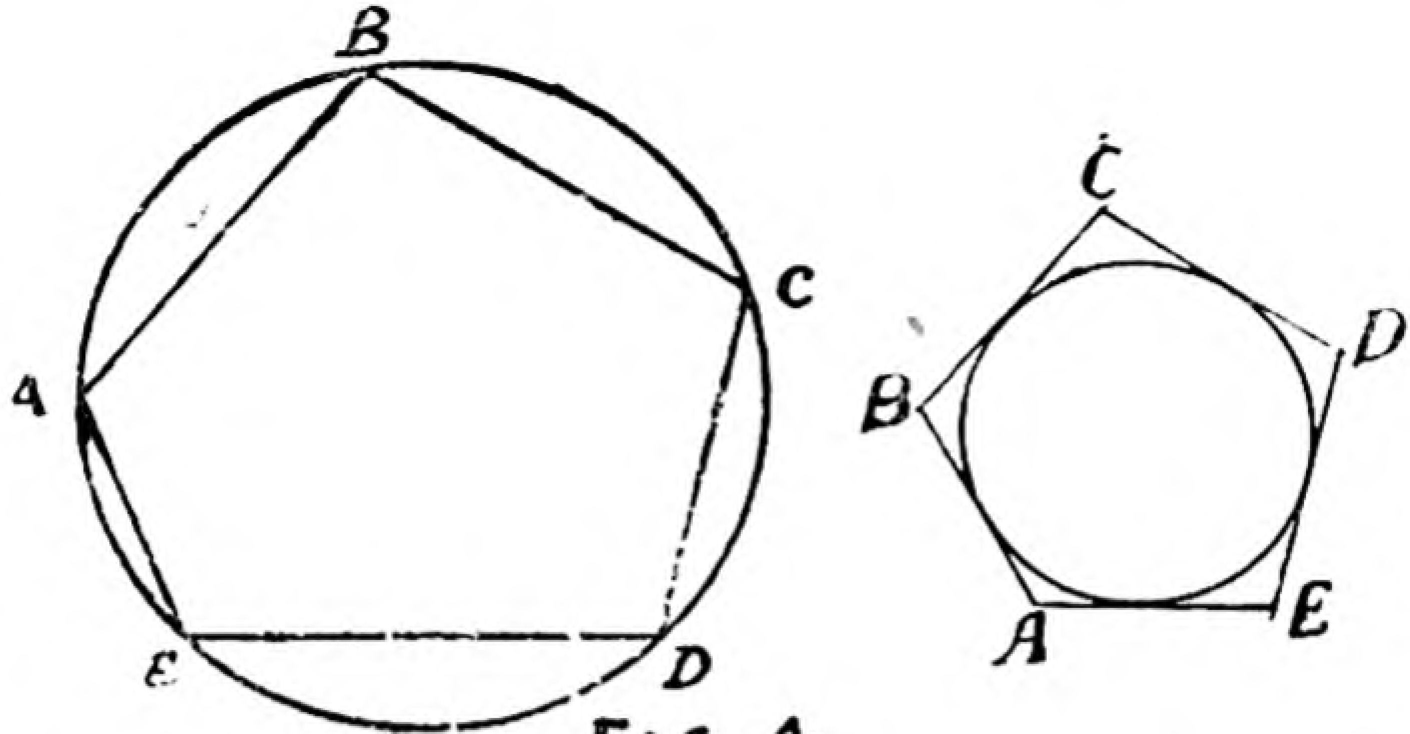


FIG 4

A figure is said to be **symmetrical** *about* (or *with respect to*) a straight line, when on being folded about the line, the parts of the figure on either side of it can be made to coincide with one another. The straight line about which symmetry takes place is called the **line of symmetry** or **axis of symmetry**.

Study closely the figures, given below, if they are folded about the line LQ, in each case the one

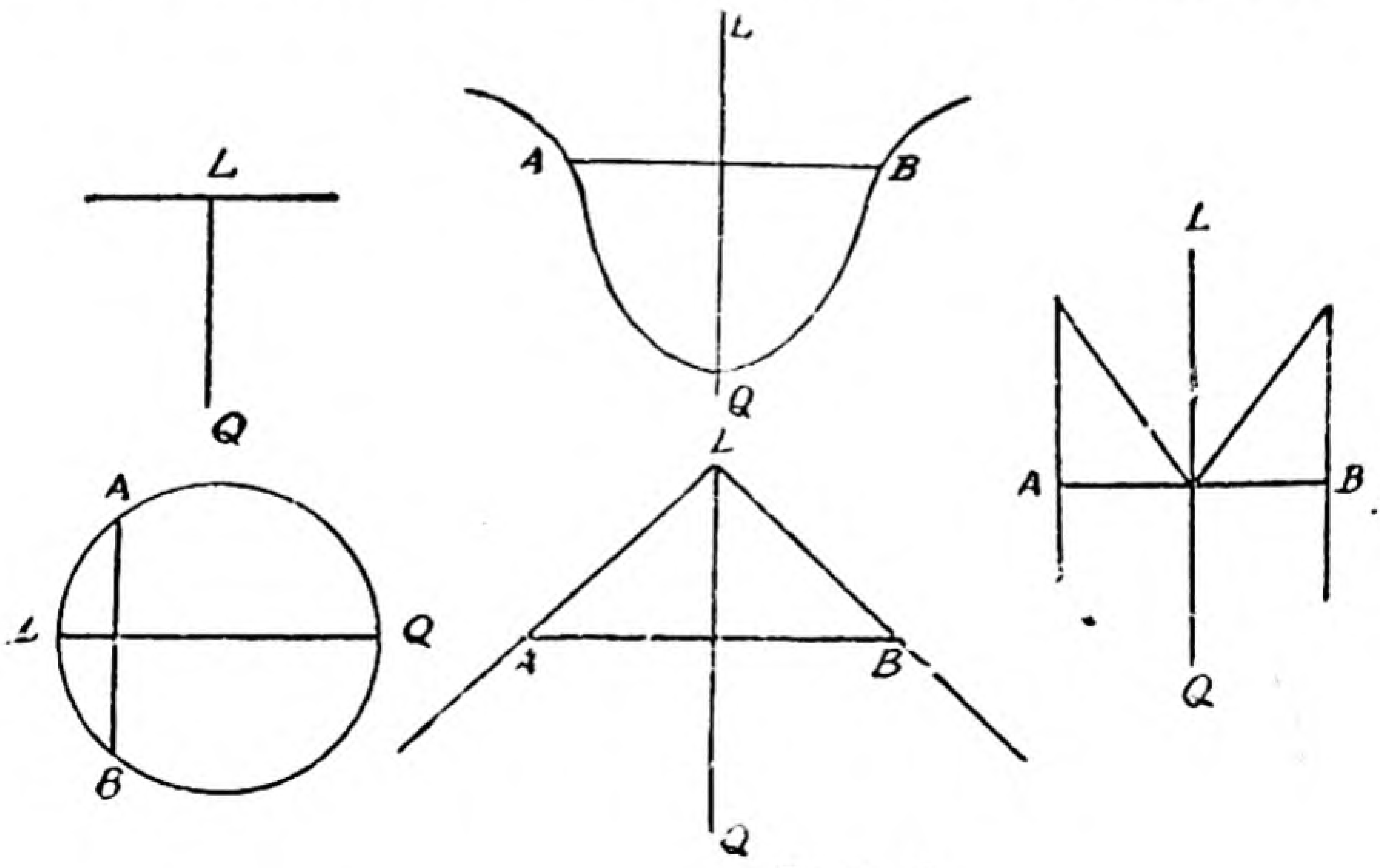


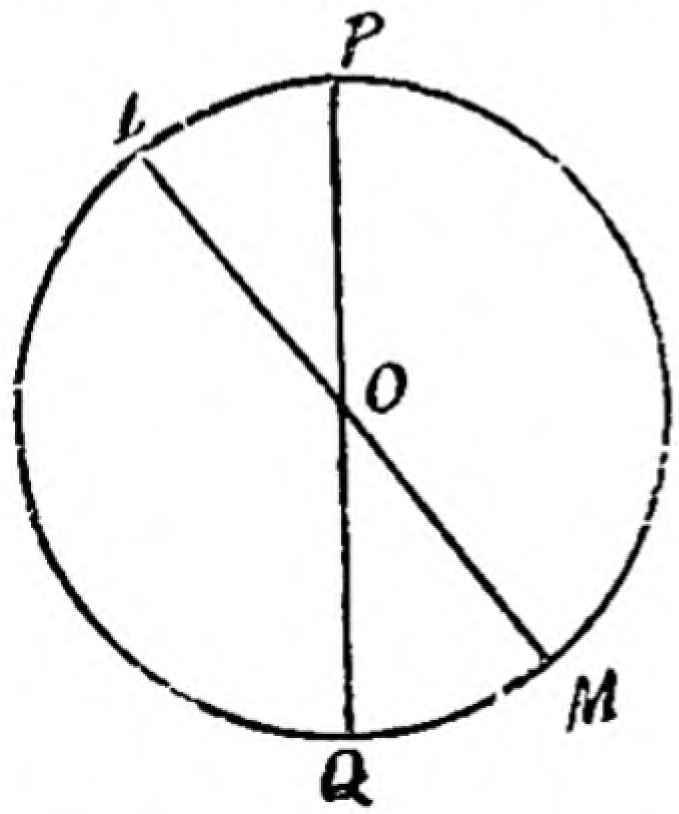
FIG 6

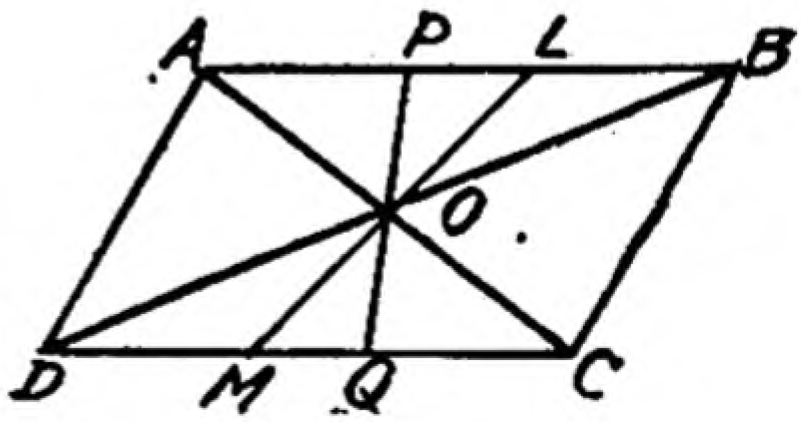
half will exactly coincide with the other half. All these figures are symmetrical and the line LQ is the axis of symmetry in each figure.

It is evident from these illustrations that if any line is drawn ⊥ to the axis of symmetry it will meet the figure on both sides at points which are equidistant from the axis. Such points are called the **Corresponding points** in the figure. Each of them is also said to be the **image** or **reflexion** of the *other* in the line of symmetry. In fig. 6, A and B are corresponding points, each being the *image* or *reflection* of the other in the axis of symmetry.

A circle is symmetrical about a diameter. For if it is folded about any diameter the two parts exactly fall, one on the other, otherwise there would be points on two halves which are *not* equally distant *from* the centre.

A figure is said to be **symmetrical about** (or **with respect to**) a point, if the point bisects every straight





line drawn through it to meet the boundary of the figure in both directions. Thus a circle is symmetrical

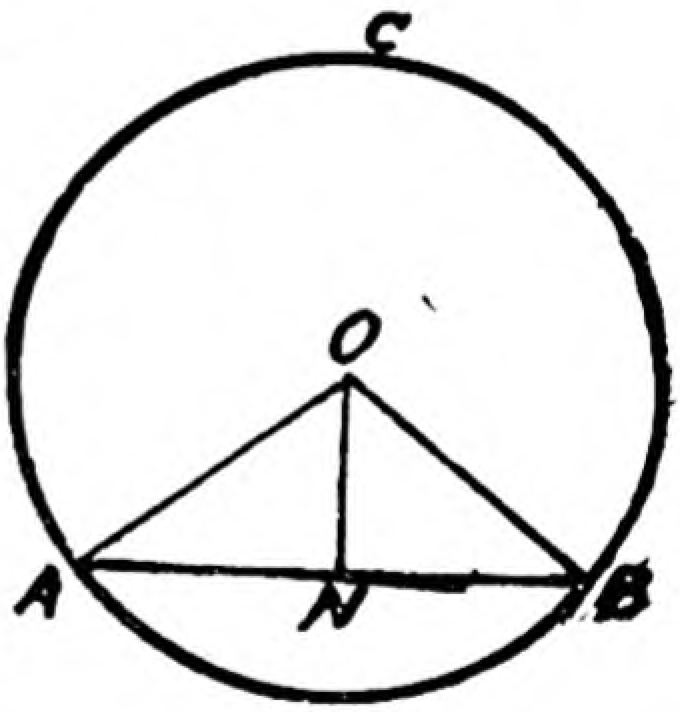
about its *centre* and a parallelogram about the *point of intersection* of its diagonals for, as is clear from the figures, every line drawn through these points so as to terminate at the boundary of the figures is bisected at these points.

Proposition 62. (*Theorm*).

A straight line drawn from the centre of a circle to bisect a chord (which is not a diameter), is at right angles to the chord.

Given :—A st. line ON is drawn from O, the centre of circle ABC to N the mid-point of a chord AB.

Required :—To prove that ON $\perp$ AB.



Construction :— Join OA, OB.

Proof :—In the $\triangle$s OAN, OBN

$$\because \begin{cases} OA=OB & \text{(radii)} \\ ON=ON & \\ AN=BN & \text{(Given)} \end{cases}$$

$\therefore$ the $\triangle$s are congruent.

$\because$ $\angle$ONA $= \angle$ONB.

But these are adj. Supp. $\angle$s.

$\therefore$ ON $\perp$ AB. **Q.E.D.**

Converse of the above Theorem.

The perpendicular drawn from the centre of a circle to a chord, bisects the chord.

Given :—ON is a st. line drawn from O, the centre of circle ABC, to meet the chord AB at rt. angles in N.

Required :—To prove that AN=BN.

Construction :—Join OA, OB.

Proof :—In the rt. angled $\triangle$s OAN and OBN.

Hypotenuse OA=hypotenuse OB (radii)

side ON=side ON.

$\therefore$ the $\triangle$s are congruent.

$\therefore$ AN=BN. **Q. E. D.**

Cor. 1. The perpendicular bisector of any chord of a circle passes through the centre.

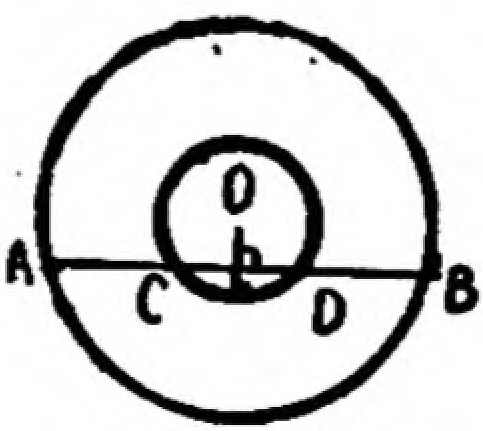
Cor. 2. A straight line cannot cut a circle in more than two points.

[For if it cuts in three points, say A, B and C, the $\perp$ bisectors of AB, BC which should meet in the centre, would in this case be $\parallel$ and would not meet.]

Exercises.

1. The locus of the centres of all circles which pass through two given points is the right bisector of the join of the points.

2. *The intercepts of a chord between two concentric circles are equal.*



Hint .—Prove that AC=BD.

3. *A circle is symmetrical about any diameter.*

4. The straight line joining the middle points of two parallel chords of a circle $\perp$ to both of them and passes through the centre.

5. The middle point of a chord being given, draw the chord.

6. The perpendicular bisectors of the sides of an inscribed quadrilateral pass through one point.

7. Two chords of a circle which are not *both* diameters cannot bisect each other.

8. *The locus of the midpoints of parallel chords of a circle is the diameter perpendicular to the chords.*

9. *Show that parallel st. lines through the points of intersection of two intersecting circles are equal.*

10. A chord whose length is 120′ is placed in a circle of radius 61′. Find how far the chord is from the centre.

11. In a circle of radius 13′ a chord is placed at a distance of 5′ from the centre. Find the radius of the circle.

12. A chord 48′ in length is placed in a ⊙ at a distance of 7′ from the centre. Find the radius of the circle.

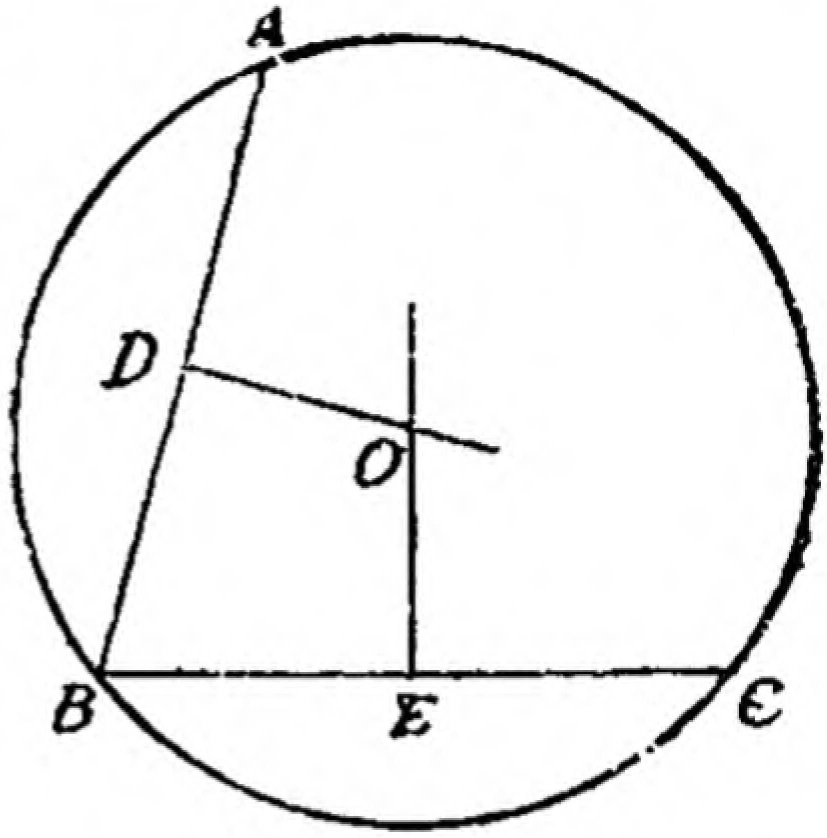
Proposition 63 (*Theorem*).

There is one circle and one only, which passes through three points not in a st. line.

Given :—A, B, C are three points not in a st. line.

Required :—To prove that one circle and one only can be drawn to pass through A, B and C.

Construction:—Join AB,BC.



Draw DO, EO the perpendicular bisectors of AB, BC and let them intersect at O.

Proof :—∵ DO is the perpendicular bisector of AB.

∴ OC=OB.

and ∵ EO is the perpendicular bisector ofBC.

∴ OB=OC.

∴ OA=OB=OC,

Hence a circle whose centre is O and radius OA will pass through A, B and C.

Again since the perpendicular bisectors DO and EO can intersect in one point only, therefore O is the only point equidistant from A,B, and C.

Hence one circle ABC only can be drawn through three given points, A, B and C not in a st. line.

Q. E. D.

Cor. 1. Circles which have three points common coincide.

Cor. 2. Two circles cannot cut each other at more than two points) (For, if they did, they would be identical circles).

Note 1.—The condition "three points not in a straight line" is absolutely essential. For if the joins A, B, C are in a st. line, the right bisectors of AB and BC will be parallel and cannot meet in a point. Therefore no centre will be found.

Note 2.—If a circle passes through three given points its size and position are fully determined. This is why a circle is generally named by three letters denoting three points on the circumference.

Exercises.

1. If from any point within a circle three equal st. lines can be drawn to the circumference, that point must be the centre of the circle.

2. Show that a circle can be described to pass through the angular points of any rectangle.

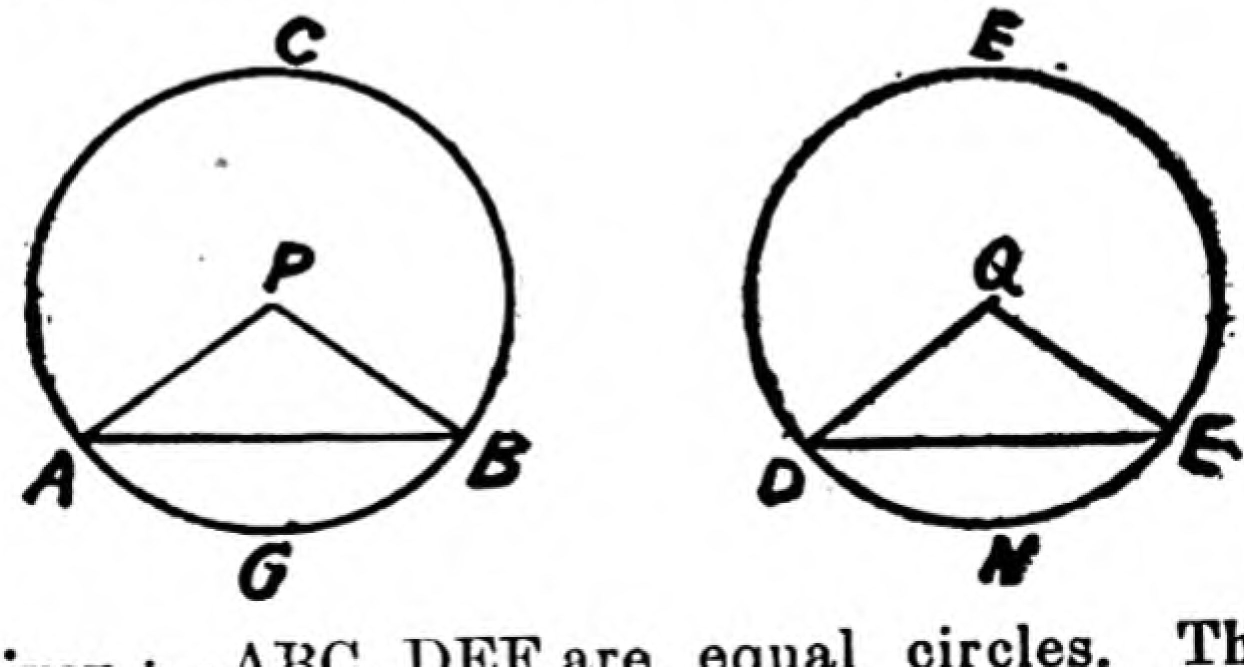
3. Show that a circle can be described about an isosceles trapezium.

Hint.—The bisector of parallel sides is an axis of Symmetry.

4. *Complete a circle of which an arc is given.*

Proposition 64. (*Theorem*).

In equal circles (or, in the same circle), if two arcs subtend equal angles at the centre they are equal.



Given :—ABC, DEF are equal circles. The arcs AGB, DHE, subtend equal angles APB, DQE, at the centres P, Q.

Required :—To prove that arc AGB=arc DHE.

Proof :—Place the circle DEF on the circle ABC, so that centre Q may fall on centre P, then

$\because$ the circles are equal, the circumference will coincide.

Now make circle DEF revolve about the centre Q till QD falls along PA. Then since $\angle DQE = \angle APB$, QE falls along PB and since the circumferences coincide, D coincides with A and E with B ($\therefore$ radius QD $=$ radius PA and QE$=$PB).

$\therefore$ Arc DHE coincides with arc AGB.

$\therefore$ Arc DHE$=$arc AGB.

Q. E. D.

Converse of the above theorem

In equal circles (or in the same circle), if two arcs are equal they subtend equal angles at the centres.

Given :—In equal $\odot$ ABC and DEF, arc AGB$=$ arc DHE.

Required :—To prove that $\angle$s APB and DQE subtended by these arcs at the centres P and Q are equal.

Proof :—Apply $\odot$DEF to $\odot$ABC, so that centre Q may fall on centre P.

Since the $\odot$s are equal, the circumferences coincide.

Now make DEF revolve about the centre Q till QD coincides with PA. Then since arc DHE$=$arc AGB.

E coincides with B.

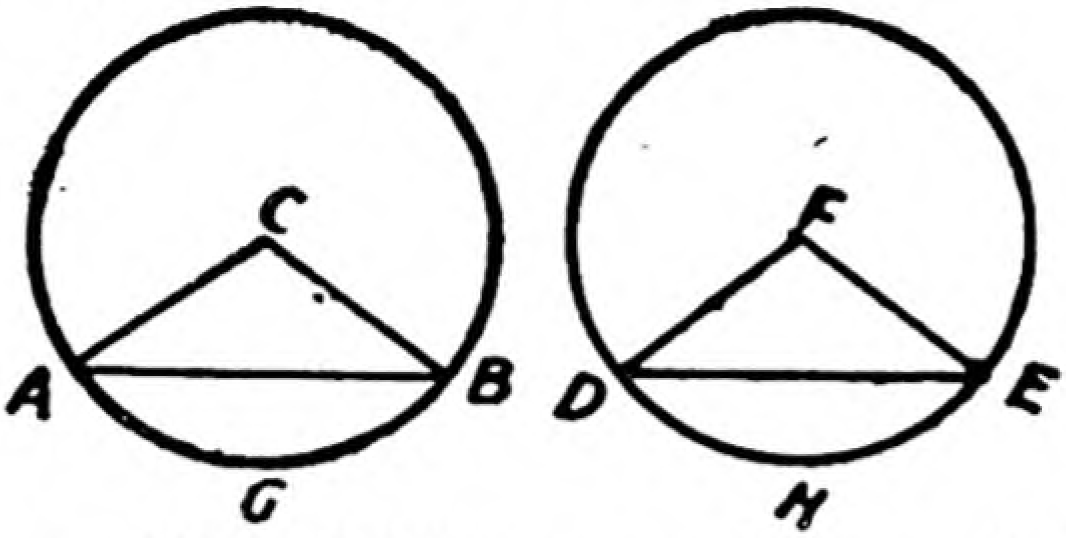
$\therefore$ QD coincides with PA, and QE with PB.

$\therefore$ $\angle DQE = \angle APB$.

Q. E. D.

Proposition 65. (*Theorem*)

In equal circles (or in the same circle), if two chords are equal, they cut off equal arcs.



Given :—ABG, DEH are equal ⊙s ; their centres are C and F. Chord AB=Chord DE.

Required :—To prove that arc AGB=arc DHE.

Construction :—Join CA, CB ; FD and FE.

Proof :—In the △s ABC and DEF.

$$\because \left\{ \begin{array}{ll} AB=DE & \text{(Given)} \\ AC=DF & \text{(Radii of equal ⊙s)} \\ BC=EF & \text{(Radii of equal ⊙s)} \\ \therefore \text{ the } \triangle\text{s are congruent.} & \end{array} \right.$$

$\therefore \angle ACB = \angle DEF.$

$\therefore$ in equal ⊙s arcs AGB and DHE subtend equal ∠s ACB and DFE at the centres.

$\therefore$ arc AGB=arc DHE.

Q. E. D.

Converse of the above theorem.

In equal circles (or in the same circle), if two arcs are equal, the chords of the arcs are equal.

Given :—In equal circles ABG and DEH with centre C and F ; arc AGB=arc DHE.

Required :—To prove that chord AB=chord DE.

Construction :—Join CA, CB ; FD, FE.

Proof :—In equal circles ABG and DEH.

∴ arc AGB = arc DHE.

∴ ∠ACB = ∠DFE.

Again in △s ACB and DFE

∵ AC = DF (Radii of equal ⊙s)

BC = EF (,, ,,)

and included ∠ACB = included ∠DFE (Proved)

∴ the △s ACB and DFE are congruent.

∴ AB = DE.

Q. E. D.

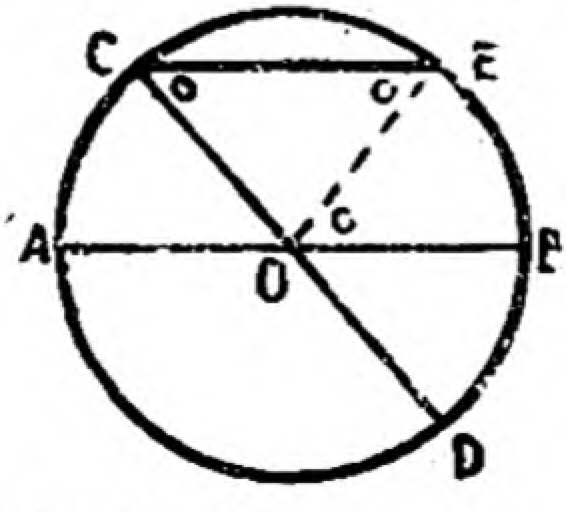
Cor. 1.—In equal circles (or in the same circle), equal chords subtend equal angles at the centres.

Exercises

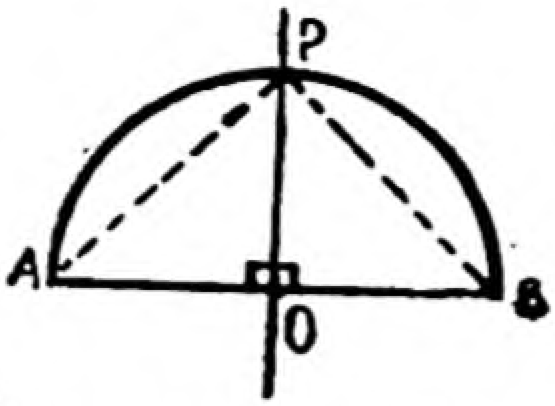
1. In equal circles or in the same circle unequal arcs subtend unequal angles at the centre, the greater angle standing on the greater arc.

2. In equal circles or in the same circle two sectors are equal, if the angles between their bounding radii are equal.

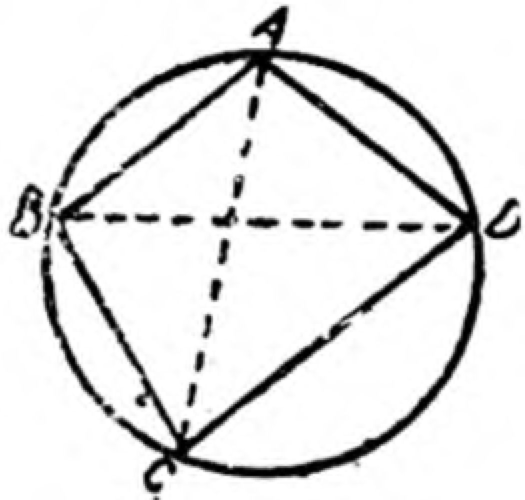
3. AB and CD are two diameters of a circle and CE is a chord parallel to AB, prove that B is the mid-point of the arc DE.



4. *The perpendicular bisector of the chord of an arc bisects the arc.*



5. *If two opposite sides of a quadrilateral inscribed in a circle are equal, its diagonals are equal.*



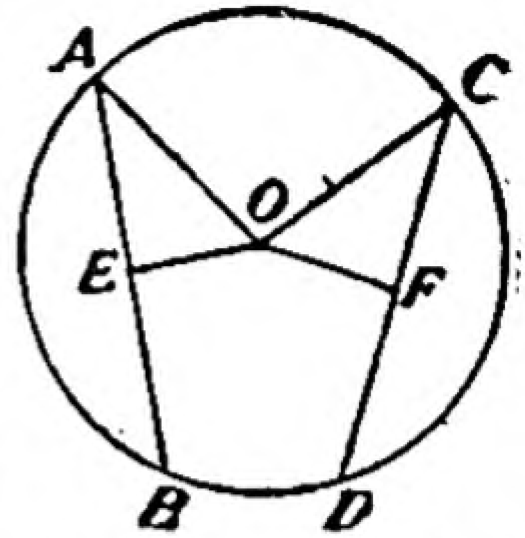
6. The straight lines joining the alternate angular points of a regular polygon inscribed in a circle are equal.

7. Triangles are inscribed in two equal circles, such that two sides of the one are equal to two sides of the other, each to each ; prove that the remaining sides are also equal.

8. *In two circles, if equal chords subtend equal ∠s at the centres, the circles are equal.*

Proposition 66 (*Theorem*)

Equal chords of a circle are equidistant from the centre.



Given :—Equal chords AB, CD of a circle with centre O, OE ⊥ AB and OF ⊥ CD.

Required :—To prove that OE=OF.

Construction :—Join OA and OC.

Proof :—∵ O is the centre of the ⊙ and OE ⊥ AB.

∴ AE=EB=half of AB.

Similarly CF=half of CD.

But AB=CD (Given)

∴ AE=CF. (Halves of equals).

Now in rt. ∠d △ AOE and COF

∵ { Hypotenuse AO=hypotenuse CO (Radii)
Side AE=side CF (Proved) }

∴ △s are congruent.

∴ OE=OF.

Q. E. D.

Converse of the above theorem.

Chords of a circle that are equidistant from the centre are equal.

Given :—In a circle with centre O,

OE ⊥ to chord AB and OF to chord CD, and OE=OF.

Required :—To prove that chord AB=chord CD.

Construction :—Join OA and OC.

Proof :—∵ OE ⊥ AB from the centre O.

∴ AE=$\frac{1}{2}$AB or AB=2 AE.

Similarly CF=$\frac{1}{2}$ CD or CD=2 CF.

Now in rt. ∠d △s AOE and COF,

{ Hypotenuse AO=hypotenuse CO (Radii)
Side OE=side OF. (Given) }

∴ $\triangle$s are congruent.

∴ AE=CF.

∴ AB=CD. (Doubles of equals).

Q. E. D.

Exerc ses.

1. *Of any two chords of a circle the one which is the greater is nearer to the centre.*

2. *Of any two chords of a circle, the one which is nearer to the centre is the greater.*

3. Of all chords that can be drawn through a given point within a circle the longest is the diameter through that point.

4. Of all chords that can be drawn through a given point within a $\odot$ the shortest is the perpendicular to the diameter through that point.

5. Two intersecting chords of a circle are equi-inclined to the diameter passing through their point of intersection. Prove that the chords are equal.

6. *Equal intersecting chords of a $\odot$ divide each other into equal segments, the greater equal to the greater and the less to the less.*

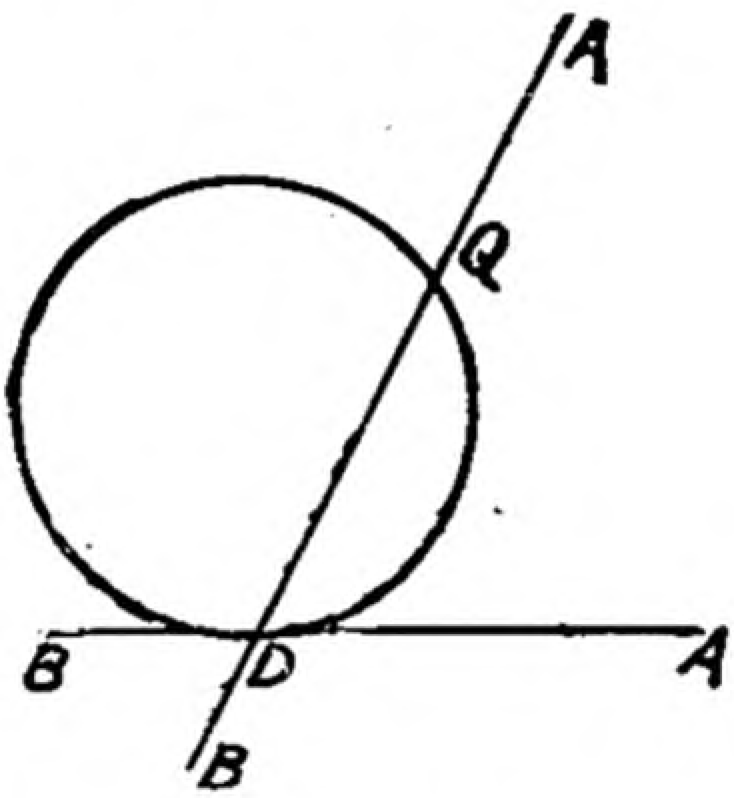
7. *The locus of the middle points of all equal chords of a circle is a concentric circle.*

8. Through two points on a diameter of a circle equidistant from the centre two parallel chords are drawn. Show that the extremities of these chords are at the angular points of a rectangle.

9. If three equal chords of a circle meet in a point, the point must be the centre of the circle.

10. *The locus of a fixed point on a chord of constant length sliding in a cirle is a concentric circle.*

Def.—Any straight line which cuts a circle is called a **secant**. In the figure below, AB cuts the circle in two points P and Q and is therefore a secant.



Let this secant AB rotate about one of its points of intersection, say P, the other point Q will travel along the arc and ultimately come very close to P. In this limiting position when P and Q become consecutive points, the secant AB will take such position as A′PB′. The line A′PB′ in this position is called the tangent to the circle.

A **tangent**, *therefore, is the limiting position of a secant which cuts the circle in two points which are so close to each other that they coalesce into one.*

The two diagrams given below are instructive and will help the student further to understand clearly how a secant is reduced to a tangent.

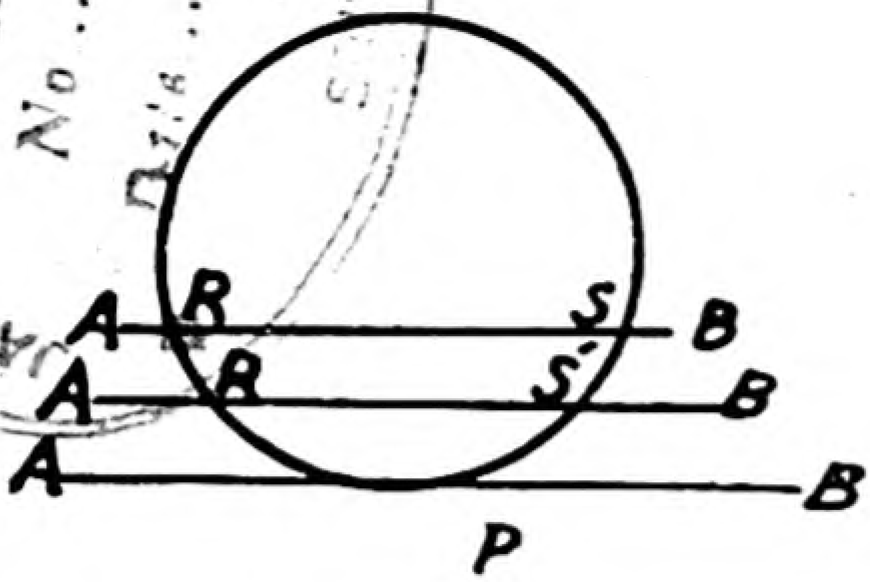


Fig. 2

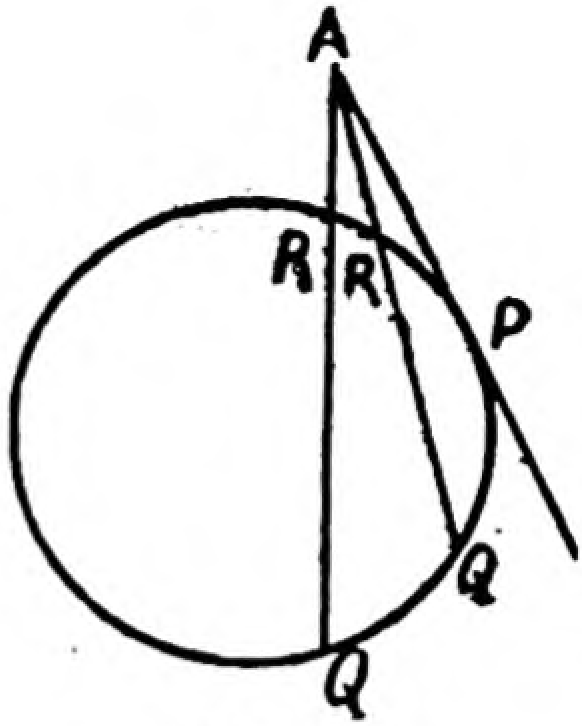


Fig. 3

In Fig. 2, a secant AB, which cuts the circle in R and S, moves parallel to itself and the points of intersection R and S approach each other so much that ultimately they coincide into one point P and the secant in that position becomes the tangent APB.

Similarly, in Fig. 3, a secant AQ turns about A and both the points of intersection R and Q finally coincide at some point P when the secant becomes the tangent AP.

The point at which a tangent touches (or cuts in two coincident points) is called the *point of contact* of the tangent.

A tangent may also be defined thus :—

A tangent is a straight line which meets a circle at one point only (i.e., where points of intersection ultimately coincide) and which when produced does not cut it again.

Students may have noticed that in Figs 2 and 3 above both the extremities of the secant have actually

moved from one position to another, but in Fig. 1' only one end Q has moved along the arcto coincide with P.

In Fig. 1 the secant becomes the tangent to the ⨀ **at** point P *lying on the circle*, but in fig. 3 the secant has become a tangent to the⨀ from a point A *'outside or external to the circle*.

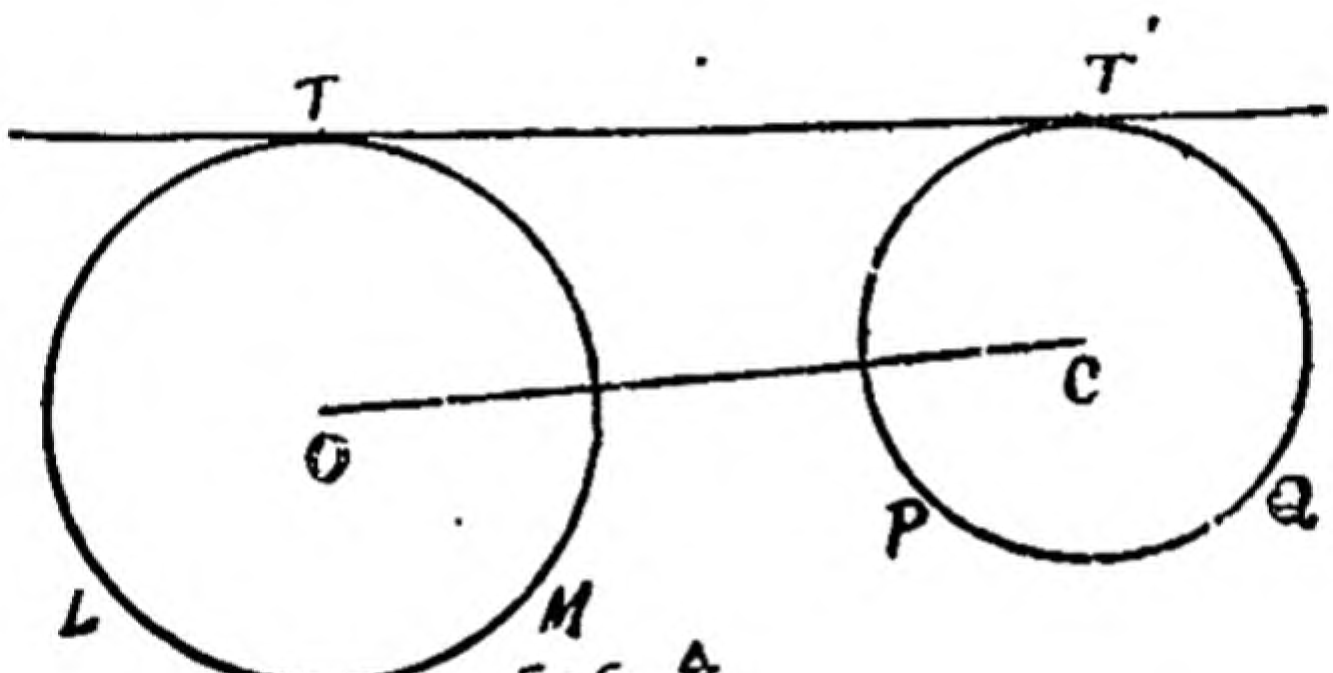


Fig. 4.

Def.—*If the same straight line touches two circles, it is called a* **common tangent**, as in Fig. 4 the line TT' is the common tangent to the circle TLM and

Def :—*If both the points of contact of the tangent with the circle lie on the same side of the line joining the centres, the tangent is called a* **direct common tangent** or **an external common tangent**. In fig 4 as both T and T' lie on the same side of OC, TT' is a direct common tangent to the circles.

Def.—*If the points of contact of the tangent with*

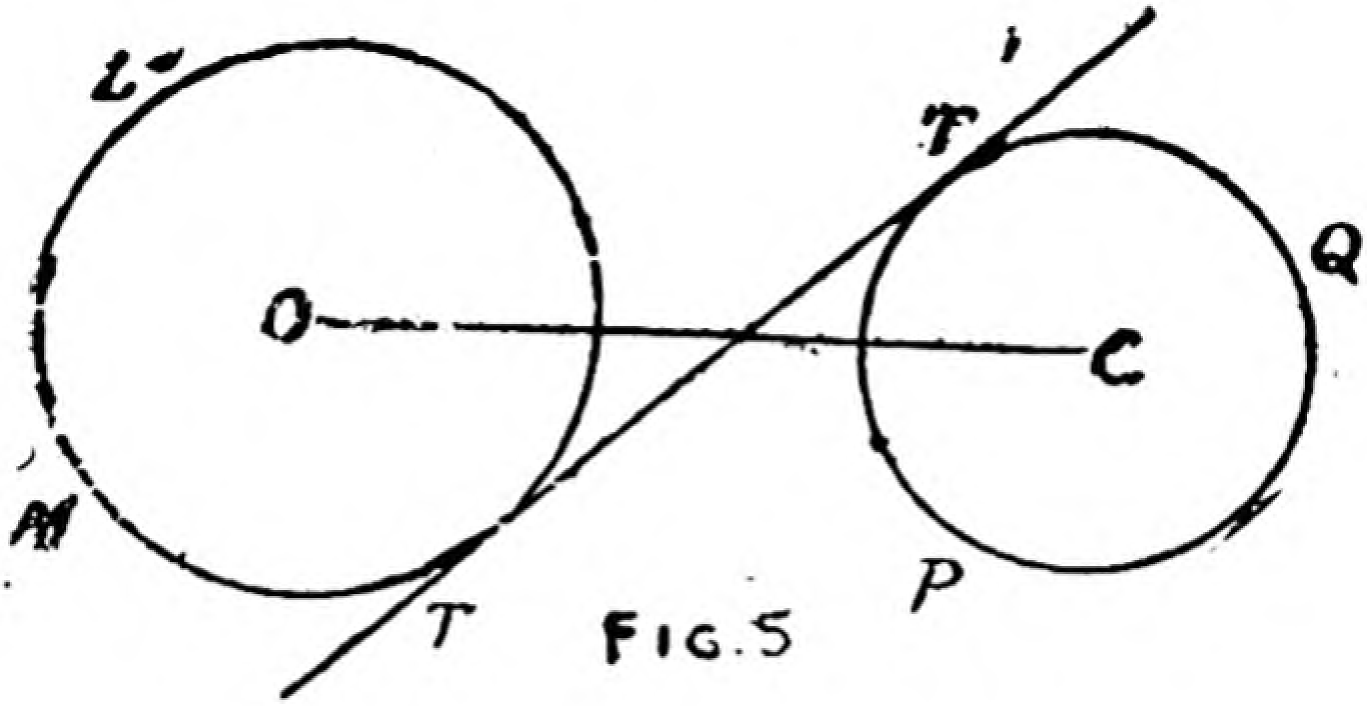


Fig. 5.

the circle lie on opposite sides of the line joining the centres, the tangent is called a **transverse common tangent** or **internal common tangent**. Thus in Fig. 5 TT′ is a transverse common tangent to the circles because T and T′ lie on opposite sides of OC, the line that joins the two centres.

Def.—If three or more points lie on the same straight line, they are called collinea points. Thus A, B, C, D, are collinear points.

ig. 6

Proposition 67 (*Theorem*).

The tangent at any point of a circle and the radius through the point are perpendicular to each other.

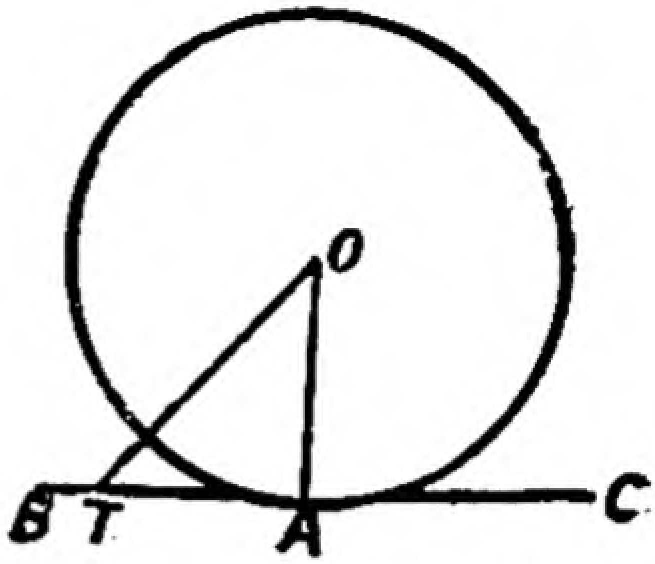
Given :—BC a tangent at A to the circle with centre O.

Required :—To prove that OA ⊥ BC.

Construction :

Take T any other point on the tangent BC.

Join OT.



Proof :—T is any point on the tangent BC other than the point of contact A.

∴ T lies outside the circle.

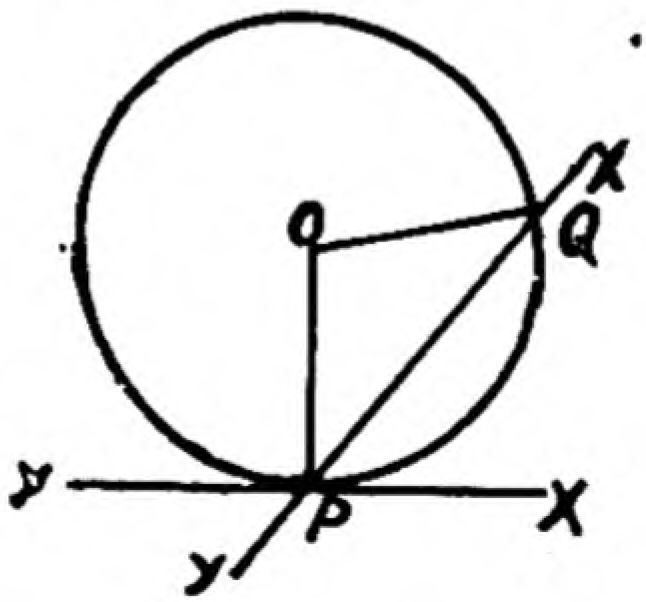
Hence OT > radius OA.

Similarly, it can be proved that every line from the centre O to BC is > OA, or OA is the shortest st. line from the point O to BC.

But of all st. lines that can be drawn to a given st. line from a given point outside it, the perpendicular is the shortest.

∴ OA ⊥ BC. **Q. E. D.**

Another proof by the method of limits.



Given.—YPX a tangent at P to the ⊙ with centre O, and OP the radius from P. the point of contact,

Required :—To prove that OP $\perp$ YPX.

Construction :—Draw XY, a secant cutting the $\odot$ in P and Q.

Join OP and OQ.

Proof : - OP=OQ (Radii)

$\therefore \angle OPQ = \angle OQP$.

$\therefore \angle OPY = \angle OQX$ [Supplements of equal $\angle$s]

Now let the secant XY turn about P The point Q will travel along the arc and ultimately coincide with P. In the position when P and Q coincide at P the secant XQPY becomes the tangent XPY and the $\angle$OPY and $\angle$OQX become equal to $\angle$OPY and $\angle$OPX respectively.

$\therefore \angle OPY = \angle OPX$.

But the sum of $\angle$OPY and $\angle$OPX is equal to two right angles.

$\therefore$ each is a right angle.

Hence OP $\perp$ YPX. **Q. E. D.**

Cor. 1. The perpendicular to a tangent at the point of contact passes through the centre.

Cor. 2. At any point on the circumference of a circle one and only one tangent can be drawn.

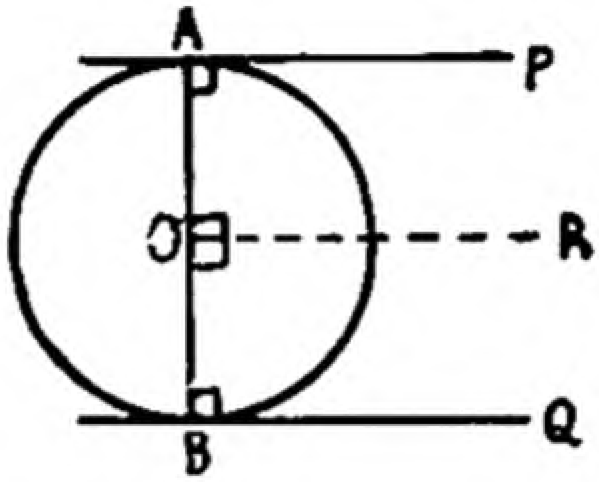
Cor. 3. The straight line perpendicular to a radius at its outer extremity is a tangent to the circle.

Exercises.

1. Tangents at the ends of a diameter of a circle are parallel.

2. The straight line joining the points of contact of any two parallel tangents is a diameter of the circle.

Hint.—Draw OR ∥ AP



3. *Two circles are concentric and a chord of the outer circle touches the inner, show that it is bisected at the point of contact.* (*Calcutta*, 1924).

4. *Two circles have the same centere. Show that all chords of the outer circle which touch the inner circle are equal.* (*Punjab*, 1911).

5. Tangents drawn at the ends of a chord of a circle are equi-inclined to the chord.

6. Draw a tangent to a circle at a point on the circumference.

7. Draw a tangent to a circle parallel to a given line.

8. Draw a tangent to a circle at right angles to a given straight line.

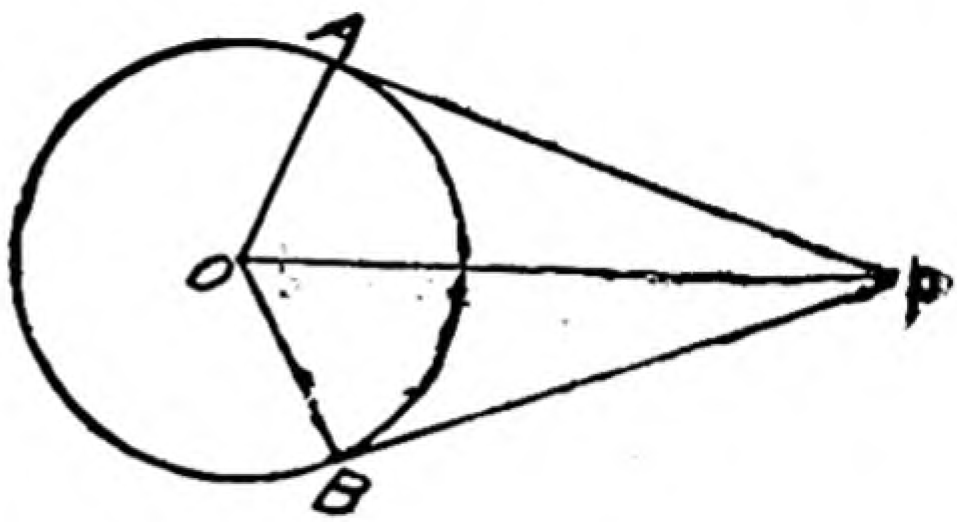
9. Draw a tangent to a circle making an angle of 30° with a fixed diameter.

10. Find the locus of the centres of circles which touch a given straight line at a given point. (Calcutta, 1916).

Propositisn 68. (*Theorem*)

If tangents are drawn frnm an external point to a circle, show that (i) they are equal, (ii) they subtend equal angles at the centre and (iii) they make equal angles with the line joining the point with the centre.

Given :—
PA and PB are tangents to the circle from P, a point outside it. O is the centre.



Required :—To prove that

(*i*) AP=BP

(*ii*) $\angle$AOP=$\angle$BOP

(*iii*) $\angle$APO=$\angle$BPO.

Proof :—As PA and PB are tangents at A and B and OA add OB are two radii.

$\therefore$ $\angle$OAP=$\angle$OPB=one right angle.

Now in right-angled triangles AOP and BOP

$\because$ $\begin{cases} \text{hypotenuse OP=hypotenuse OP} \\ \text{OA=OB} \qquad\qquad\qquad \text{(Radii)} \end{cases}$

$\therefore$ $\triangle$s are congruent.

Hence (*i*) AP=BP

(*ii*) $\angle$AOP=$\angle$BOP

(*iii*) $\angle$OPA=$\angle$OPB.

Q. E. D.

Exercises.

1. *Prove that if a circle be drawn as to touch all the sides of a quadrilateral, the sum of one pair of opposite sides is equal to the sum of the other pair.*

(Bombay and London).

2. Enunciate and prove the converse of Ex. 1.

Hint.—If the sum of one pair of opposite sides of a quadrilateral (XYZP) = the sum of the other pair XY + PZ = PX + ZY) show that a circle may be inscribed in the quad.

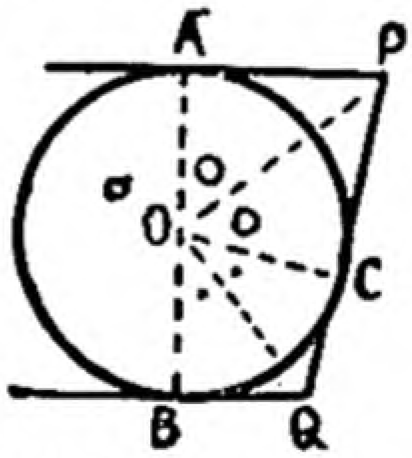
Draw a circle touching XY, YZ and ZP. If it does **not** touch PX, draw XO touching the ⊙.

∵ XY + PZ = XP + YZ (Given)
and XY + OZ = XO + YZ (Exercise 1)
∴ PZ − OZ = XP − XO.
Or PO = XP − XO, which is impossible, etc.

3. *A* ||^{m} *circumscribed about a* ⊙ *is a rhombus.*

4. *The perimeter of the triangle formed by two fixed tangents and the segment of a variable tangent intercepted between them is constant.*

5. A tangent to a circle cuts two parallel tangents in the points P and Q; prove that PQ subtends a rt. angle at the centre.



Hint.—AP and BQ || tangents meet a third tangent at **C.** Then ∠POQ is a rt. ∠ (O being the centre).

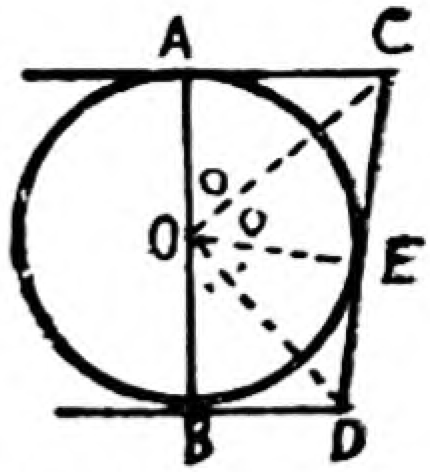
∴ tangents AP and BQ are || .

∴ AOB is a st. line. Again ∠COP = ∠AOP and ∠COQ = ∠BOQ

∴ ∠POR = $\frac{1}{2}$ st. ∠AOB = a rt. ∠.

6. *AB is a diameter of a circle with centre O and radius r, the tangents at A and B meet any other tangent at C and D. Prove that AC.BD = r^2.*

(Punjab, 1929)



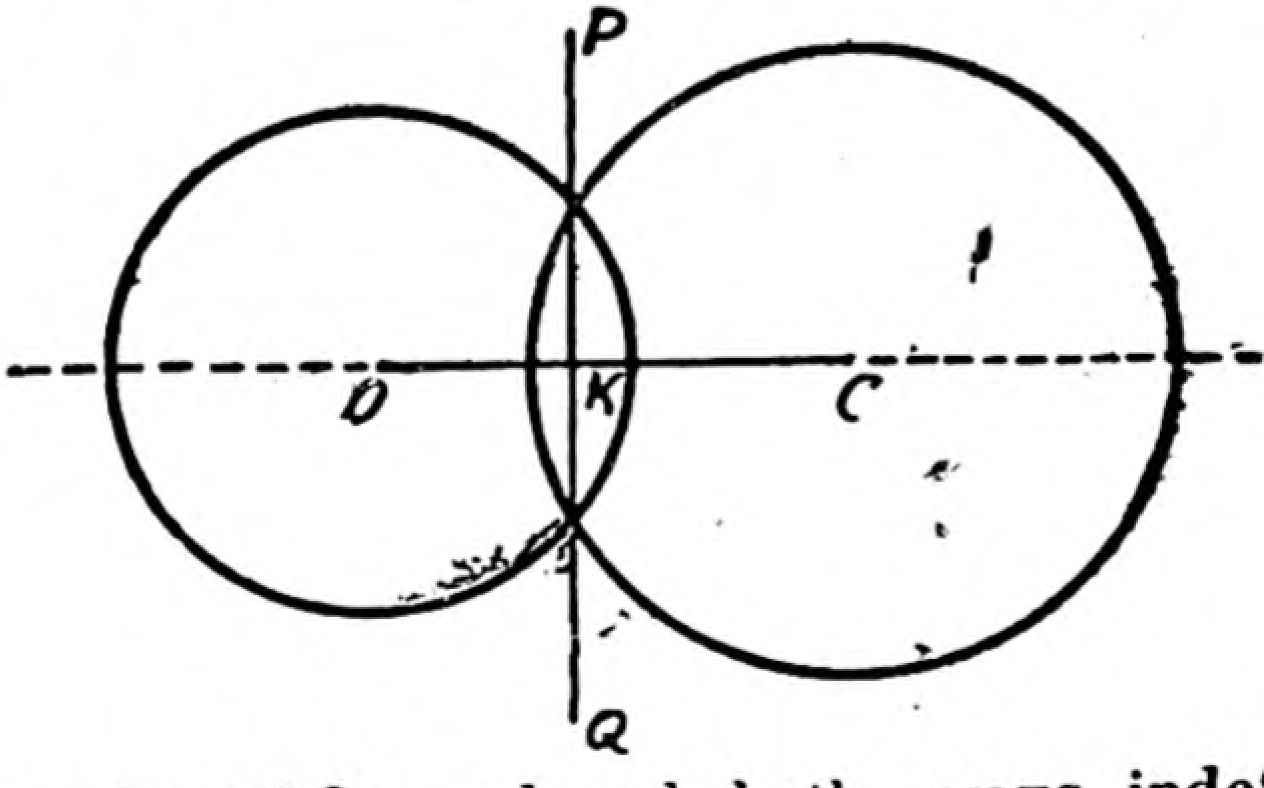
Hint.—The point of contact of third tangent is E Join OC, OE, OD. $\angle COD$ is a rt. $\angle$ (last Ex.) also $OE \perp CD$. $\therefore$ CE. $DE = OE^2$ but $CE = AC$ and $DE = BD$ $\therefore$ AC. $BD = \text{radius } OE^2$.

7. Two tangents are drawn to a circle from a fixed point and meet a third tangent. Prove that the part of the third tangent that they intercept subtends a constant angle at the centre of the circle.

(Calcutta, 1923).

Hint.—Angle$=\frac{1}{2}$ the supplement of the $\angle$ between the tangents from the fixed point.

Def.—Draw two circles cutting each other as in the figure. The points P and Q where these circles meet are called the points of intersection and the line PQ, *the chord of intersection or the common chord.*



The line PQ produced both ways indefinitely is known as the *secant of intersection* or *the common secant.*

The line joining the centres O and C of the two circles is called **the line of centres.**

Produce OC both ways indefinitely. The whole figure is obviously symmetrical about this line, as each of the two circles is symmetrical about any diameter. If the figure is folded about this line, one half will exactly coincide with the other half and the point P will fall on the point Q. Thus P and Q are the corresponding points and each is the image of the other in the line OC. Therefore PQ is at right angles to OC and is bisected by it at the point of intersection K.

Now suppose that the bigger circle slides slowly to the right. During this operation the points P and Q approach each other. Since in all positions PQ is to be bisected by OC, both the points P and Q will arrive at the line simulatneously and ultimately coalesce into one point as in Fig. 2.

In this position circles are said to be touching each other *externally* at the point P. The common secant becomes the common *tangent to the circles* at P.

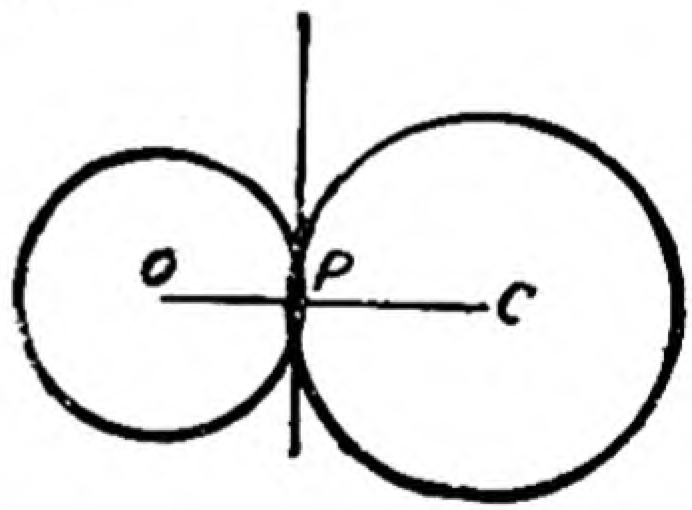


Fig. 2.

In the same way if the bigger circle moves slowly to the left, its centre remaining always on the line OC, the points P and Q will approach each other and ultimately coalesce into one single point at P. (Fig. 3.) The common secant in the limiting position becomes the common tangent to the circles at P and the circles are said to be touching each other *internally*.

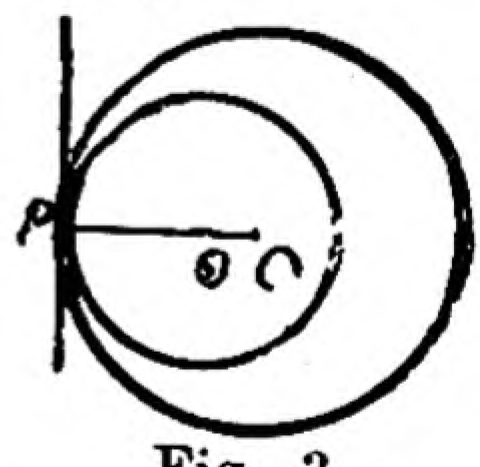


Fig. 3.

When two circles intersect each other at the point of intersection each has a tangent, separate and distinct. These tangents are always inclined to each other at some angle called the *angle of intersection* of two circles.

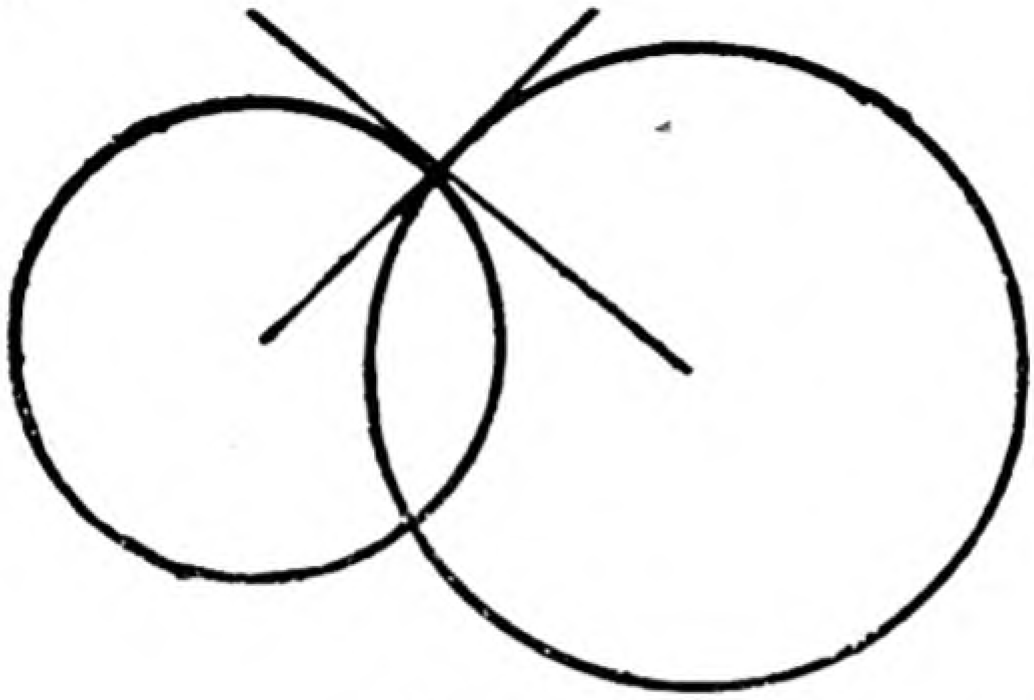


Fig. 4.

When the angle between these tangents is a right angle, the intersection is called an **orthogonal** as in

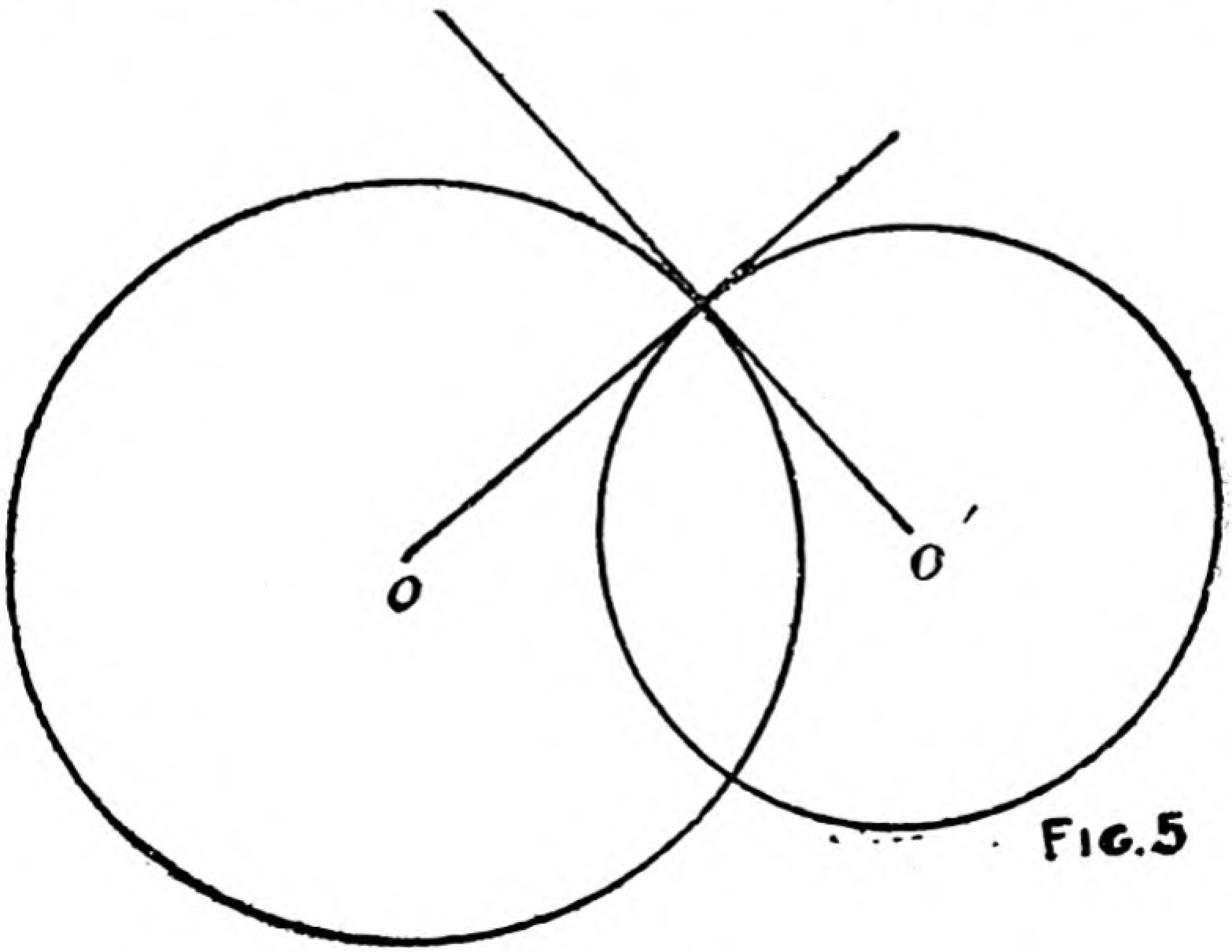


Fig. 5.

Fig. 5. In this case tangent to one passes through the centre of the other.

Proposition 69 (*Theorem*)

If two circles touch, the point of contact lies on the st. line joining the centres.

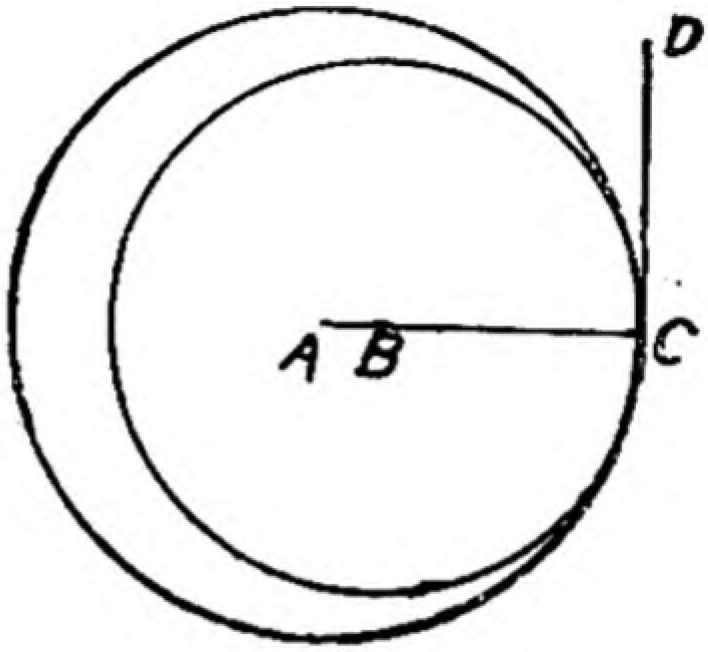


Fig 1.

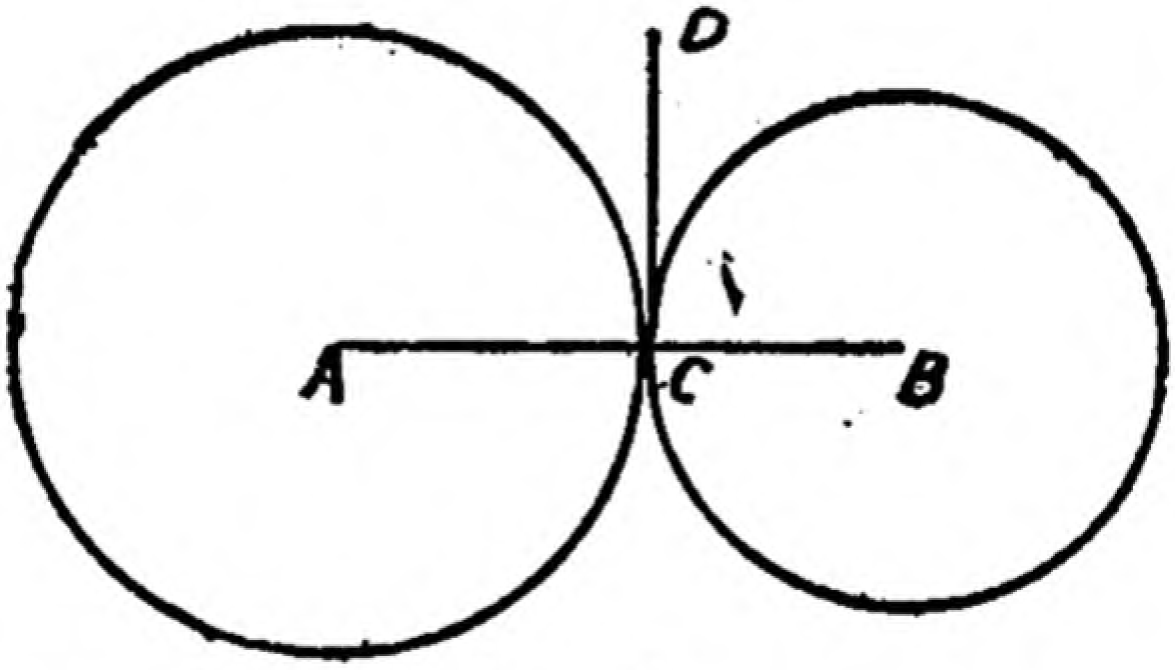


Fig. 2.

Given :—Two circles whose centres are A and B and touch at the point C.

Required :—To prove that A,B,C are in one st. line

Construction :—Suppose CD to be the common tangent to the circles. Join AC, BC.

Proof :—$\because$ CD is a tangent to a circle and AC is the radius to the $\odot$ through the point of contact.

$\therefore \angle ACD$ is a rt. angle.

Similarly $\angle BCD$ is a rt. angle.

Now in fig. 1. $\angle ACD = \angle BCD$ (rt. $\angle$s.)

And $\because$ these equal angles are on the same side of the common arm CD.

$\therefore$ CD falls along CA.

So the points A, C, B, are in one st. line.

In fig. 2 the adjacent angles ACD, BCD are together equal to two rt. angles.

$\therefore$ AC and BC are in one and the same st. line. So the points A, C, B, are in one st. line.

Hence if two circles touch (internally or externally,) the point of contact lies on their line of centres.

Cor. 1. If two circles touch one another externally, the distance between their centres is equal to the sum of their radii.

Cor. 2. If two circles touch one another internally the distance between their centres is equal to the difference of their radii.

Note.—This proposition may also be enunciated thus :—

If two circles touch each other, their centres and the point of contact are collinear.

Exercises.

1. If two circles meet at their line of centres, they must touch each other.

2. One of the three common tangents of two circles which touch one another externally bisects the other two.

3. If two circles touch each other internally or externally and two diameters are drawn ‖ to each other the point of contact and an extremity of each diameter lie on a st. line.

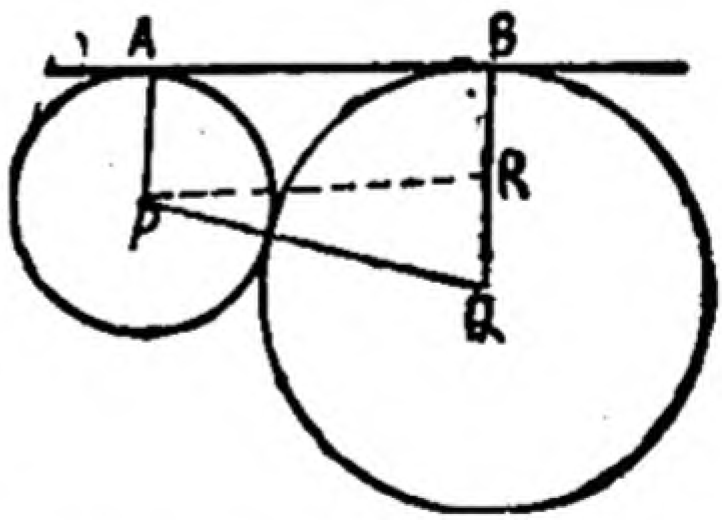
4. *Find the locus of centres of circles of radius r which touch a given st line.*

5. *Find the locus of centres of circles of radius r which touch externally a fixed circle of radius R.*

6. *What is the locus of centres of circles of radius r which touch internally a fixed circle of radius R?*

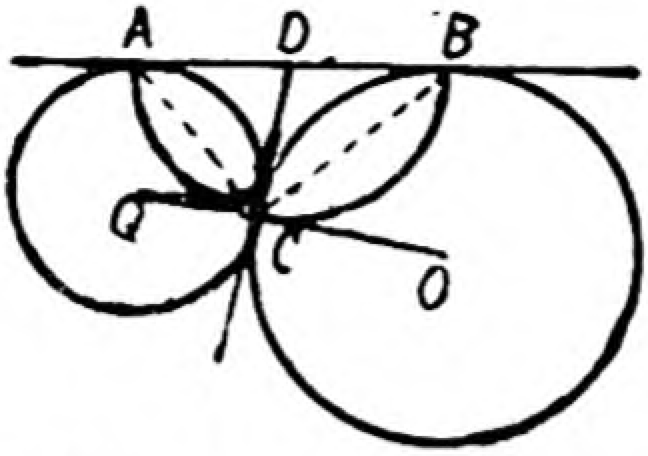
7. *Find the locus of the centres of ⊙s touching two equal ⊙s which touch one another externally.*

8. *The direct common tangent to two circles, touching externally, is a mean proportional between their diameters.*



Hint.—$PR^2 = PQ^2 - QR^2 = (PQ+QR)(PQ-QR)$
$= 2R.2r = D.d.$

9. *A and B are the points of contact of a common tangent to two circles which touch externally at C; prove that the line of centres of the circles is a tangent to the circle described on AB as diameter.* (Punjab, F. A.)

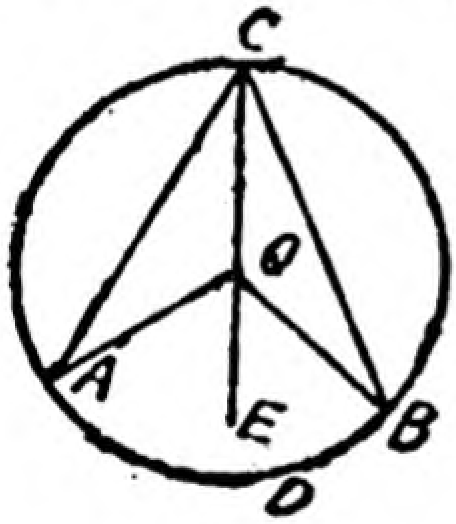


Hint —Draw CD, the common tangent at C to the circles meeting AB in D. Now BD=DC=AD and $\angle$OCD is a rt. $\angle$, hence the result.

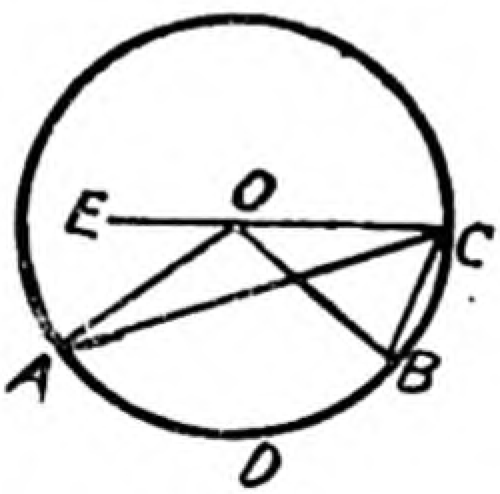
10. *Three ⊙s touch one another two by two, externally at P, Q and R. Show that the tangents pass through the same point.*

Proposition 70 (*Theorem*)

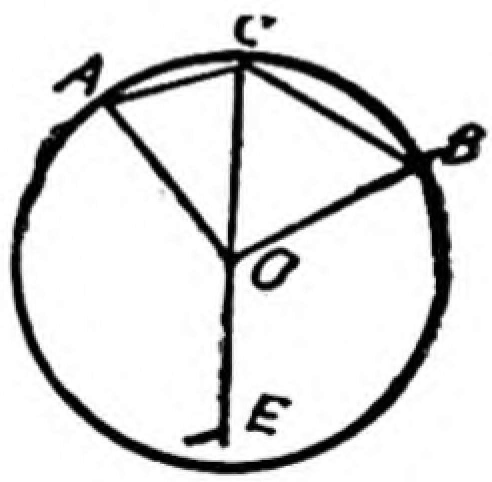
The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining point of the circumference.



(1)



(2)



(3)

Given :—The arc ADB of ⊙ ACB subtends angle AOB at the centre O, and angle ACB at C any point on the remaing part of the circumference.

Required :—To prove that $\angle$AOB=2$\angle$ACB.

Construction :—Join CO and produce it to E.

Proof :—∵ Radius OA=radius OC.

∴ $\angle$QCA=$\angle$OAC.

Also in $\triangle AOC$, ext. $\angle AOE = \angle OCA + \angle OAC$.

$= 2\angle OCA$

Similarly $\angle BOE = 2\angle OCB$.

$\therefore$ Sum (as in Fig. 13) or difference (as in Fig. 2) of $\angle$s $\angle EOA$, EOB = twice the sum or difference of $\angle OCA$ and $\angle OCB$.

$\therefore$ $\angle AOB = 2\angle ACB$.

Q. E. D.

Cor. 1.—In equal circles (or in the same ⊙) arcs which subtend equal angles on the circumferences (or circumference) are equal.

Cor. 2.—In equal circles (or in the same ⊙) angles at the circumferences (or circumference) which stand on equal arcs are equal.

Exercises.

1. Parallel chords of a ⊙ cut off equal arcs.

2. *A trapezium inscribed in a ⊙ is an isosceles trapezium.*

3. AB and CD are equal chords, prove that AC ∥ BD.

4. The internal bisector of the vertical angle of a △ inscribed in a ⊙ meets the circumcircle again in a point equidistant from the base ends.

5. Two chords of a circle intersect *within* the circle ; show that the angle between them is equal to the angle subtended by the *sum* of the intercepted arcs at the circumference or one half of the angle at the centre subtended by the sum of the intercepted arcs.

Hint .—Chord AB, CD, intersect internally at O. Draw BE ∥ CD ∴ arc BC=arc DE ∴ arc ADE=arc AD +arc DE=arc AD+arc BC. Now ∠AOD=∠ABE which stands on arc ADE.

(*ii*) Evidently ∠AOD=$\frac{1}{2}$ the angle at the *centre* standing on the arc=sum of intercepted arcs.

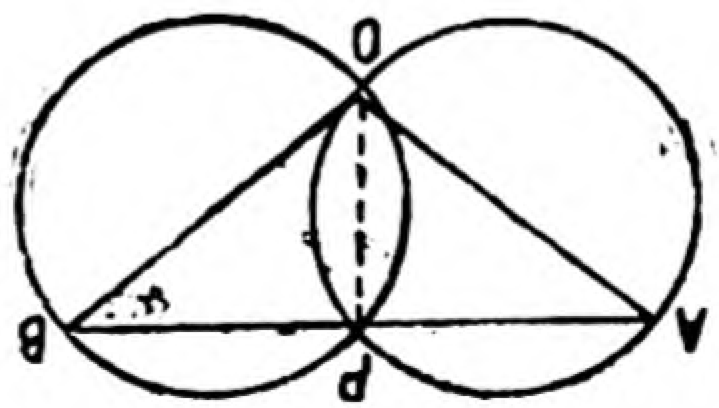
6. If two chords of a circle intersect *outside* the circle, the angle between them is equal to the angle subtended by the *difference* of the intercepted arcs at the circumference or half of the angle subtended by the difference of the intercepted arcs at the centre.

Hint .— Chords AB, DC, intersect *externally* at O. Draw BE ∥ CD ∴ arc BC=arc DE ∴ arc AE=arc AD —arc BC. Now ∠AOD=∠ABE on arc AE.

(*ii*) Also ∠AOD=∠ABE ($\frac{1}{2}$ the angle at the centre)

7. *The internal bisector of an angle at the circumference of a circle bisects the arc on which it stands and the external bisector bisects the arc of the segment which contains the angle.*

8. Two equal ⊙s intersect in P and Q. Any st. line through P meets the ⊙s in A and B. Show that QA=QB.

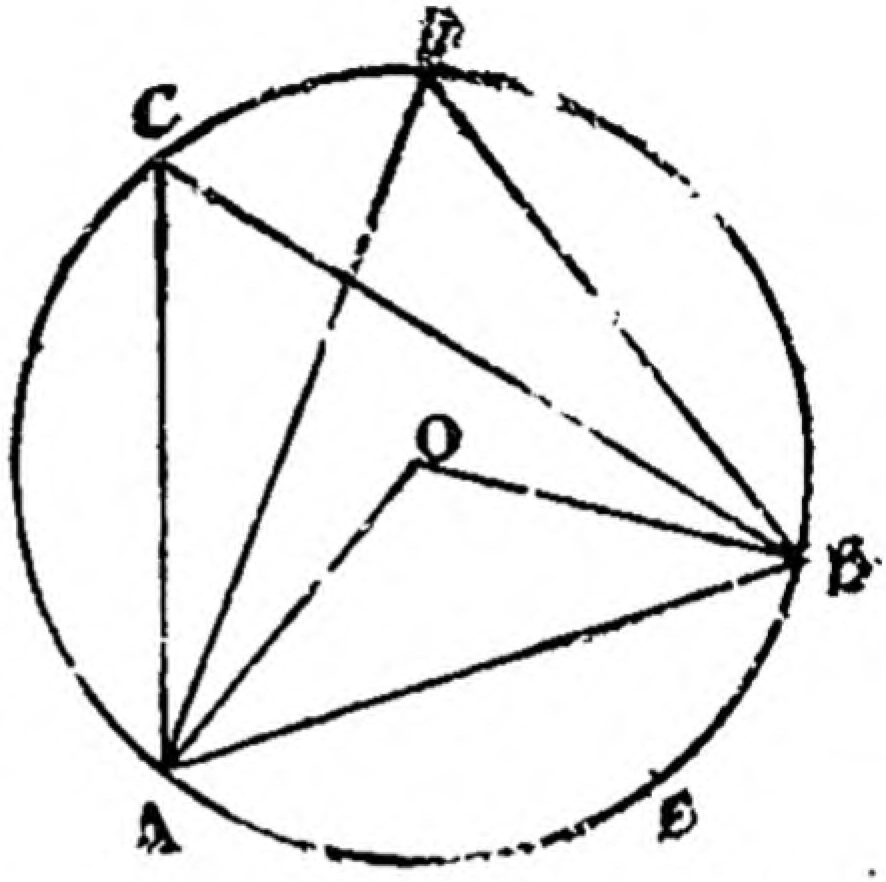


Def.—Four or more points are said to be **concyclic** if a circle can be drawn through them.

Proposition 71. (*Theorem*).

Angles in the same segment of a circle are equal.

Given :—Angles ACB and ADB are in the same segment of a circle with centre O.



Required :— To prove that $\angle ACB = \angle ADB$.

Construction :— Join OA, OB.

Proof :—$\because$ Arc AEB subtends $\angle AOB$ at the centre and $\angle ACB$ at a point on the remaining part of the circumference.

$\therefore \angle ACB = \frac{1}{2} \angle AOB$.
Similarly $\angle ADB = \frac{1}{2} \angle AOB$
$\therefore \angle ACB = \angle ADB$.

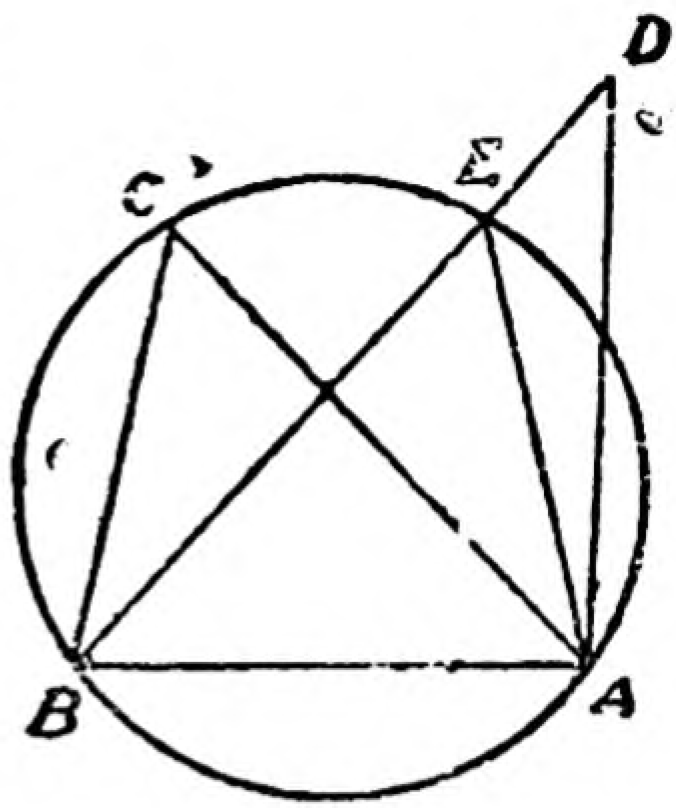
Q. E. D.

Converse theorem :

If the line joining two points subtend equal angles at two other points on the same side of it, the four points lie on a circle.

Given :—The line joining A and B subtends equal angles ACB and ADB at C and D on the same side of AB.

Required :—To prove that A, B, C, D lie on a circle.



Construction :—Suppose, a ⊙ is drawn through A, B, C

If this circle does not pass through D, let it cut BD (or BD produced) at E.

Join AE.

Proof :—Angles BCA and BEA are in the same segment of a ⊙.

$\therefore \angle BCA = \angle BEA$

But $\angle BCA = \angle BDA$ (Given)

$\therefore \angle BDA = \angle BEA$.

That is, an ext. $\angle$ of $\triangle$ ADE = an int. opp. $\angle$ which is impossible.

Hence circle ABC must pass through D.

$\therefore$ A, B, C and D lie on a ⊙.

Q. E. D.

Exercises

1. The locus of a point on one side of a given straight line at which that line subtends a constant angle is an arc of which that line is the chord.

2. *The base and vertical angle of a triangle being given, the locus of the vertex is an arc of a circle.*

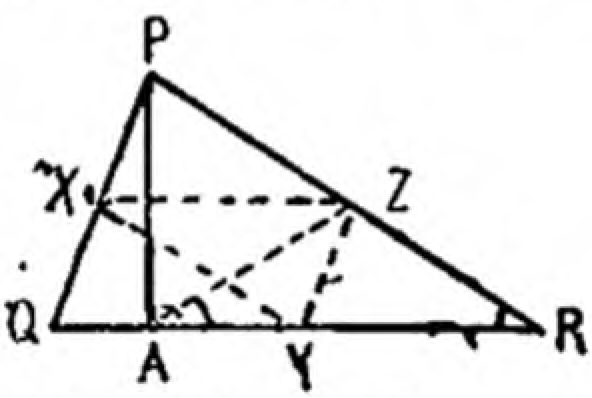
3. If a straight line subtends equal angles at any number of points on the same side of it, all these points are concyclic.

4. PQ and SR are two chords intersecting at O. Show that $\triangle$s POS and QOR are equiangular.

5. In an acute-angled triangle perpendiculars are drawn from two of the angular points on the opposite sides ; prove that the feet of the perpendiculars and these two angular points are concyclic.

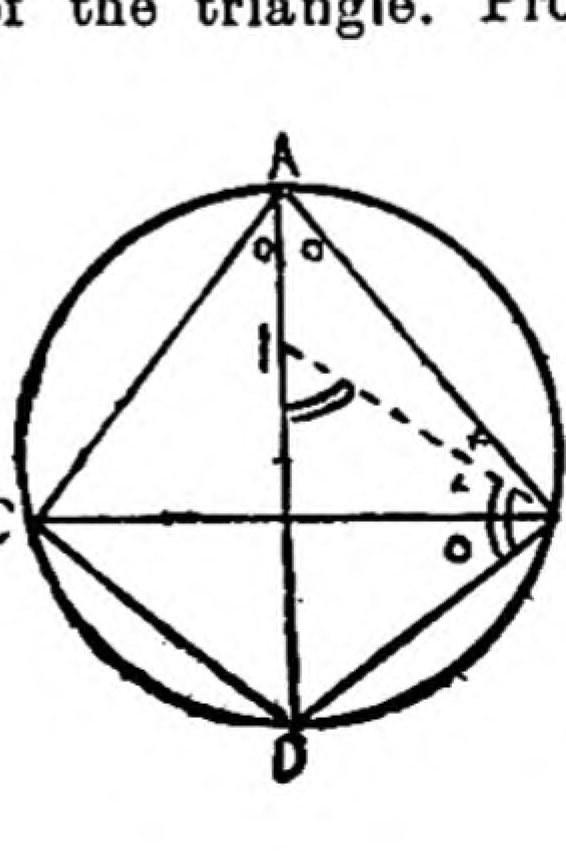
6. *The straight lines, which bisect the vertical angles of all triangles on the same base and on the same side of it, and having equal vertical angles all intersect at the same point.*

7. *Prove that the middle points of the sides of a triangle and the foot of the perpendicular drawn from the vertex to the opposite side lie on a circle.*



Hint :—In $\triangle PQR$, X, Y, Z are the mid-points of PQ, QR and PR and $PA \perp QR$; join XY, XZ, ZA. $ZA = ZR$ $\therefore$ $\angle ZAR = \angle ZRA = \angle YXZ$ ($\because$ YRZX is $\|^{m}$) $\therefore$ $\angle ZAY = \angle YXZ$, $\therefore$ A, Y, Z, X lie on a circle.

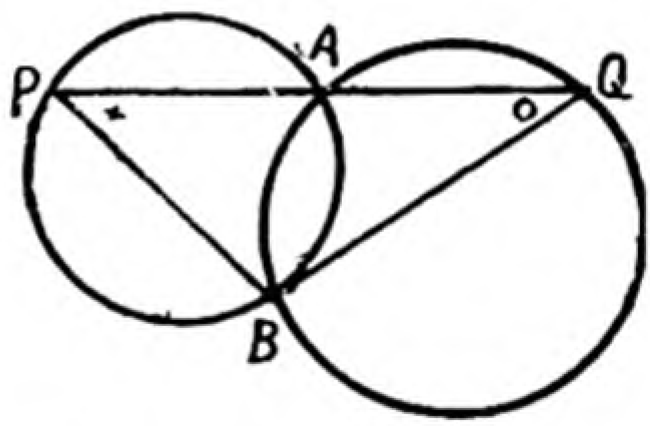
8. **The bisector of the vertical angle A of a triangle ABC meets the circumcircle again at D, I is the in-centre of the triangle. Prove that $DI = DB$ CD.**



Hint.—Since I is the in-centre $\therefore \angle BAI = \angle CAI \therefore$ $DB = DC$. $\because$ I is she in-centre $\therefore$ $\angle IBA = \angle IBC$ and $\angle IAB = \angle DAC \therefore \angle BID = \angle IBA + \angle IAB$

$$= \angle IBC + \angle DAC$$
$$= \angle IBC + \angle DCC = \angle IBD$$
$$\therefore DB = DI \therefore DI = BD = CD.$$

9. Two circles cut in A, B ; a line through A cuts the circles again in P, Q. Prove that $\angle PBQ$ is constant.



Hint.—As the line PQ varies, $\angle APB$ remains constant ($\angle$s in the same segment).

Similarly, $\angle AQB$ remains constant.

But $\angle APB + \angle AQB + \angle PBQ = 2$ rt. $\angle$s.

$\therefore$ $\angle PBQ$ is constant

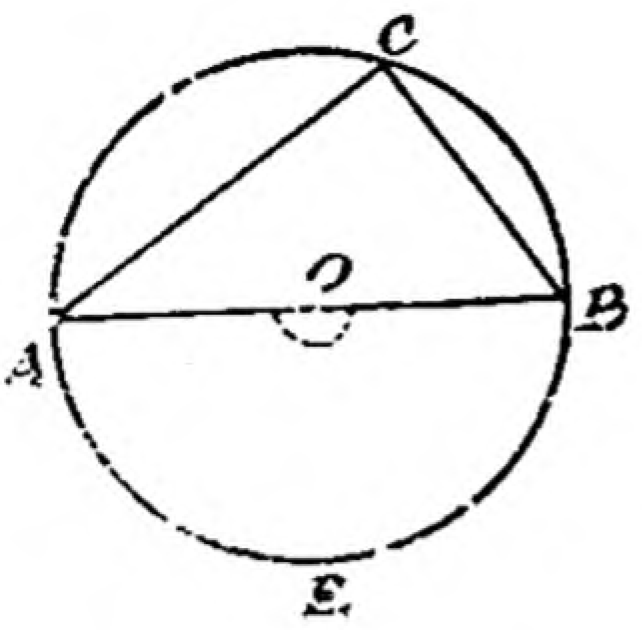
10. Given three points on the circumference of a circle, find a fourth point *without finding the centre.*

Proposition 72. (*Theorem*)

(*i*) *The angle in a semi-circle is a right angle.*

Given :—The segment ACB is as semi-circle.

Required :—To prove that $\angle ACD$ is a rt. angle.



(i)

Proof :—∵ The arc AEB subtends $\angle AOB$ at the centre and $\angle ACB$ at the circumference.

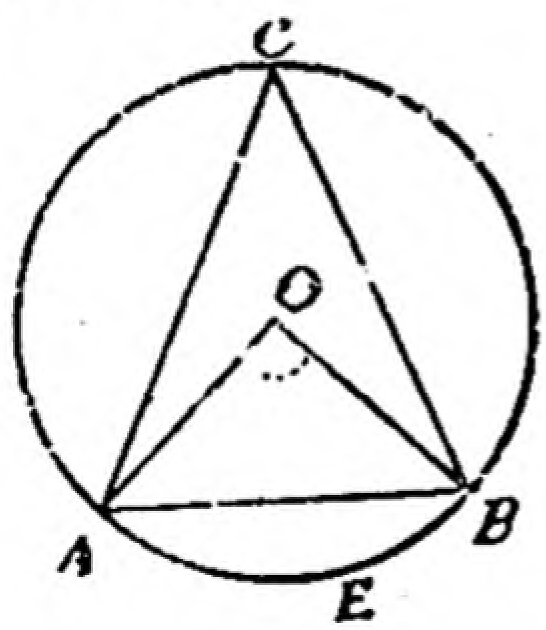
$\therefore \angle ACB = \frac{1}{2} \angle AOB$.

But $\angle AOB = 2$ rt. $\angle$s.

$\therefore \angle ACB =$ half of 2 rt. $\angle$s $=$ a rt $\angle$.

Q E. D.

(ii) *The angle in a segment greater than a semi-circle is less than a right angle.*



(2)

Given :—Segment ACB is $>$ a semi-circle.

Required :—To prove that $\angle ACB$ is less than a rt. $\angle$.

Construction :—Joint the centre O with A and B.

Proof :—∵ The arc AEB subtends ∠AOB at the centre and ∠ACB at the circumference.

∵ $\angle ACB = \frac{1}{2} \angle AOB$.

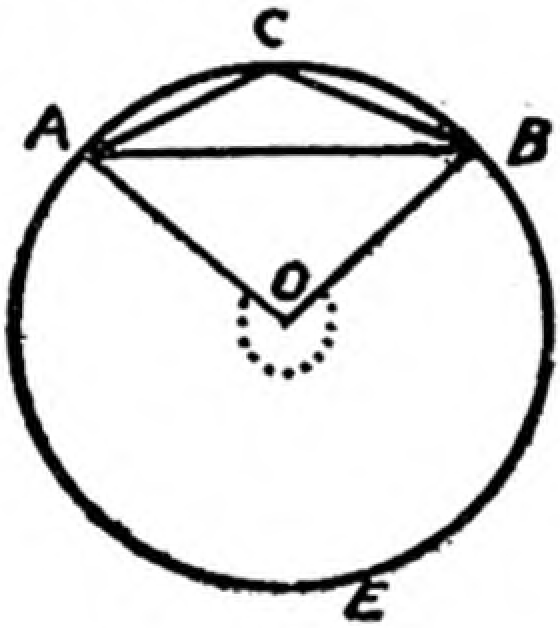
But ∠AOB, being an angle of a △ is less than 2 rt. ∠s.

∴ ∠ACB is less than half of two rt. ∠s. that is,

∴ ∠ACB is less than one rt. angle.

Q. E. D.

(*iii*) *The angle in a segment less than a semi-circle is greater than a right angle.*



(3)

Given :—Segment ACB is less than a semi-circle.

Required :—To prove that ∠ACB is greater than a rt. ∠.

Construction :—Join the centre O with A and B.

Proof :—The arc AEB subtends ∠AOB at the centre and ∠ACB at the circumference.

∴ $\angle ACB = \frac{1}{2} \angle AOB$.

But ∠AOB is greater than two rt. ∠s, being the difference of 4 rt. ∠s. and an interior ∠ of a △.

∴ ∠ACB is greater than a rt. ∠.

Q. E. D.

Note :—Proposition 72 can also be enunciated thus :—

An angle in a semi-circle is a right angle, the angle in a major segment of the circle is an acute angle and the angle in a minor segment is an obtuse angle.

Cor.—The circle described on the hypotenuse of a right-angled triangle, as diameter, passes through the right angle.

Exercises

1. The ⊙s described on the two sides of a △ as diameters intersect at the base.

2. *Prove that the circles described on the four sides of a rhombus as diameters pass through a fixed point.*

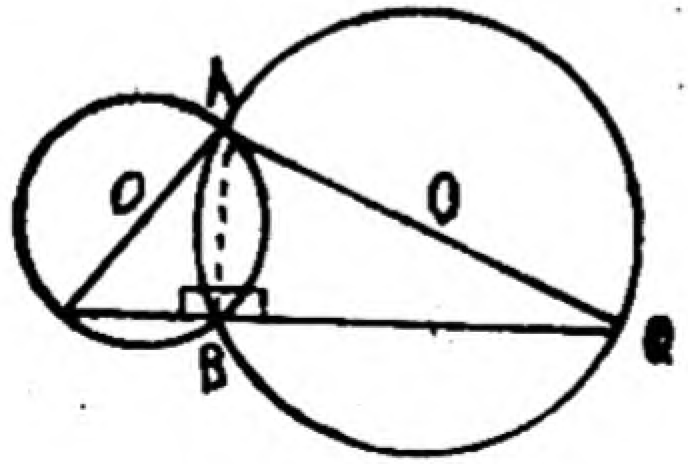
(*Punjab*, **1918**).

Hint.—The fixed point is the point of intersection of the diagonals.

3. If a circle is described on one of the equal sides of an isosceles triangle as diameter, prove that it passes through the mid-point of the base.

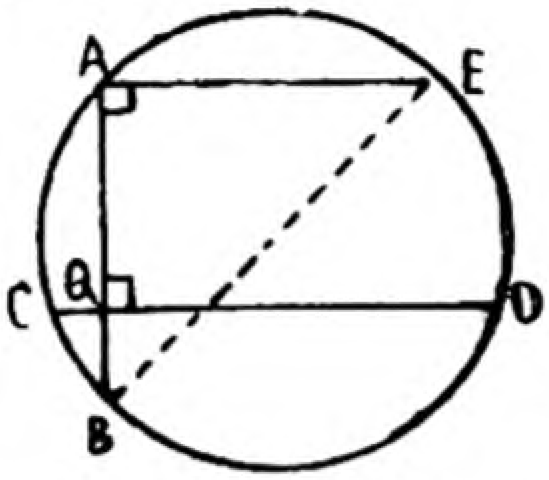
(**Allahabad, 1923**).

4. *If two circles cut one another and from one of the points of intersection two diameters are drawn, their extremities and the other point of intersection will be in one st. line.*



5. The mid-point of the hypotenuse of a right angled triangle is equidistant from three vertices.

6. AB and CD are chords of a circle at right angles at Q. Show that (*i*) arc AC+arc BD=a semi-circumference : (*ii*) $QA^2+QB^2+QC^2+QD^2$ is constant.

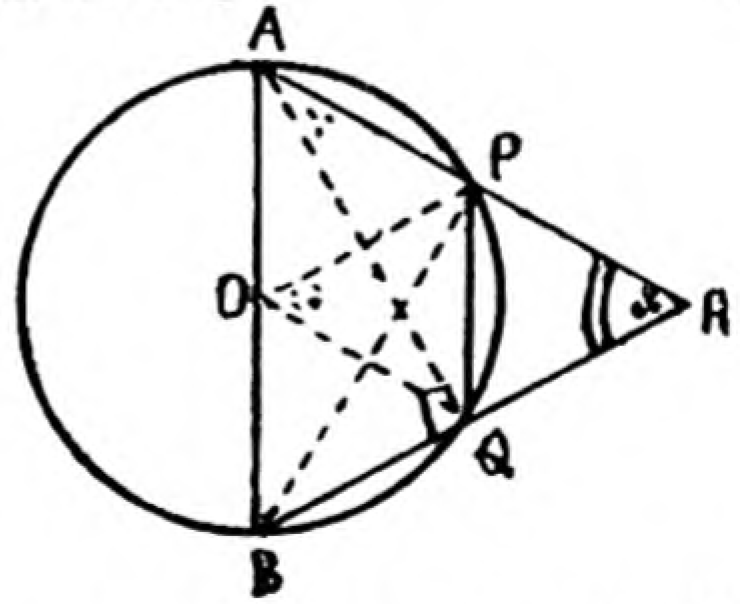


Hint :—(*i*) Draw AE ∥ CD. ∴ arc AC=arc DE. $\angle BAE=\angle BQD$=a rt. $\angle$ ∵ BE is a diameter ; arc AC+arc BD=arc DE+arc BD=arc BDE=semi-circumference. ∵ BE is a diameter.

(*ii*) $QA^2+QC^2+QB^2+QD^2=AC^2+BD^2=DE^2+BD^2$ ∵ DE=AC.

$=BE^2(\angle BDE=\text{a rt. }\angle)=(\text{diameter})^2$.

7. AB is a fixed diameter of a given circle and PQ is any chord whose length is equal to the radius ; AP, BQ are joined and produced to meet in R. Prove that the angle ARB is of constant magnitude whatever the position of PQ. (Punjab. 1916).



Hint :—Let O be the centre, join PO, OQ. △ OPQ is equilateral ∴ $\angle POQ=60°$

∴ $\angle RAQ=\frac{60°}{2}=30°$ and $\angle AQB=90°$

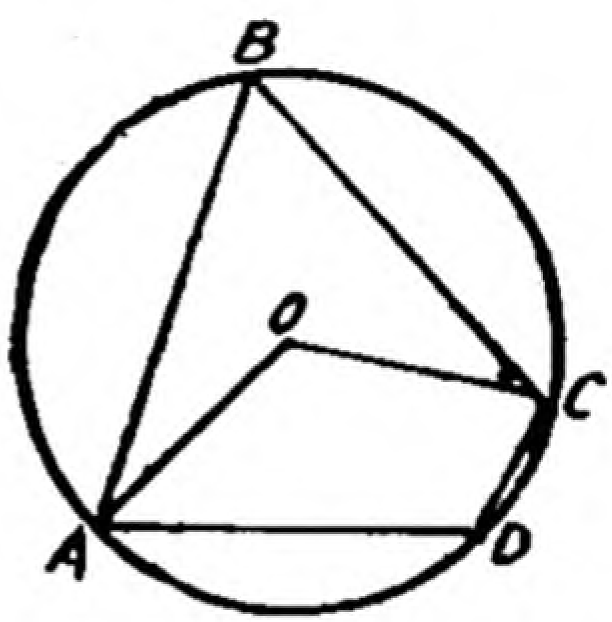
∴ $\angle R=90°-30°=60°$=constant.

8. Find the locus of the points of contact of tangents drawn from the same point to concentric circles.

Def.—A quadrilateral is **cyclic**, if its vertices lie on the circumference of a circle.

Proposition 73. (*Theorem*)

The opposite angles of a quadrilateral inscribed in a circle are supplementary.



Given :—ABCD is a quadrilateral inscribed in a ⊙ ABCD whose centre is O.

Required :—To prove that $\angle B + \angle D = 2$rt. $\angle$s.

Also $\angle A + \angle C + 2$rt. $\angle$s.

Construction :—Join OA, OC.

Proof :—$\angle B$ at the circumference$=\frac{1}{2}$ $\angle AOC$ at the centre (subtended by the same arc).

Also $\angle D$ at the circumference$=\frac{1}{2}$ reflex $\angle AOC$. at the centre (subtended by the same arc.)

$\therefore$ $\angle B + \angle D = \frac{1}{2}(\angle AOC + \text{reflex} \angle AOC)$.

But $\angle AOC +$ reflex $\angle AOC = 4$rt. $\angle$s. ($\angle$s round a pt.)

$\therefore \angle B + \angle D = \frac{1}{2}$ of 4 rt. $\angle s = 2$ rt. $\angle$s.

Similarly $\angle A + \angle C = 2$ rt. $\angle$s.

Q. E. D.

Converse of the above Theorem.

If the opposite angles of a quadrilateral are supplementary, it can be inscribed in a circle.

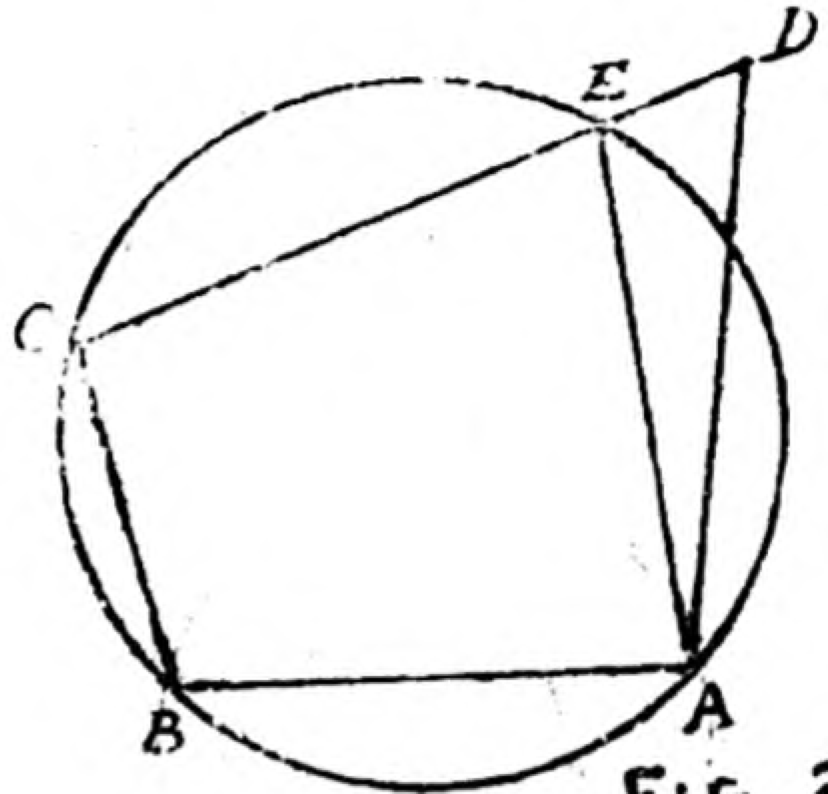


FIG. 2

Given :—In the quad. ABCD, the opposite angles. ABC, ADC are supplementary.

Repuired :—To prove that the quad. ABCD is cyclic.

Construction :—A circle can be described to pass through points A, B and C. If this circle does not pass through D, let it cut CD or CD produced in some point E. Join AE.

Proof :— $\because$ ABCE is a cyclic quad.

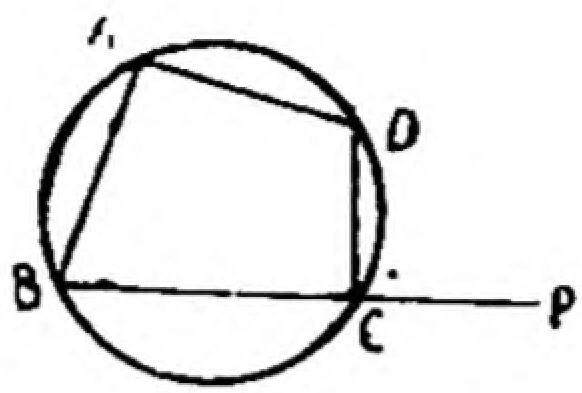
$\therefore \angle B + \angle AEC = 2$ rt. $\angle$s.

But $\angle ABC + \angle ADC = 2$ rt. $\angle$s. (Given)

$\therefore \angle B + \angle AEC = \angle ABC + \angle ADC.$

$\therefore \angle AEC = \angle ADC.$

That is, an ext. $\angle$ of $\triangle$ ADE = an int. opp $\angle$ which is impossible.



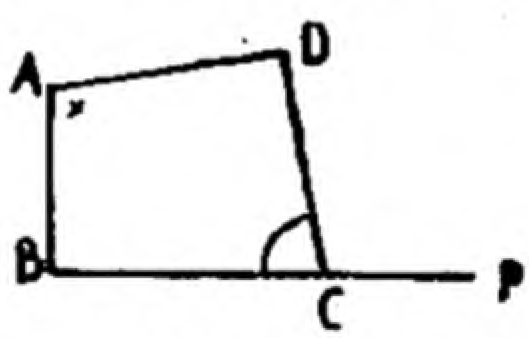
Hence the ⊙ through A, B, C must pass through D, *i. e.*, the quad. ABCD is cyclic.

Q. E. D.

Cor. 1. If a side of a cyclic quadrilateral be produced the exterior angle is equal to the interior opposite angle.

Cor 2. If one side of a quadrilateral be produced and the exterior angle so formed is equal to the interior opposite angle, the quadrilateral is cyclic.

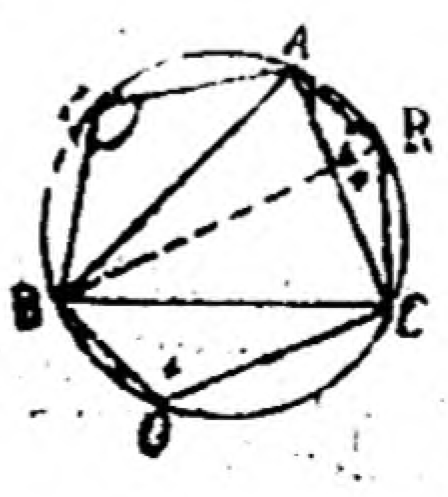
Cor 3. Every rectangle is a cyclic quadrilateral.



Exercises.

1. A ‖ᵐ inscribed in a ⊙ is a rect.

2. The sum of the angles in the exterior segments cut off by the sides of an inscribed triangle is equal to four right angles.



Hint :—$\angle P + \angle ARB = 2$ rt. $\angle$s
$\angle Q + \angle CRB = 2$ rt $\angle$s.

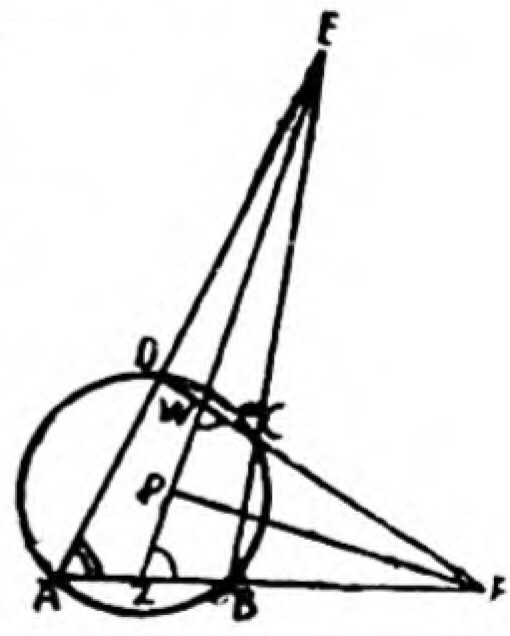
3. *Prove that the sum of the angles in the four segments exterior to a cyclic quadrilateral is equal to six right angles.* (*Punjab*, 1929)

4. In a cyclic hexagon two pairs of opposite sides are respectively parallel to each other; prove that the remaining pair of sides are also parallel.

5. Non-parallel sides of a cyclic trapezium are equal.

6. *Show that the bisectors of the angles formed by producing opposite pairs of sides of a cyclic quadrilateral are at right angles.* (*Punjab*, 1920)

Hint :—Sides AB, DC of a cyclic quad. intersect at F and BC and AD intersect at E. EP and FP bisectors of $\angle$s E and F intersect at P; then EPF is a rt $\angle$.



Produce EP to cut AB at Z, EP cuts CD at W.

$\angle ECW = \angle A$; $\angle CWZ = \angle ECW + \frac{1}{2}\angle E$

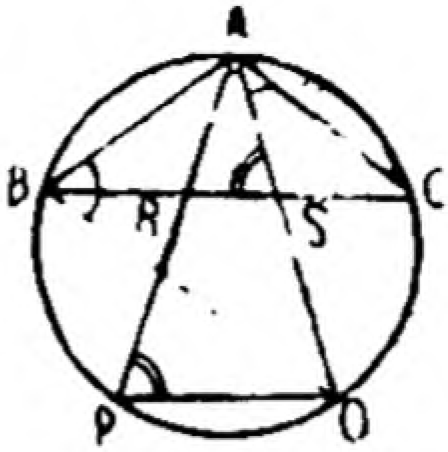
$= \angle A + \frac{1}{2}\angle E$ and $\angle BZW = \angle A + \frac{1}{2}\angle E$.

$\therefore \angle CWZ = \angle BZW$; $\angle WFP = \angle ZFP$

$\therefore$ $\angle$s at P are equal, hence rt $\angle$s.

7. AB, AC are equal chords of a $\odot$ and AP and AQ are any two other chords which cut BC in the points R and S respectively. Prove that P, Q, R, S, are concyclic. (Punjab, 1921)

Hint.—Join PQ. $\angle BSA = \angle SAC + \angle SCA$.
But $\angle SAC = \angle QBS$ and $\angle SCA = \angle ABS$ ($\because AB = AC$.)
$\therefore \angle BSA = \angle QBS + \angle ABS = \angle ABQ = \angle APQ$.
$\therefore$ P, Q, R, S are concyclic.



8. The quad. formed by the external bisectors of the angles of any quad. is cyclic.

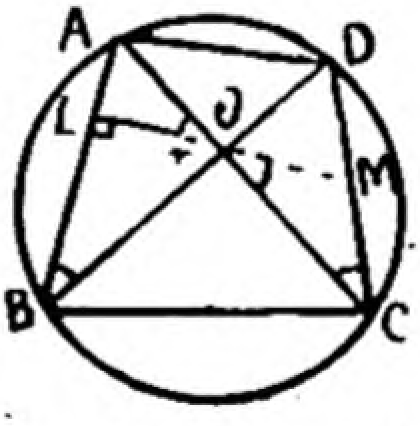
9. Four ⊙s touch one another, two by two, show that the pts. of contact are concyclic.

10. *If the diagonals of a cyclic quadrilateral are at right angles, the perpendicular from their intersection on any side, being produced, bisects the opposite side.*

(*Punjab*, 1932).

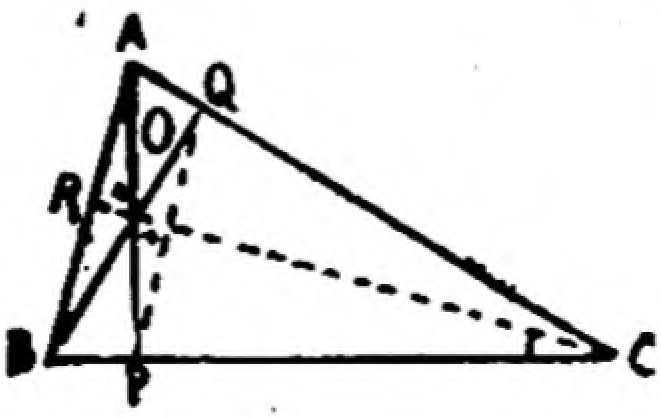
Hint.—AC, BD, cut at rt. ∠s at O ; OL ⊥ AB, then LO produced bisects CD at M.

∠COM = ∠AOL = complement of ∠LOB = ∠OBL or ∠ABD = ∠ACD or ∠OCM. ∴ OM = MC. Similarly OM = MD. ∴ M is the mid-point of the hypotenuse CD.



11. *The three altitudes of a triangle are concurrent.*

(Bombay, 1927).

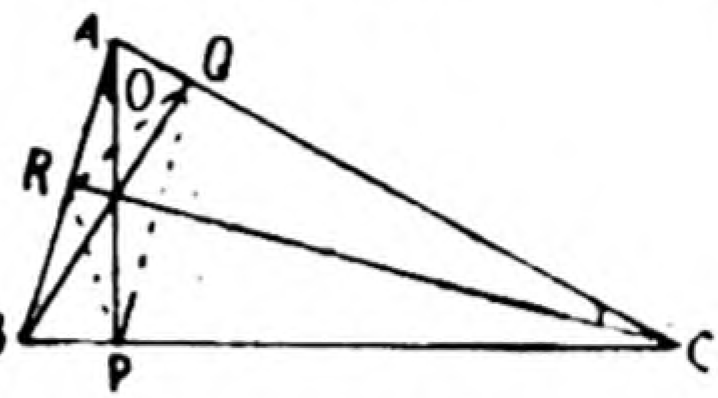


Hint.—In $\triangle ABC$, draw AP, BQ $\perp$ s BC, AC to intersect at O. Join CO and produce it to meet AB at R. Prove that CR $\perp$ AB. Join PQ.

Now $\angle BAP = \angle BQP$ or $\angle OQP$ ($\because$ AQPB is cyclic) $= \angle OCP$ ($\because$ OQCP is cyclic).

$\angle ROA = \angle POC$ $\therefore$ $\angle ARO = \angle OPC =$ a rt $\angle$.

12. *The angles of the pedal triangle are bisected by the altitudes of the original triangle.* (*Punjab*, 1932).

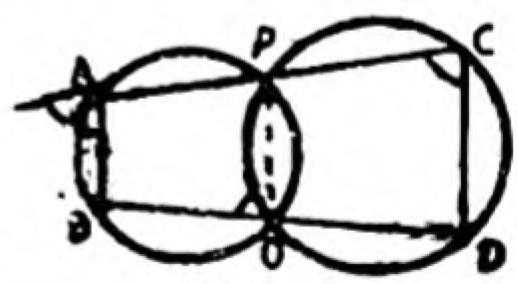


Hint.— $\angle OPR = \angle OBR$ $\because$ OPBR is cyclic.
$= \angle OCQ$ $\because$ QRBC is cyclic.
$= \angle OPQ$ $\because$ OPCQ is cyclic.
$\therefore$ AP bisects $\angle QPR$, etc.

13. *The sides of the pedal triangle meeting in a side of the original triangle make equal angles with it.*

Hint.—(In fig. of Ex. 11) $\angle APB = \angle APC$ rt $\angle$s.
$\angle OPR = \angle OPQ$ (Proved in Ex. 12).
$\therefore$ $\angle RPB = \angle QPC$.

14. *Through each of the points of intersection of two circles, straight lines are drawn cutting one circle in A and B and the other in C and D. Prove that AB $\parallel$ CD.*

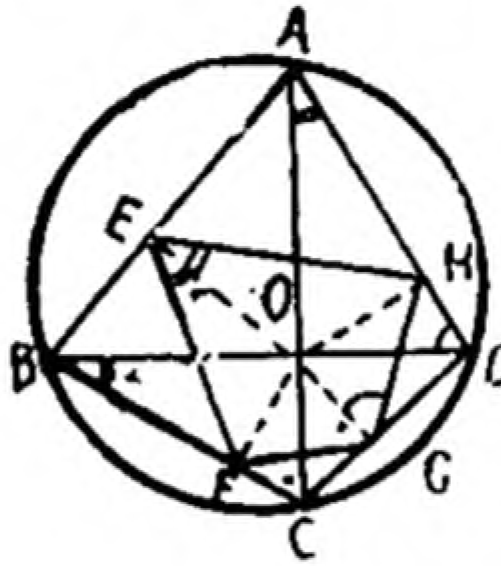


Hint.—Through P, APC and through Q, BQD are drawn ($\odot$s intersect at P, Q), produce CA to X. Now $\angle PCD = \angle PQB = \angle BAX$. $\therefore$ AB $\parallel$ CD.

15. If the diagonals of a quadrilateral are at right angles, then the feet of the perpendiculars from

the point of intersection of the diagonals on the sides are concyclic. (Bombay, 1922).

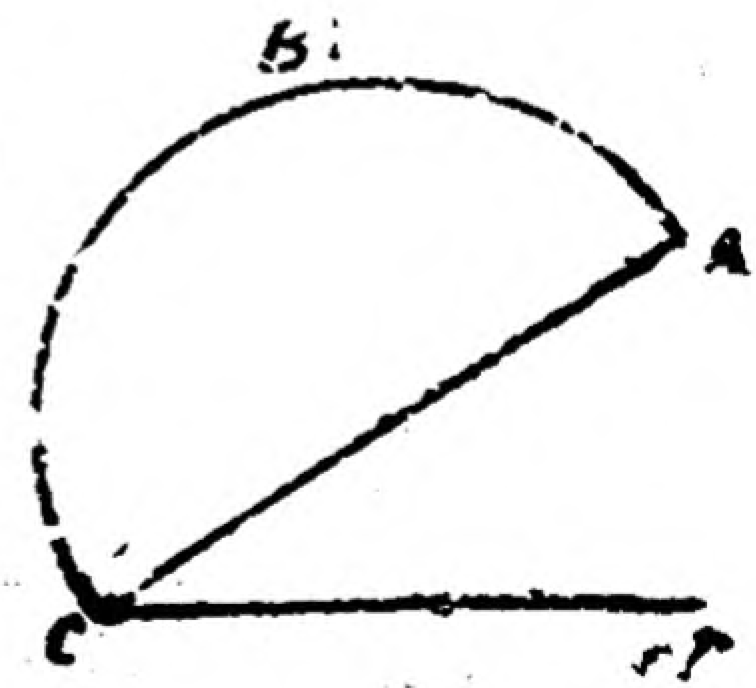
Hint.—Diagonals AC, BD, of a quad. intersect at rt. $\angle$s at O, from O, OE, OF, OG, OH are drawn $\perp$s to AB, BC, CD, AD. $\angle HGO = \angle HDO$ $\therefore$ HDGO is cyclic. $\angle OGF = \angle OCF$ $\therefore$ $\angle HGF = \angle HDO + \angle OCF$.



Similarly $\angle HEF = \angle OAH + \angle OBF$. Hence $\angle HGF + \angle HEF = \angle HDO + \angle OCF + \angle OAH + \angle OBF = 2$ rt. $\angle$s.
$\therefore$ H, G, F, E are concyclic.

16. *The feet of the $\perp$s from any pt. on the circum ⊙ of a $\triangle$ on its sides are collinear.*

Def.—If ABC be a segment of a circle on the chord AC, and ACD be an angle such that B, D are on opposite sides of AC, then the segment ABC is said to be **alternate** to the angle ACD.



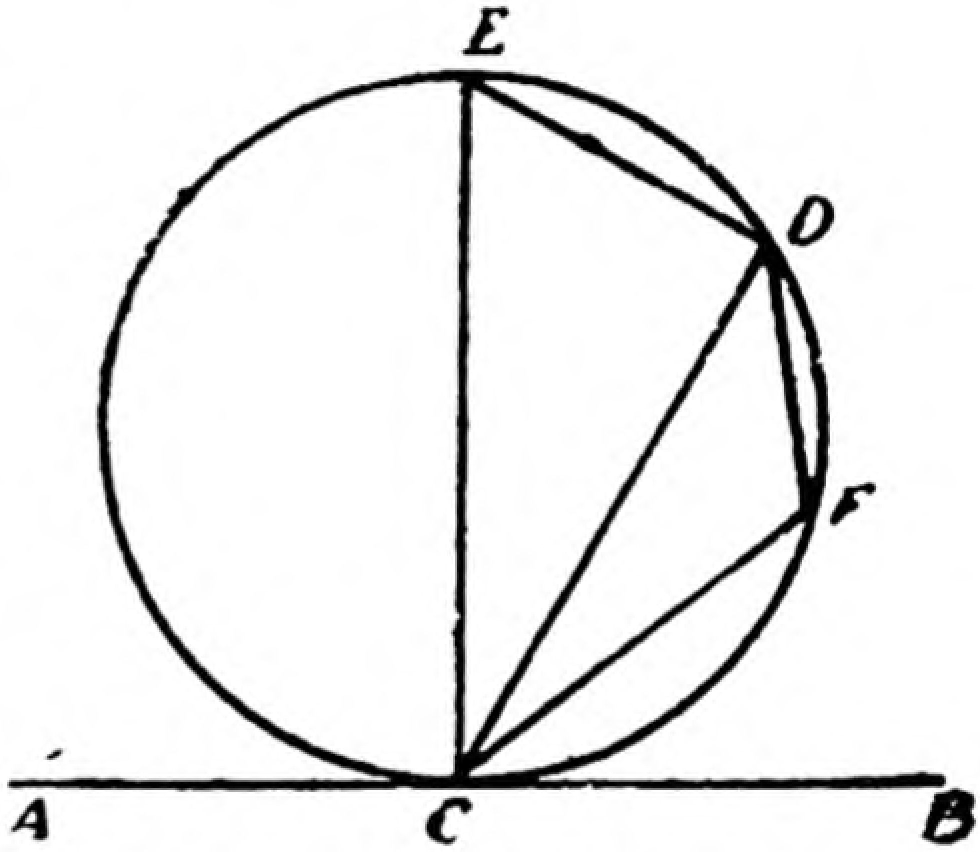
Proposition 74 (*Theorem*)

If a st. line touches a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

Given :—AB a tangent at C to the circle CEDF and CD a chord through C.

Required :—To prove that (*i*) $\angle$BCD = any angle in the alternate segment DEC.

(*ii*) $\angle$ACD = any angle in the alternate segment DFC.



Construction :—Through C draw CE the diameter of the circle. Take any point F on the arc DFC. Joint CF, DF, DE.

Proof :—(*i*) Since AB is a tangent and CE a diameter.

$\therefore \angle$ECB is a rt. $\angle$.

Again $\because \angle$EDC is a rt. $\angle$ (being in a semi-circle)

$\therefore$ in $\triangle$ECD, $\angle$CED + $\angle$DCE = 1 rt. $\angle$

Also $\angle$ECB = $\angle$BCD + $\angle$DCE = 1 rt. $\angle$.

$\therefore \angle$CED + $\angle$DCE = $\angle$BCD + $\angle$DCE

$\therefore \angle$BCD = $\angle$CED

= any $\angle$ in the alt. segment CED.

(*ii*) $\because$ CFDE is a cyclic quad.

$\therefore \angle CED + \angle CFD = 2$ rt. $\angle$s.

But $\angle BCD + \angle ACD = 2$ rt. $\angle$s. $\because$ DC stands on AB.

$\therefore \angle BCD + \angle ACD = \angle CED + \angle CFD$.

But $\angle BCD = \angle CED$. (Proved in (i))

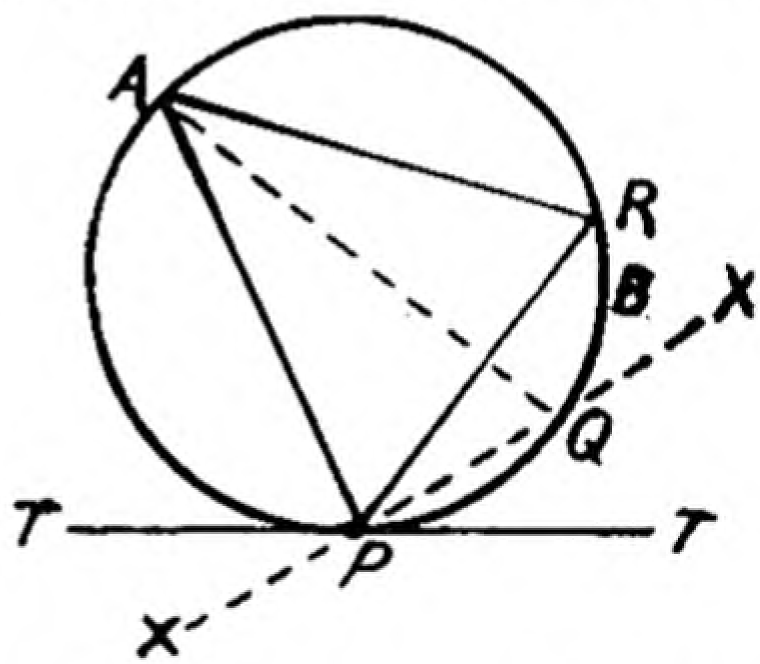
$\therefore \angle ACD = \angle CFD$

$=$ Any $\angle$ in the alt. segment CFD.

Q. E. D.

Alternate proof by the Method of Limits.

Given :—TPT′ a tangent touching the circle PARB at the point P, PR a chord through P. A and B two other points on the ⊙ce.



Required :—

To prove that

$\angle RPT' = \angle PAR$

and $\angle RPT = \angle RBP$.

Construction :—Draw XX′ any line through P meeting the circle again in Q. Join AQ.

Proof :—Whatever be the position of P and Q $\angle RAQ = \angle RPQ$.

Now let the line XX′ revolve round P so that Q approaches P and ultimately coincides with it. Then XX′ will coincide with the tangent TT′.

The $\angle RAQ$ becomes the $\angle RAP$.

and $\angle RPQ$ becomes the $\angle RBT'$

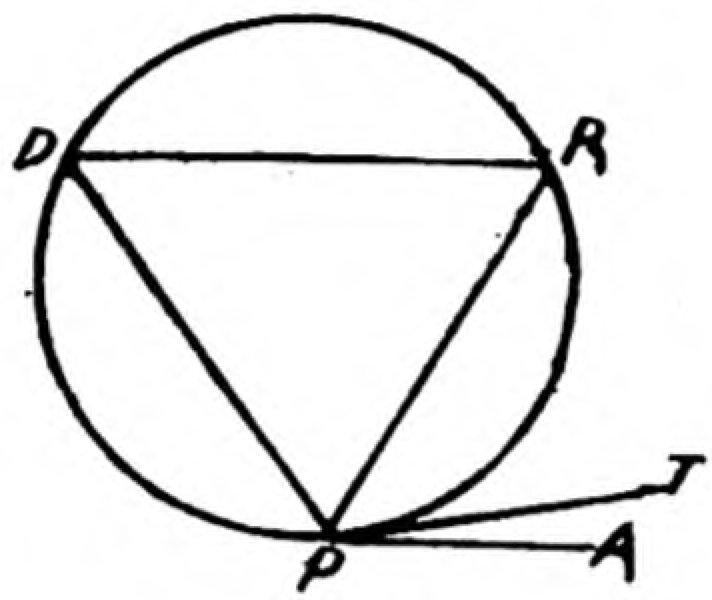
$\therefore \angle RAP = \angle RPT'$.

Similarly we can prove that $\angle RPT = \angle RBP$.

Q. E. D.

Converse of the Above Theorem

If through an extremity of a chord of a circle a straight line be drawn such that either of the angles it makes with the chord is equal to the angle in the alternate segment, the straight line is a tangent to the circle.



Given :—PT is drawn through P, the extremity of the chord PR such that $\angle RPT = \angle RDP$ in the alternate segment.

Required :—To prove that PT is a tangent to the $\odot$.

Construction :—If PT is not a tangent, then draw another line PA as tangent at P.

Proof :—$\because$ PA is a tangent and PR a chord of the circle through the pt. of contact P,

$\therefore \angle RPA = \angle RDP$.

But $\angle RPT = \angle RDP$ (Given)

$\therefore \angle RPT = \angle RPA$

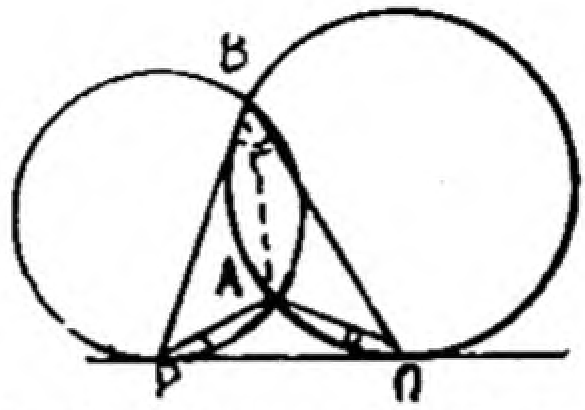
This is possible only when PT and PA coincide.

Hence PT is a tangent to the circle.

Q. E. D.

Exercises.

1. *If two circles intersect, the angles subtended by a common tangent at the points of intersection are supplementary.*



Hint:—Two circles intersect at A and B, PQ a common tangent nearer A; prove that $\angle PBQ+\angle PAQ=$ 2rt. $\angle$s. Join AB, AP, AQ BQ, and BP.

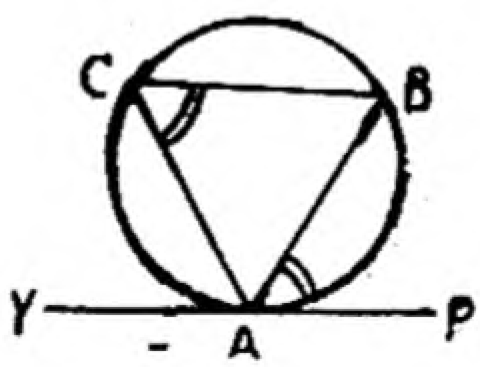
$\angle APQ=\angle ABP$, $\angle AQP=\angle ABQ$ $\therefore$ $\angle APQ+\angle AQP=\angle ABP+\angle ABQ=\angle PBQ$. Add $\angle$ PAQ.

$\therefore \angle PAQ+\angle PBQ=$ sum of $\angle$s. of $\triangle$ APQ $=2$ rt. $\angle$s.

2. If two straight lines are drawn through the point of contact of two touching circles, the chords joining their extremities are parallel.

3. Prove also that the tangents drawn at the extremities of the chords in Ex. 2 are parallel two by two.

4. *Show how to draw a tangent to a circle without using its centre.* (Punjab, 1919).



Hint.—Given a point A on a circle. Take any other two points B and C on the $\odot$. Join AB, AC, BC. At A make $\angle BAP=\angle ACB$ and produce PA to Y. PAY is the reqd tangent.

5. *If AC and BD be the diagonals of a parallelogram intersecting at O, show that their circumscribing circles of the* $\triangle$*s AOB and COD touch one another at O.*

6. Any straight line drawn through the point at which two circles touch each other, cuts off similar segments from the circle.

7. PR and SQ are two perpendicular chords of a circle ; prove that the tangents at P, Q, R, S form a cyclic quadrilateral. (*Punjab*, 1918).

Hint .—PR, SQ cut at O. Through P, AD and through R, BC, tangents are drawn to cut other tangents at A, B, C, D. Join PQ, QR, RS and SP. $\because$ $\angle APS=\angle PQO$, $\angle CQR=\angle QPO$ $\therefore$ $\angle APS+\angle CQR=\angle PQO+\angle QPO=$ a rt. $\angle$.

Similarly $\angle ASP+\angle CRQ=$ a rt. $\angle$. $\therefore$ $\angle A+\angle C=2$ rt. $\angle$s.

$\therefore$ ABCD is cyclic.

8. If the angle ACB of a triangle ABC is bisected by CE cutting AB in E, and another point D is taken in AB produced such that $\angle ECD=\angle CED$, show that CD touches the circumcircle of ABC.

(*Bombay*, 1919)

Hint :—$\angle BCD=\angle ECD-\angle ECB=\angle CED-\angle ACE$.

But $\angle BAC=\angle CED-\angle ACE$ $\therefore$ $\angle BCD=\angle BAC$.

$\therefore$ CD tonches the $\triangle ABC$.

9. ABC is a triangle. Tangents are drawn at A, B, C to the circumcircle of the $\triangle$s so as to form another triangle PQR. Show that the angles of this triangle are respectively supplementary to double the angles of $\triangle ABC$. (*Bombay*, 1921).

Hint :—$\angle PBA=\angle BCA$, $\angle PAB=\angle ACB$.

But $\angle PBA+\angle PAB=$ supplement of $\angle APB$.

$\therefore$ $\angle BCA+\angle ACB$ or $2\ \angle ACB=$ supplement of $\angle APB$.

10. AB is a diameter of a circle with centre O, AC and BD two chords on the same side of AB intersect at E; show that OC is a tangent to the circle passing through C, D, E.

Hint :— $\angle OCA = \angle OAC = \angle CDE$ $\therefore$ OC is tangent to the circle CDE (Prop. 74. converse).

11. AB is a diameter of a circle and BC is a chord equal to the radius. If tangents at A and C meet at D, prove that ACD is an equilateral triangle.

Hint :—Let O be the centre, $\triangle$ OBC is equilateral. $\angle DAC = \angle DCA = \angle OBC = 60^\circ$ $\therefore \triangle ACD$ is equilateral

12. A circle is inscribed, in a $\triangle$ ABC touching the sides BC, CA, AB in the points D, E, F; show that the angles of the triangle DEF are respectively the complements of half the corresponding angles of the triangle ABC. (*Bombay*, 1926).

Hint :— $\angle AEF = \angle AFE = \angle D$ $\therefore \angle A + 2\angle D = 2$rt. $\angle$s. $\therefore \frac{1}{2}\angle A + \angle D = 1$ rt. $\angle$.

13. PQ and PR are a chord and a diameter respectively of a circle, PS is drawn $\perp$ to the tangent at Q. Prove that PQ bisects $\angle SPR$.

(*Bombay*, 1927)

Hint :—In $\triangle$s PRQ, PSQ, $\angle PSQ = \angle PQR$ (rt. $\angle$s). $\angle PQS = \angle PRQ$ (alt. seg.)
$\therefore \angle SPQ = \angle RPQ$ $\therefore$ PQ bisects $\angle SPR$.

Proposition 75. (*Theorem*).

If two chords of a circle intersect either inside or outside the circle, the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

Given :—Chords AB and CD intersect at O.

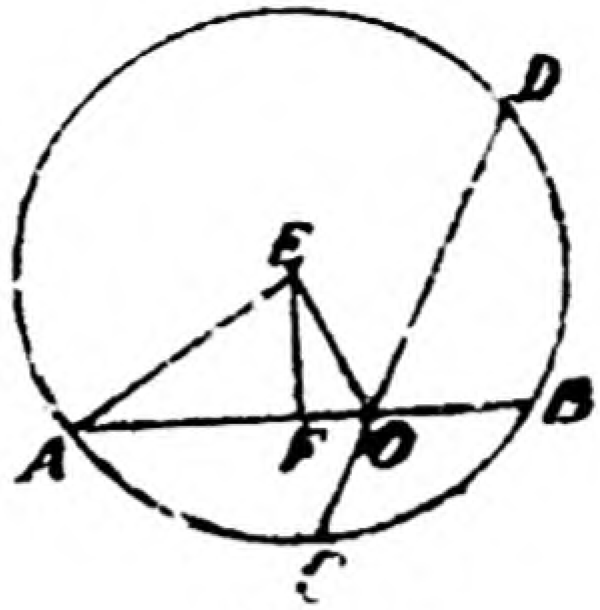


Fig. 1.

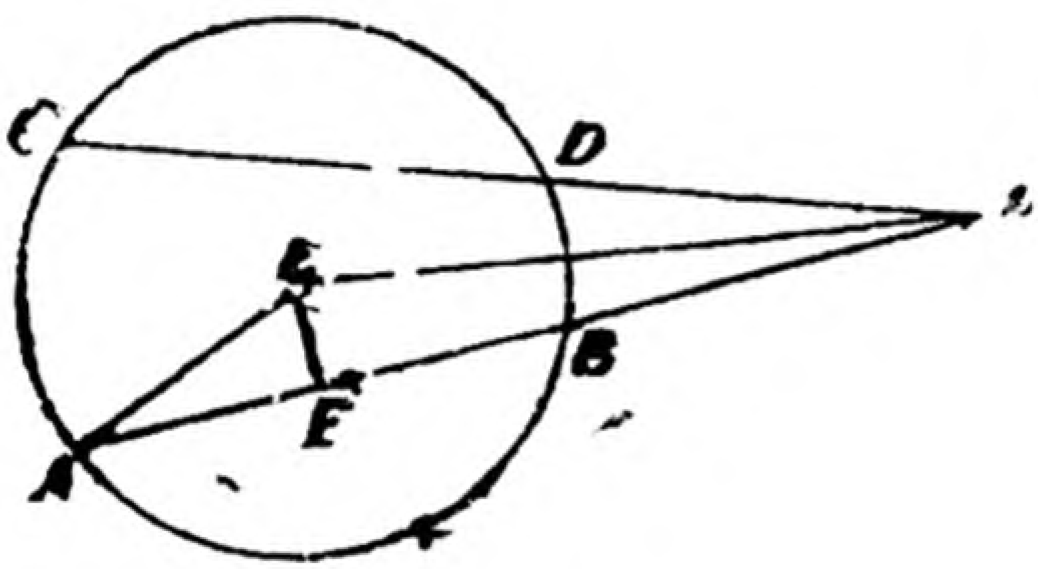


Fig. 2.

Required :—To prove that AO. BO=CO . OD

Construction :—From centre E draw EF $\perp$ AB. Join AE and OE.

Proof :—(*i*) When O, the point of intersection, is inside the circle. (Fig. 1.)

$\because$ EF $\perp$ AB.

$\therefore$ AF=BF.

Now $AO.OB=(AF+OF)(BF-OF)$

$=(AF+OF)(AF-OF)$ ($\because$ AF=BF)

$=AF^2-OF^2$

$=(AF^2+EF^2)-(OF^2+EF^2)$

$=AE^2-OE^2$ ($\because$ $\angle$s at F are rt. $\angle$s.)

$=(\text{radius})^2-OE^2$.

Similarly CO. OD$=(\text{radius})^2-OE^2$

$\therefore$ AO. OB$=$CO. OD.

(*ii*) When O, the point of intersection. is outside the $\odot$. (Fig. 2).

$\because$ EF $\perp$ AB, $\therefore$ AF$=$BF.

Now AO, OB$=$(OF$+$AF) (OF$-$BF).

$=(OF+AF)(OF-AF)$. ($\because$ AF$=$BF)

$=OF^2-AF^2$

$=(OF^2+FE^2)-(AF^2+EF^2)$.

$=OE^2-AE^2$. ($\because$ $\angle$s at F are rt. $\angle$s.)

$=OE^2-(\text{radius})^2$

Similarly CO.OD$=OE^2-(\text{radius})^2$.

$\therefore$ AO.OB$=$CO.OD.

Q. E. D.

Alternative Proof

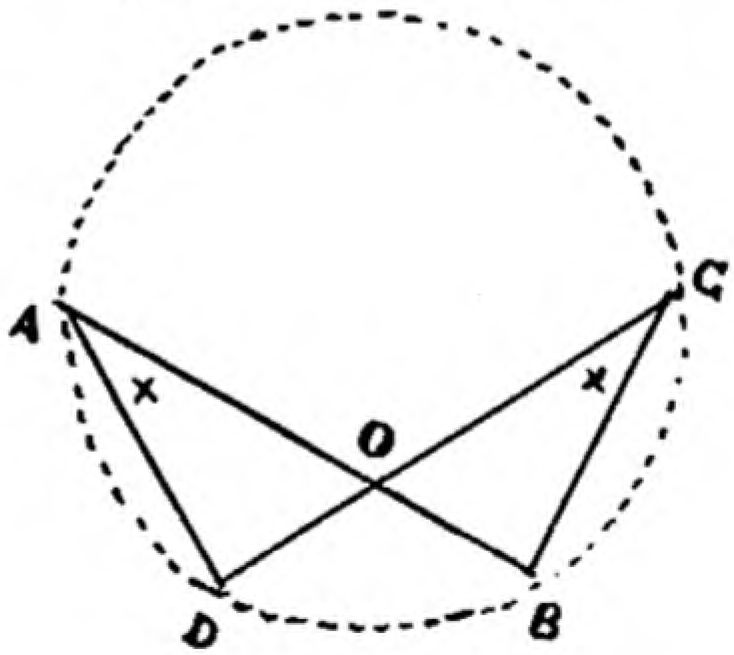


Fig. 1.

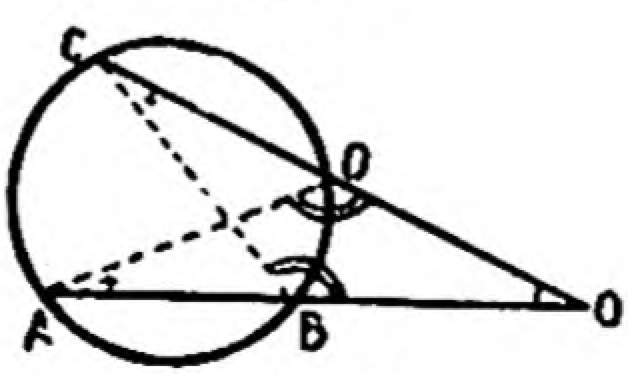


Fig. 2.

Given :—Two chords AB,CD of a ⊙ intersect at O.

Required :—To prove that AO.OB=CO.OD.

Construction :—Join AD.CB.

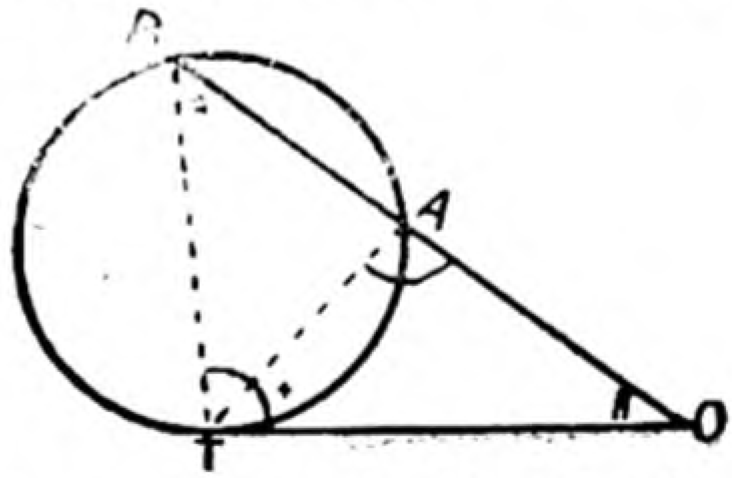
Proof :—In △s AOD,COB.

∠OAD=∠OCB, (∠s in the { same segment of a ⊙)
and ∠AOD=∠BOC { ver. opp. ∠s in fig. I.
Common in fig. II.)

∴ the △s are equiangular.

$$\text{Hence } \frac{AO}{OC}=\frac{OD}{OB}\ i.e.,\ AO.OB=OC.OD.$$

Cor. 1. If from any point without a circle a secant and a tangent be drawn to the circle, the rectangle contained by the secant and its external segment is equal to the square on the tangent.



Given :—From any pt. O outside a ⊙, a secant OAB and a tangent OT are drawn to the ⊙.

Required :—To prove that $OA.OB=OT^2$.

Construction :—Join TA,TB.

Proof :—In △s OTB, OTA

∠O=∠O (Common)

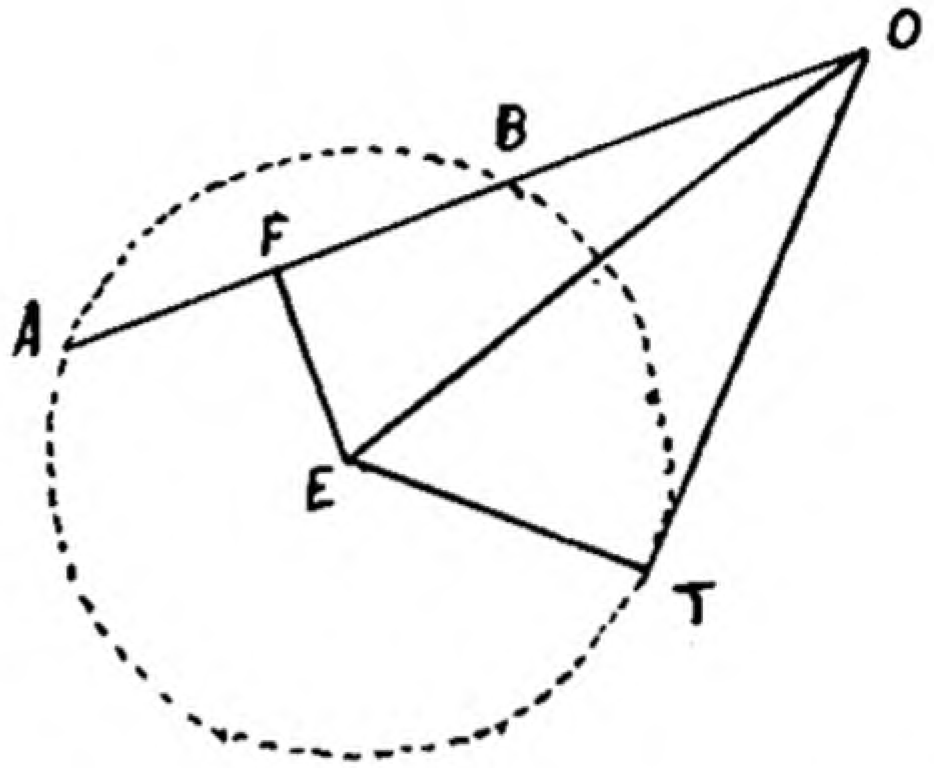
and ∠OBT=∠OTA. (∠s in the alt. segment of a ⊙)

∴ the △s are equiangular

$$\text{Hence } \frac{OB}{OT}=\frac{OT}{OA}\ i.e.\ OA.OB=OT^2$$

Or Alternatively

Given :—OBA a secant and OT a tangent drawn to the circle from the same point O, outside the circle.



Required ;—To prove that $OA.OB = OT^2$.

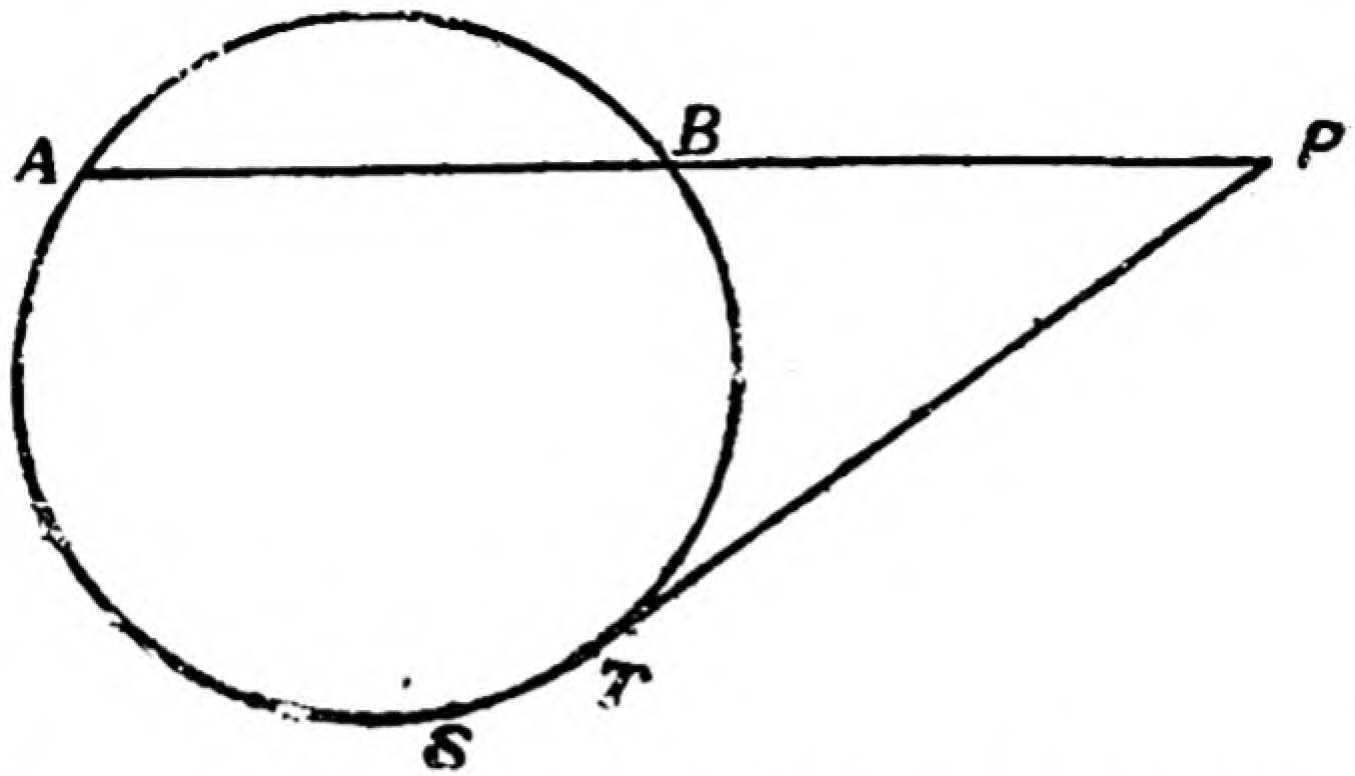
Construction :—Draw $EF \perp AB$. Join EA, EO, ET.

Proof :—$OA.OB = OE^2 - (\text{radius})^2$ (Proved)
$= OE^2 - ET^2$. ($ET = \text{radius}$.)
$= OT^2$, (OTE is a right angle).

Q E. D.

Cor. 2.—If from any point P without a circle two straight lines be drawn, one cutting the circle in B and A and the other meeting it at T so that $PA, PB = PT^2$, then PT touches the circle at T.

Given :—A line PBA cutting the circle at B and A and another PT meeting it at T such that $PA.PB = PT^2$.



Required :—To prove that PT is a tangent at T.

Construction :—If PT is not a tangent, it must cut the circle in another point, say S.

Proof :—PBA and PTS are two intersecting chords.

$\therefore$ PB.PA=PT.PS.

But PA.PB=PT^2. (Given)

$\therefore$ PS=PT.

This is not possible unless T and S coincide.

Hence PT touches the circle at T.

Q. E. D.

Note.—Cor. 2 is the converse of Cor. 1.

Converse of the above Theorem

If two straight lines cut one another internally or externally so that the rectangle under the segments of the one is equal to the rectangle under the segments of the other, the four extremities of the lines lie on a circle.

Given :—Two lines PQ and P′Q′ cutting each other at O, internally in Fig. 1 and externally in Fig. 2 **PO.OQ=P′Q.OQ′.**

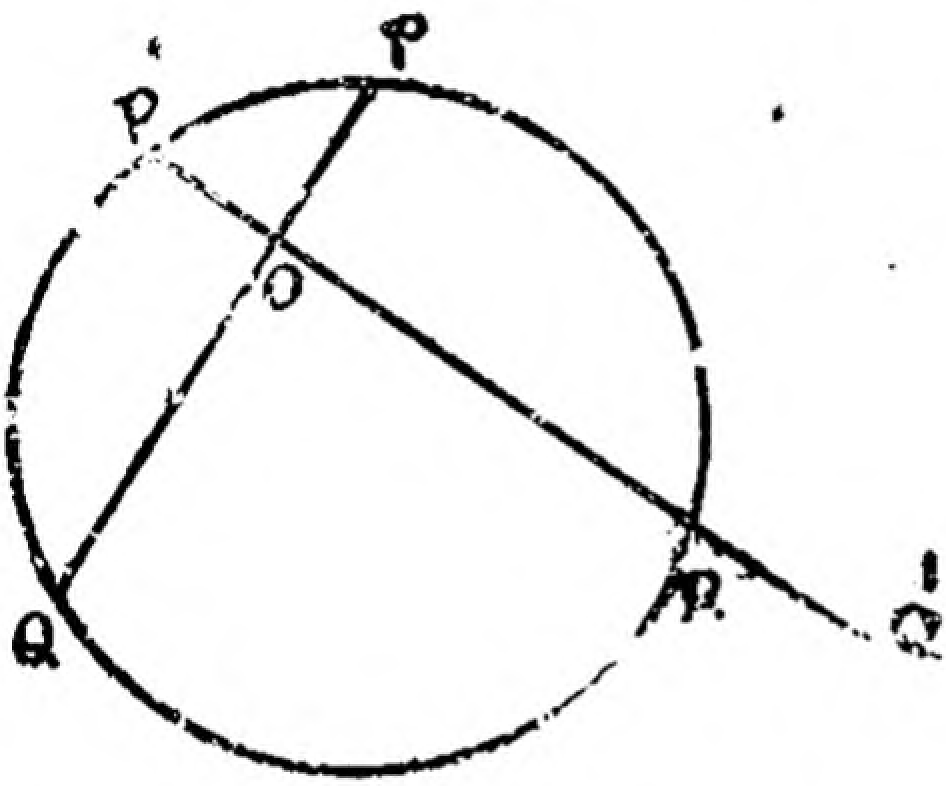


Fig. 1

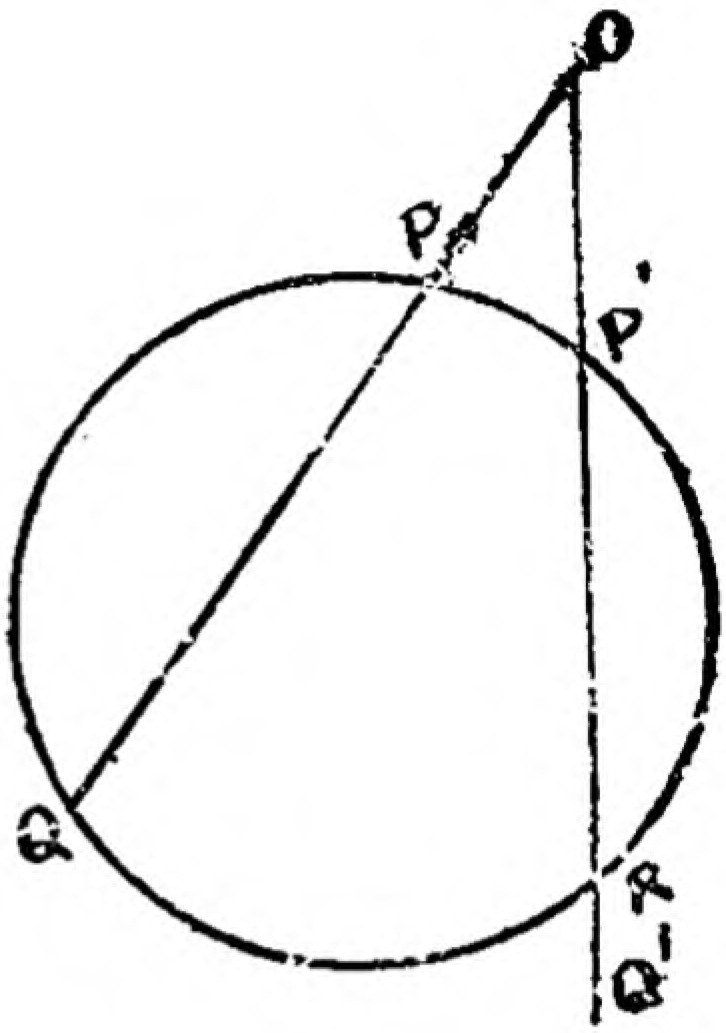


Fig. 2.

Required :—To prove that P, P′, Q and Q′ lie on a circle.

Construction :—Draw a circle passing through P, P, Q′. If this does not pass through Q, let it cut P′Q′ at some point R..

Proof :—$\because$ PQ and PR are two intersecting chords in a circle.

$\therefore$ PO.OQ=P′O.OR.

But PO.OQ=P′O.PQ′. (Given)

$\therefore$ P′O.OR=A′O.OQ′.

Or OR=OQ′.

This is not possible unless R and Q′ coinoide.

Hence the circle through P, P′, Q passes through Q′ as well ;

or the points P, P′, Q, Q′ lie on a circle.

Q.E D.

Exercises.

1. *If two tangents be drawn to a circle from the same point they are equal.*

2. *If from any point on the common chord of two intersecting circles tangents are drawn to the circles, these tangents are equal.*

3. *If the tangents from any point to two intersecting circles are equal, the point lies on the common chord of the circles.*

4. *The common chord of two intersecting circles bisects both their common tangents.*

5. The common chords of three intersecting circles, taken two by two, are concurrent.

6. One of the three common tangents of two circles which touch each other externally, bisects the other two. (Punjab, 1909).

7. If from a point on the common chord (produced) of two intersecting circles, two secants are drawn one to each circle their four points of intersection are concyclic.

8. XY is a fixed straight line perpendicular to the diameter AB of a circle. A straight line through A meets the circle in P and XY in Q. Prove that the rectangle AP.AQ is constant. (Calcutta.)

Hint :—AB cuts XY at O, join PB. $\angle BOY$ is a rt. $\angle$ and $\angle BPA$ = a rt. $\angle$ (in a semi$\odot$).

$\therefore$ BOQP is cyclic. Chords BO, PQ being produced cut outside the second circle at A. $\therefore$ AP.AQ = AO.AB = a constant quantity as AB is a diameter.

9. If two chords AB, AC are drawn from any point A on a circle and are produced to D and E respectively so that the rectangle AE, AC = rect. AB.AD ; then show that AO $\perp$ DE, O being the centre of the circle. (Bombay, 1925).

Hint :—Join AO and produce it to meet DE in P cutting the $\odot$ in X. Since AC.AE = AB.AD,

$\therefore$ C, B, D, E are concyclic.

$\therefore$ $\angle PEC = \angle ABC = \angle AXC$ $\therefore$ C, X, P, E, are concyclic.

Hence $\angle XPE = \angle ACX$ = a rt. $\angle$ (in a semi. $\odot$).

10. A straight line AB is divided at any point C. On AC as diameter a circle is described and chord AP is drawn in it. If AP is produced to Q so that AP. AQ = AB. AC, show that the point Q lies on a fixed straight line.

Hint.—Since AP.AQ = AB.AC $\therefore$ P, C, B, Q are concyclic.

Hence $\angle B = \angle APC$ = a rt. $\angle$ (in a semi. $\odot$) $\therefore$ Q lies on a st. line $\perp$ AB.

11. ABC is a $\triangle$ obtuse-angled at A, and D, E are the middle points of BC, CA respectively ; if O is the circumcentre of the triangle ABC and OE intersect BC in P, show that PD.PC = OP.PE.

Hint.— Since O is the circumcentre and D and E the mid-points of sides of the $\triangle$ $\therefore$ $\angle ODC$ = a rt. $\angle = \angle OEC$.

$\therefore$ O, D, E, C are concyclic $\therefore$ DP.PC = OP.PE.

12. ABC is a $\triangle$ right-angled at C. From any point P in AC, PQ is drawn perpendicular to AB. Prove that AP. AC=AQ.AB.

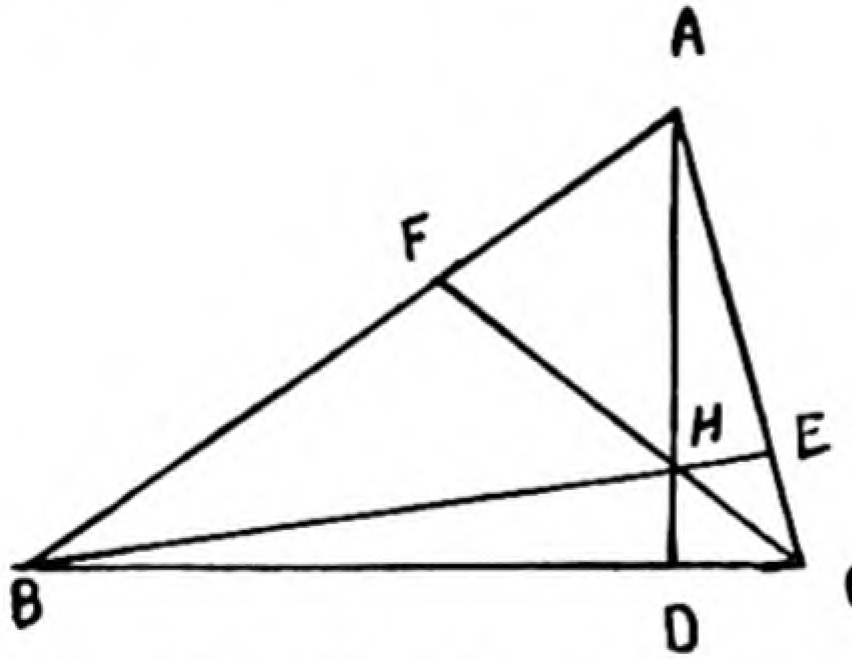
13. ABC is a triangle, AD, BE, CF are perpendiculars from the angular points to the opposite sides of the triangle meeting in the orthocentre H. Prove that

(*i*) AH.HD=BH.HE.=CH.HF.

(*ii*) AD.AH=AC.AE=AB.AF.

(*iii*) AD.HD=BD.DC.

Hint :—(*i*) $\because \angle AEB = \angle ADB$ (rt. $\angle$s.) $\therefore$ A, E, D, B are concyclic $\therefore$ AH.HD=BH.HE. Similarly E, F, B, C are concyclic $\because$ BH.HE=CH. HF, hence AH.HD=BH.HE=CH.HF.



(*ii*) Again $\angle D + \angle E = 2$ rt. $\angle$s $\therefore$ EHDC is a cyclic quad.

$\therefore$ AD.AH=AC.AE.

Similarly AD.AH=AB.AF,

$\therefore$ AD.AH=AC.AE=AB.AF.

(*iii*) $\because$ EHDC is a cyclic quad.

$\therefore$ BH.BE=BD.BC.

or BH (BH+HE)=BD (BD+CD).

or $BH^2 + HB.HE = BD^2 + BD.CD$.

or $BD^2+HD^2+HB.HE=BD^2+BD.CD.$

[$\because BH^2=BD^2+HD^2$

or $HD^2+HB\ HE=BD.CD.$

or $HD^2+AH.HD=BD.CD.$ [$\because HB.HE=AH.HD.$

or $HD\ (HD+AH)=BD.CD.$

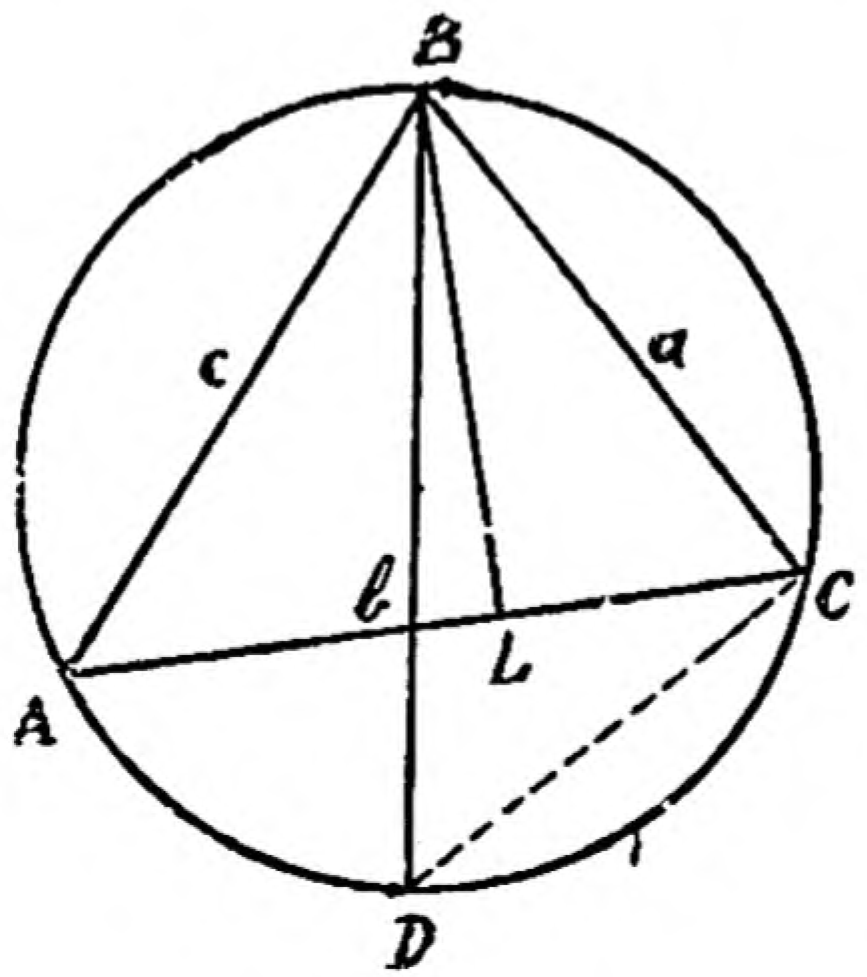
or $HD.AD=BD.CD$ [$\because AD=HD+AH.$

Q.E.D.

14. In a circle of radius r, a chord is drawn through a point within the circle whose distance from the centre is c; show that the area of the rectangle under the segments of the chord is (r^2-c^2).

15. If from the vertex of a triangle a perpendicular is drawn to the base, the triangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the circumdiameter of the triangle.

Given:—ABC is a triangle with ABCD its circumcircle.



BD is diameter of the circle and BL$\perp$t base AC of the triangle.

Required:—To prove that
AB.BC=BL.BD.

Const:—Join CD.

Proof:—In the $\triangle$s ABL and DBC

$\angle$BAL=$\angle$BDC (in the same segment)

also $\angle$BLA=$\angle$BCD (rt. angles)

$\therefore$ $\triangle$s are equiangular

$$\therefore \frac{AB}{BL}=\frac{BD}{BC}$$

or AB.BC=BL BD.

Prove that the circumradius of a triangle is given by the formula $R=\dfrac{abc}{4\triangle}$ where a, b, c and $\triangle$ are used in the usual sense and R stands for the circumradius.

From exercise 11 we have

AB.BC=BL.BD.

$\therefore$ ca=BL 2R.

or $b.\ c.\ a=b.$ BL. 2R.

$$=2 \text{ area of the triangle} \times 2R$$

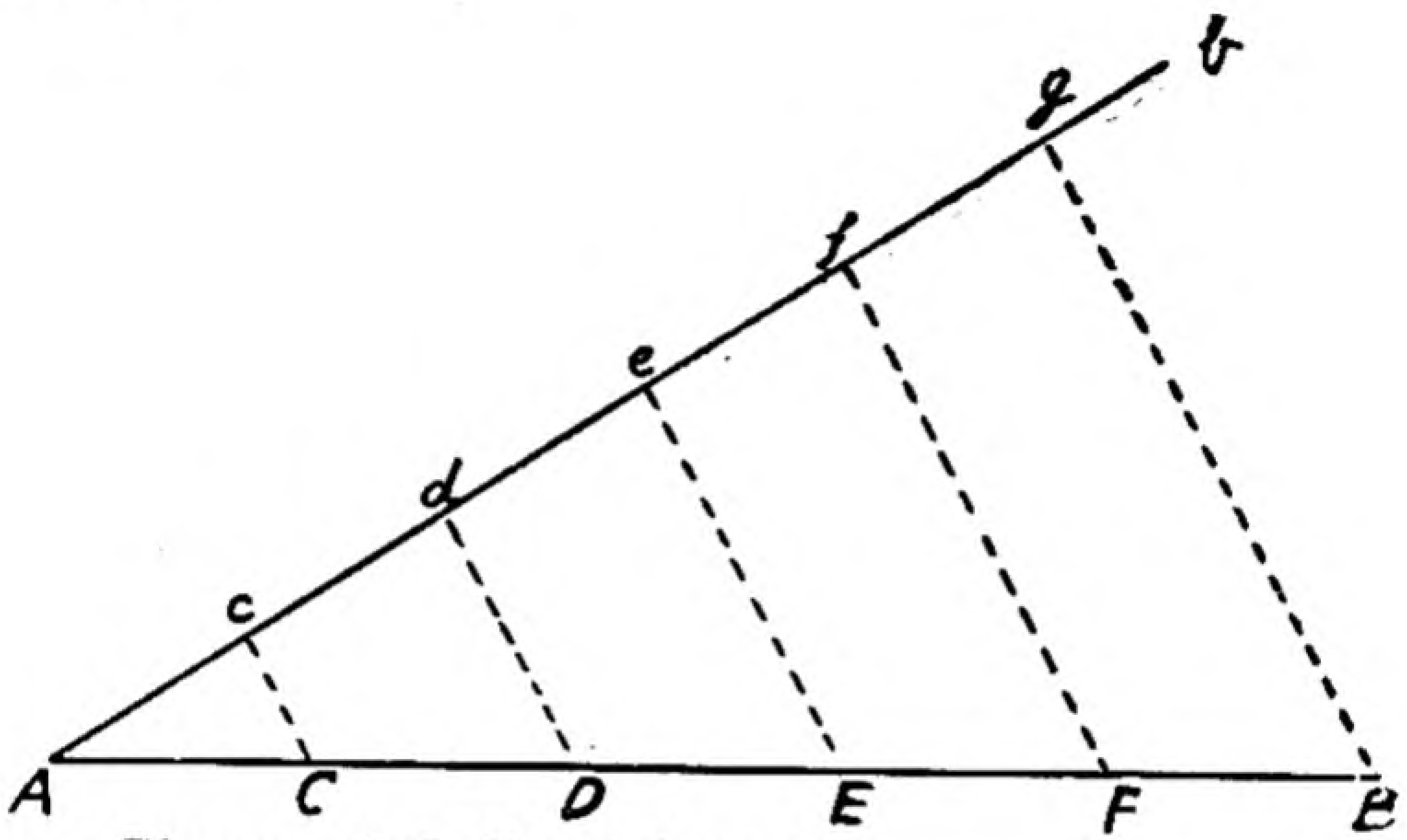
$$=2\triangle.2R$$

$$=4\triangle.R$$

$$\therefore \frac{abc}{4\triangle}=R$$

Proposition 76. (*Problem*)

To divide a given finite st. line into any number of equal parts.



Given :—A finite st. line AB.

Required :—To divide it into any number of equal parts (say five).

Construction : 1. Draw A b making any angle with AB.

2. On Ab mark off equal distances Ac, cd, de, ef and fg of any convenient length.

3. Join Bg.

4. From c, d, e and f draw lines parallel to Bg, meeting AB in C, D, E and F respectively.

Then shall AC=CD=DE=EF=FB.

Proof :—∵ cC, dD, eE, fF and Bg are parallel and Ab and BA meet them in such a way that the intercepts on Ab are equal. (Const.)

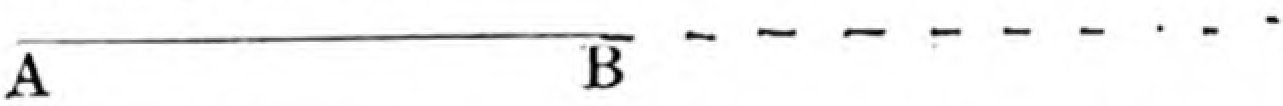
∴ AC=CD=DE=EF=FB.

Q. E. F.

Note.—If line AA (figure of the proposition) is to be divided into two parts in the ratio of 2 : 3, mark off 5 (=2+3) equal portions on Ab. Join gB and draw dD a parallel to it through the second point of division. The line AB is then divided at D in the ratio of 2 : 3. That is 2 parts lie on one side of D and 3 on the other.

Def.—The line AB is said to be divided internally at C in the ratio $a : b$, if $\frac{AC}{CB} = \frac{a}{b}$ and the point C lies in the line AB.

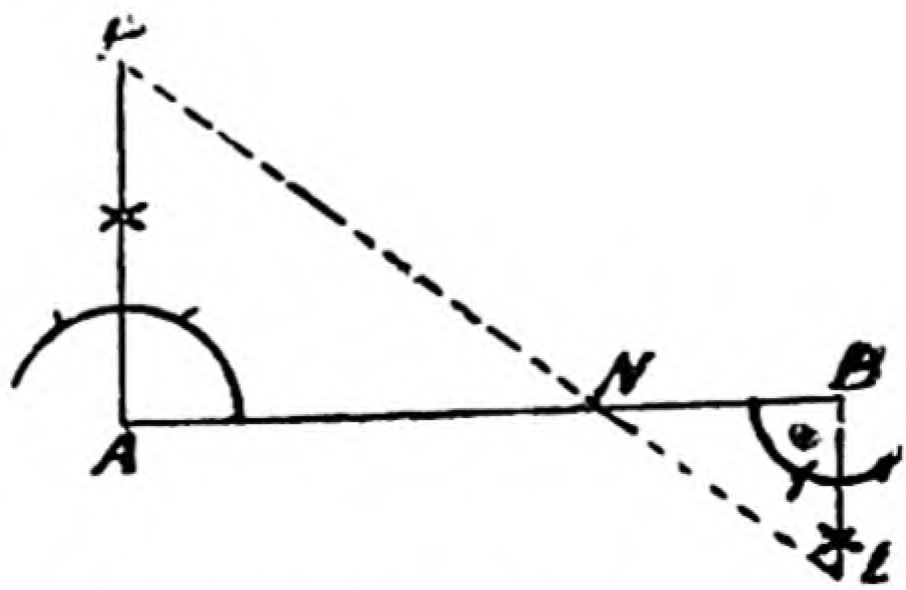
The line AB is said to be divided externally at C in the ratio $a : b$, if $\frac{AC}{CB} = \frac{a}{b}$ and the point C lies on the line AB *produced.*



The portion AC and CB (in both the above lines) are said to be the segments of the line AB.

Proposition 77. (*Problem*)

(*a*) *To divide a given straight line internally in the given ratio a : b.*



Given :—A straight line AB.

Required :—To divide it internally in the ratio of $a : b$.

Construction:—At A and B erect perpendiculars on opposite sides of the line AB and cut off AP equal to a units of length and BL equal to b units of length. Join PL cutting AB in N. Then the line AB is divided in the ratio $a : b$ at N *i.e.* $\frac{AN}{NB}=\frac{a}{b}$

Proof:—In the triangles PAN and LBN

$\angle PAN = \angle LBN$ (being right angles)

$\angle PNA = \angle LNB$ (being vertically opposite angles).

$\therefore$ Third $\angle APN$ = third $\angle BLN$.

Hence the triangles are equiangular.

$\therefore \frac{AP}{BL} = \frac{AN}{BN}$

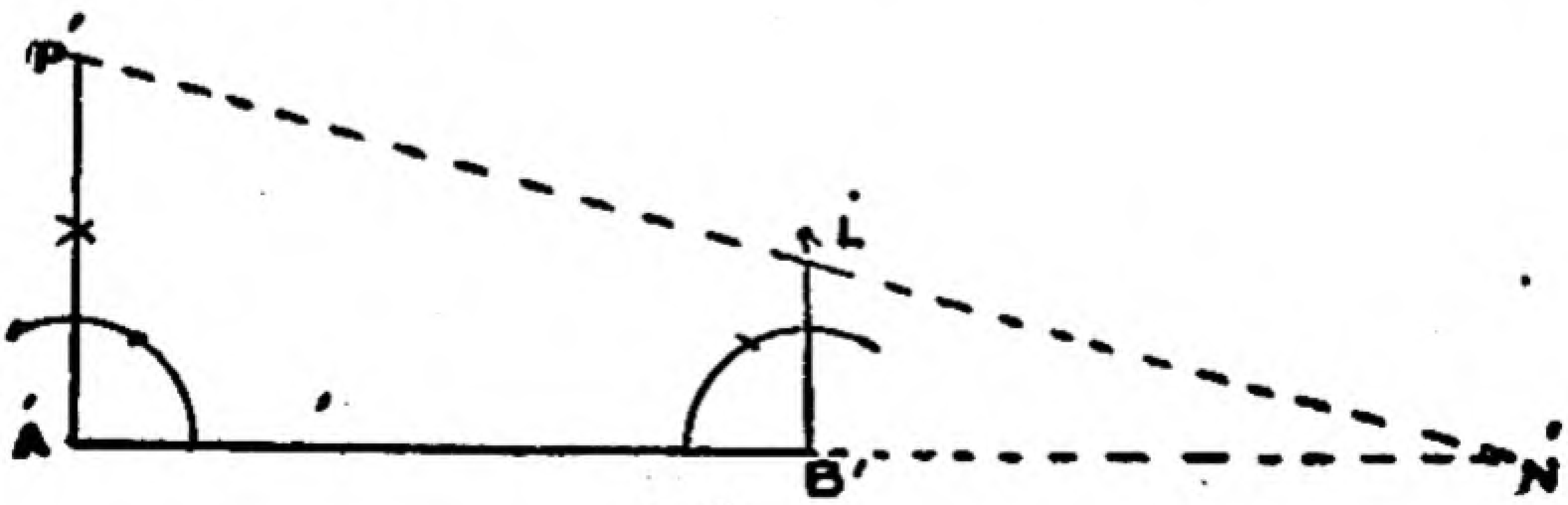
But $\frac{AP}{LB} = \frac{a}{b}$ (Const)

$\therefore \frac{AN}{BN} = \frac{a}{b}$

Hence AB is divided internally at N in the ratio $a : b$.

Q. E. D

(*b*) *To divide a given straight line externally in the given ratio a : b.*



(2)

Given:—A straight line $A'B'$.

Required:—To divide it externally in the ratio $a : b$.

Construction:—At A' and B' erect perpendiculars on the same side of the line $A'B'$. Cut off AP' equal to a units of length and $B'L'$ equal to b units of length Join $P'L'$ and produce it to meet the line $A'B'$ produced in N'. Then the line $A'B'$ is divided at N' externally in the ratio $a : b$ so that $\frac{A'N'}{B'N'} = \frac{a}{b}$

Proof:—$\triangle$s $P'A'N'$ and $L'B'N'$ are equiangular.

$\therefore \frac{P'A'}{B'L'} = \frac{A'N'}{B'N'}$

But $\frac{A'P'}{B'L'} = \frac{a}{b}$ (Const.) $\therefore$ $\frac{A'N'}{B'N'} = \frac{a}{b}$

Hence $A'B'$ is divided externally at N' in the ratio $a : b$.

Q. E. F.

Exercises.

1. Divide a st. line 3·5″ long into 5 equal parts and check the result by measurement.

(*Punjab*, 1900, 1912).

2. *Divide a line 2·5″ long in the ratio of 2 : 3 and check the result by measuremet and calculation.*

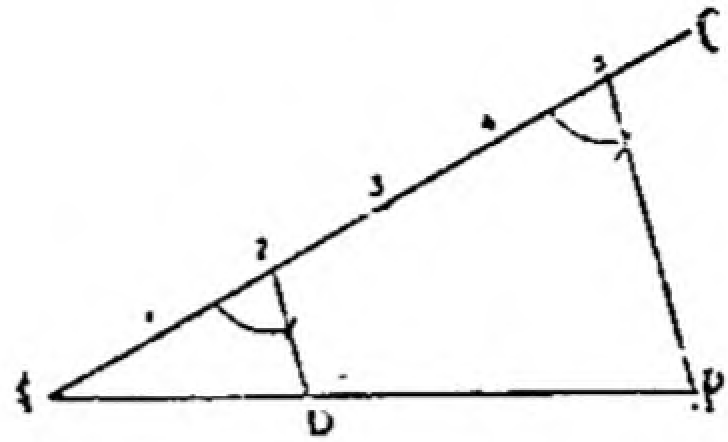
(*Punjab* 1914, 1916)

Take a st. line $AB = 2{\cdot}5''$.

Draw a st. line AC making an angle with AB.

Along AC mark off $(2+3) = 5$ equal parts.

Join 5B and through 2 draw 2P parallel to 5B meeting AB in P.



Then AP : PB : : 2 : 3.

By measurement AP$=1''$ and PB$=1{\cdot}5''$.

By calculation, AP$=\frac{2{\cdot}5}{5}\times2=1{\cdot}0''$.

and PB$=\frac{5}{5}\times3=1{\cdot}5''$.

3. *Divide a straight line 4·8″ long in the ratio of 3 : 4 : 5. Construct a △ with these parts as sides. Measure its greatest angle.* (*Punjab*, 1916, 1920)

4. *Find graphically* $\frac{3}{5}$ *of* 4·5 *cm.*

Hint.—Take a line 4·5 cm. long, divide it into 5 equal parts and take 3.

5. *Divide geometrically AB, 3·6″ long into 2 parts such that one part may be* $\frac{2}{7}$ *of the other.*

Hint.—Divide the line into two parts in the ratio of 2 : 7.

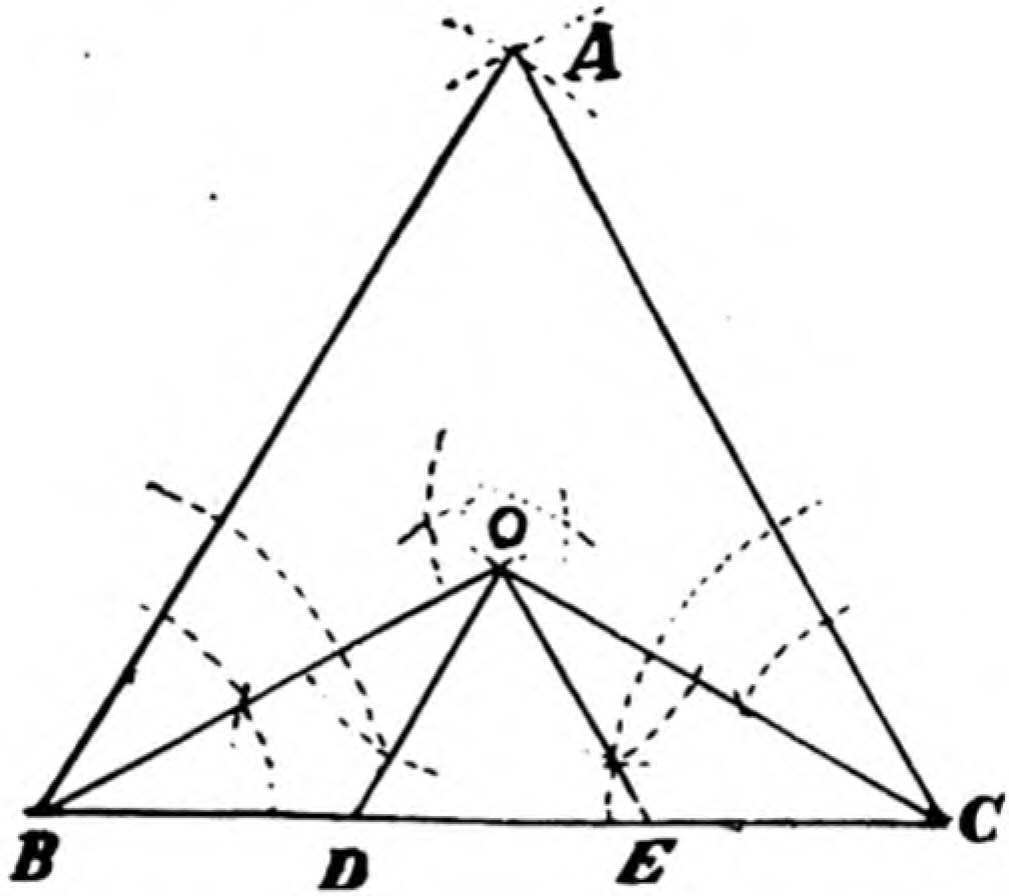
6. Divide a straight line 4 cm. long internally in the ratio of 5 : 3.

7. Draw a st. line 3″ long and divide it externally in the ratio of 2 : 3. (Punjab 1919)

What would be the result if the line were to be divided externally in the ratio 2 : 2 (Why ?)

8. *Draw a line AB equal to 5·6 cm. Divide it externally in the ratio of 3 : 4.*

9. Apply properties of an equilateral triangle to trisect a given line BC.



Hint.—On BC describe an equilateral $\triangle$ABC. Let the bisectors of $\angle$s B and C meet in O. Draw OD $\parallel$ AB and OE $\parallel$ AC. Then BC is trisected at D and E.

Proposition 78. (*Problem*)

To construct a fourth proportional to three given lines.

Given :—Three lengths l, m and n.

Required :—To construct a fourth proportional to l, m and n.

Let $\frac{l}{m}=\frac{n}{x}$ where x is the required fourth proportional.

Construction :—Take OX and OY any two lines inclined at any angle. Cut off from OX, OA equal to

l units of length. AB equal to *m* units of length and from OY, OC equal to *n* units of length. Join AC and through B draw BD parallel to AC. Then CD is the required fourth proportional.

Proof :—$\because$ In $\triangle$OBD, AC $\parallel$ BD.

$$\therefore \quad \frac{OA}{AB}=\frac{OC}{CD} \text{ or } \frac{l}{m}=\frac{n}{x}.$$

Hence CD is the required fourth proportional to these three given lengths.

Q. E. F.

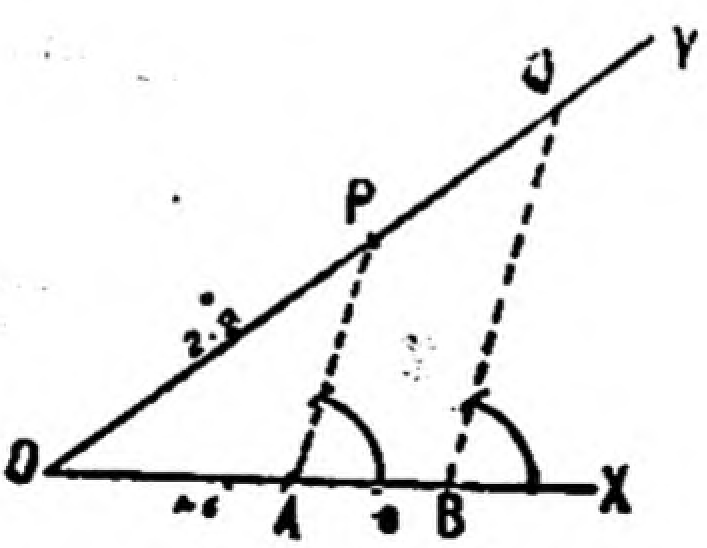
Note.—In the above construction all the lengths could have been measured from O.

Exercises

1. *Find by construction a fourth proportional to three lines whose lengths are 1·6″, ·8″ and 2·2″.*

Hint :—Draw two lines OX, OY inclined to each other at an $\angle$. On OX take OA=1·6″ and AB=8″.

Let $\frac{1\cdot6}{\cdot8}=\frac{2\cdot2}{X}$ where X is the fourth proportional.



On OY take OP=2·2″.

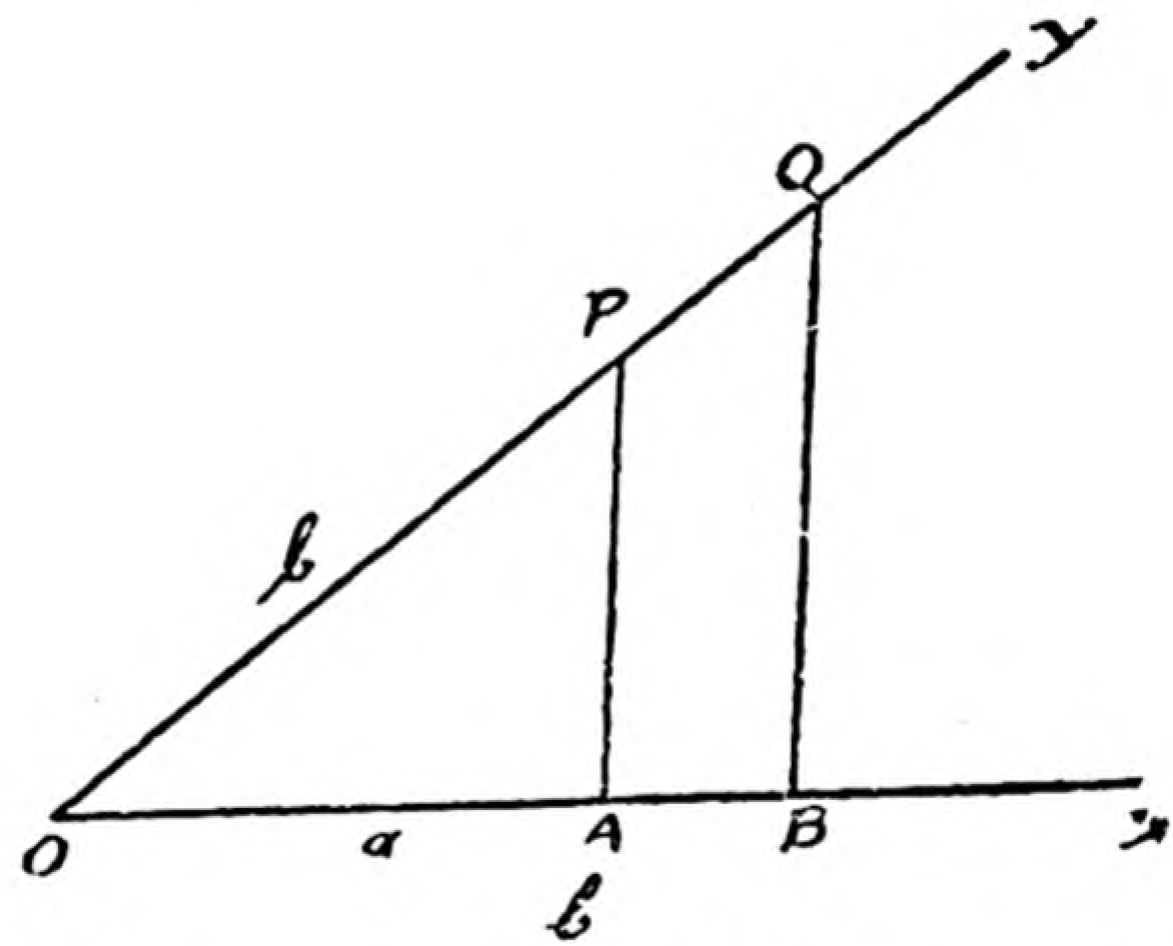
Join AP and through B draw BQ $\parallel$ AP cutting OY in Q. Then PQ is the required fourth proportional=X.

By measurement PQ=1·1″.

2. Find graphically a line which is a fourth proportional to the three lines whose lengths are 1·7", 2·3", 2·9". Measure its length and verify your result by calculation. (Punjab, 1912, 1928).

3. *Find a third proportional to two given lines a and b.*

Hint.—Let $\frac{a}{b}=\frac{b}{X}$ where X is the third proportional.



Draw two lines OX and OY inclined at an angle
From OX cut off OA=a units.
AB=b units.
From OY cut off OP=b units.
Join AP and through B draw BQ ∥ AP cutting OY in Q. Then PQ=X the required third proportional to a and b.

4. Find the third proportional to 1·8" and 1·2".

5. *Find geometrically the approximate value of* $\frac{5\cdot2\times3\cdot5}{6\cdot5}$ *and verify the result by simplifying the fraction.*

(Punjab, 1912)

Hint.—Let $\frac{5\cdot2\times3\cdot5}{6\cdot5}=\frac{x}{1}$ or $\frac{6\cdot5}{5\cdot2}=\frac{3\cdot5}{x}$.

Find a fourth proportional to 6·5, 5·2, 3·5.

6. *Find geometrically the value of* :—

(*i*) $\frac{(1\cdot2)^2}{1\cdot8}$ (*ii*) $\frac{6\cdot84}{2\cdot13}$

Verify your result by calculation.

Hint.—(*i*) $\frac{(1\cdot2)^2}{1\cdot8}=\frac{(1\cdot2)(1\cdot2)}{1\cdot8}$.

(*ii*) $\frac{6\cdot84}{2\cdot13}=\frac{1\times6\cdot84}{2\cdot13}$.

7. *Find a line equal in length to* $(1\cdot4)^2$ *inches.*

(Punjab, 1918)

Hint.—$(1\cdot4)^2=\frac{1\cdot4\times1\cdot4}{1}$

8. Find x, if $4\cdot1 : x :: 3\cdot2 : 2\cdot1$.

Take 1″ as unit of length and check the result by arithmetic.

9. Find P if P, 2·4 and 1·6 are in continued proportion.

10. *Find grahically the value of* $\frac{4\cdot8}{3\cdot6\times\cdot8}$.

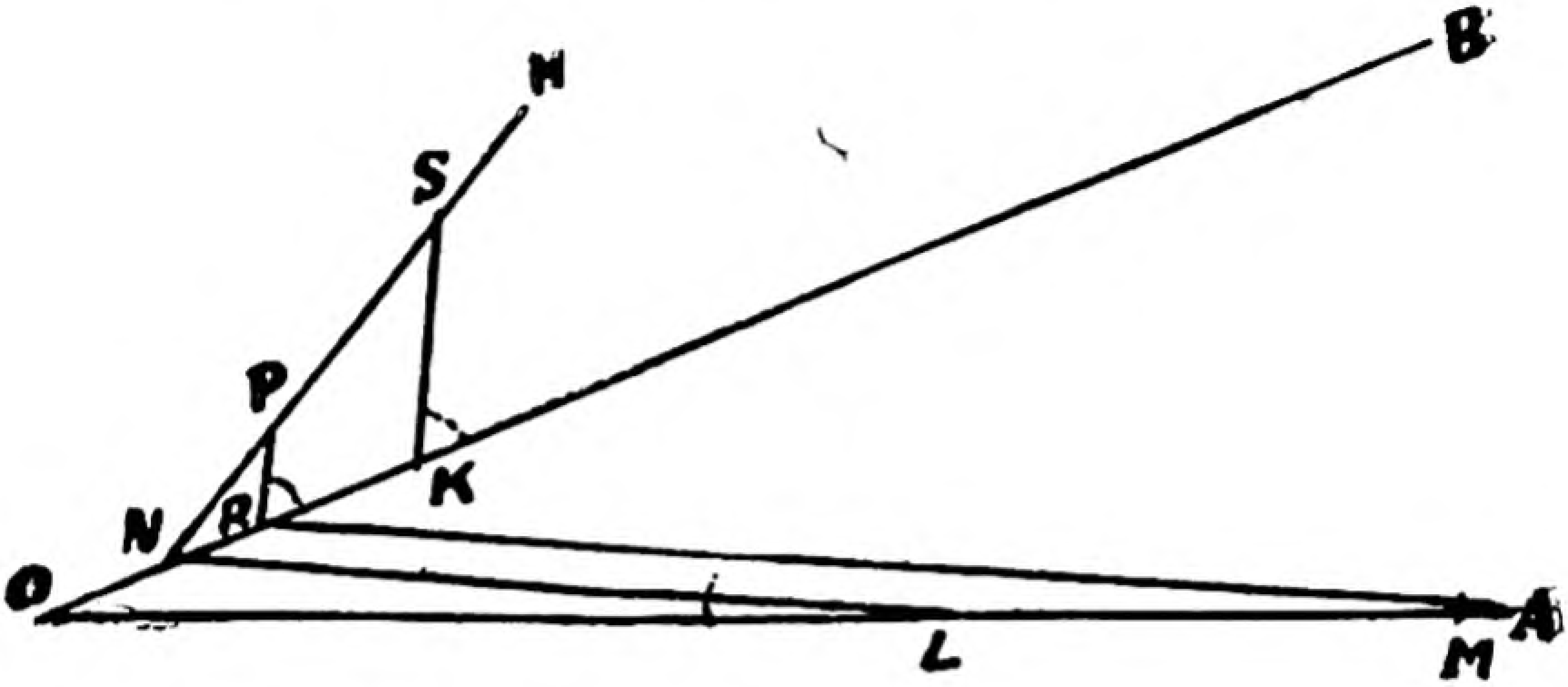
(Punjab, 1922).

Hint.—Let $\frac{4\cdot8}{3\cdot6\times\cdot8}=\frac{1}{x}$

or $4.8\times x=3\cdot6\times\cdot8$

or $\frac{4\cdot8}{3\,6}=\frac{\cdot8}{x}$ [x must take the fourth place.]

Take two lines OA and OB inclined at any angle.



Cut off OL=4·8, LM=3·6, ON=·8.
Join LN and draw MR ∥ NL,
Then NA=x
Now draw NH inclined at any angle with NB.
Cut off AK=1 and NP=1.
Join AP and draw KS ∥ AP.

Then PS=$\frac{1}{x}$=the given fraction.

Q. E. F.

11. Construct a rectangle whose sides are 1·2 cm. and 2.3 cm. Construct another rectangle of equivalent area but having one side=1·7 cm.

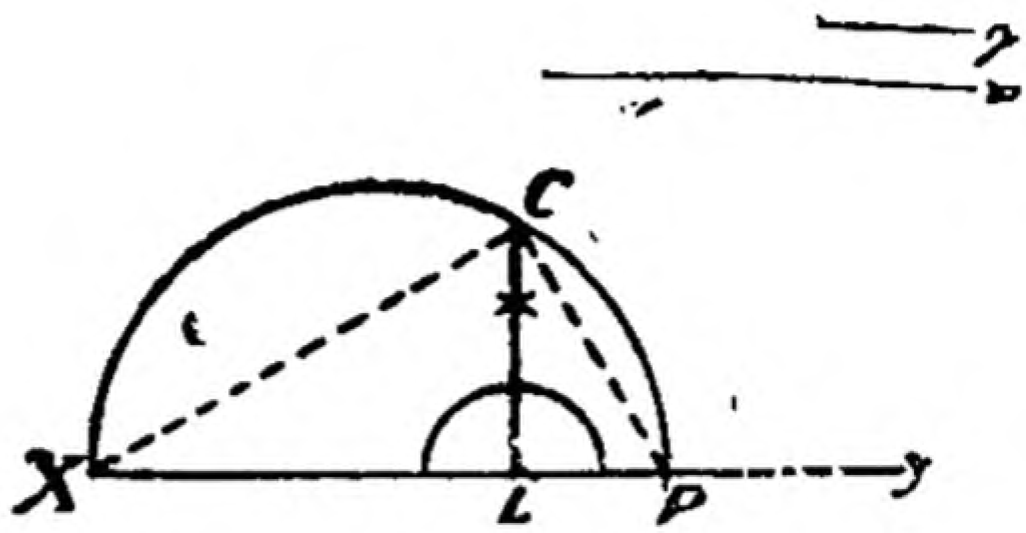
12. *Take any straight line OX. From it cut off OA=a. Draw AC ⊥ OX and make AC=b. Join OC. Draw CB ⊥ OC, cutting OX at B. Prove that AB is the third proportional to a and b.*

Hint:—Triangle OCB is a right-angled △.

13. Evaluate graphically a third part of $(1\cdot5)^2$.

Proposition 79. (*Problem.*)

To find a mean proportional to two given straight lines.



Given:—Two lengths a and b.

Required:—To construct a mean proportional to a and b.

Construction:—Take XY any line and cut off XL equal to a units of length and LP equal to b units of length. On XP as diameter describe a semi-circle. At L erect LC perpendicular to XP meeting the semi-circle in C.

Then LC is the required mean proportional.

Proof:—Join XC and PC. $\because$ Triangle XLC and PLC are equiangular.

$$\therefore \frac{XL}{LC}=\frac{LC}{LP}, \qquad \text{or } \frac{a}{LC}=\frac{LC}{b}$$

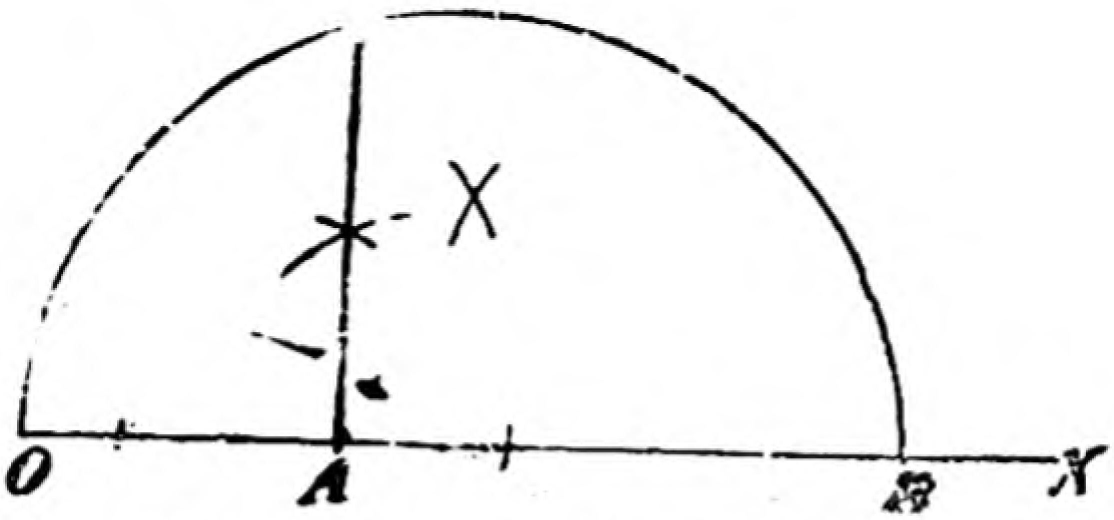
Hence LC is the required mean proportional to a and b. *i.e,* $LC^2=ab$.

Q. E. F.

Exercise.

1. *Find by a Geometrical construction a mean proportional between* ·9″ *and* 1·6″. *Test the result by arithmetic.*

(Punjab 1916)



Note:—From any st. line OX cut off OA=.9″ and AB=1·6″.

On OB as diameter describe a semi-circle. At A erect AP ⊥ to AB meeting the circle at P.

Then AP is the required mean proportional between ·9″ and 1·6″.

On measurement AP=1·2″.

By calculation $AP=\sqrt{OA.AB}=\sqrt{.9\times 1\cdot 6}$
$=\sqrt{1\cdot 44}=1\ 2$.

2. Find graphically the approximate value of $\sqrt{\cdot 91\times 6}$.

3. *Draw a line equal in length to* $\sqrt{7}$ inches.

(1917).

Hint.—$\sqrt{7}=\sqrt{7\times 1}$ ∴ Find the mean proportional between 7 and 1.

Or $\sqrt{7}=\sqrt{\frac{7}{2}\times 2}=\sqrt{3\cdot 5\times 2}$. ∴ Find the mean proportional between 3·5 and 2.

4. Find graphically the values of :—(*i*) $\sqrt{23}$, (*ii*) $\sqrt{10}$ (*iii*) $\sqrt{14}$, (*iv*) $\frac{1}{\sqrt{3}}$ (*v*) $2\sqrt{5}$.

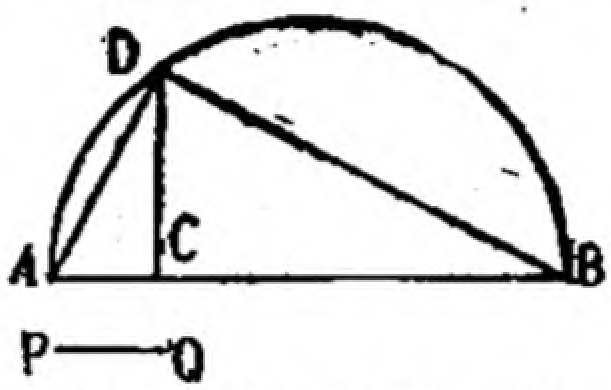
Hint :—Choose such factors for these numbers as convenient ; *e.g.* $\sqrt{23}=\sqrt{\frac{23\times 10}{10}}=\sqrt{2\cdot 3\times 10}$.

5. *Find the side of a square whose area is 7 sq. inches.* (1919) [*Hint.—Draw a line*$=\sqrt{7}$ *inches.*]

6. Find graphically the unknown term in the proportion :—$x : 16 = 25 : x$. Let 1″ represent 10. Check the result by arithmetic.

7. *Justify the following const. for finding the mean proportional between AB and PQ.*

Const :—On AB, the larger of the two lengths draw a semi ⊙ ADB. Cut off AC=PQ. Draw CD ⊥ AB meeting the semi ⊙ in D. Join AD. Then AD is the mean proportional between AB and PQ.

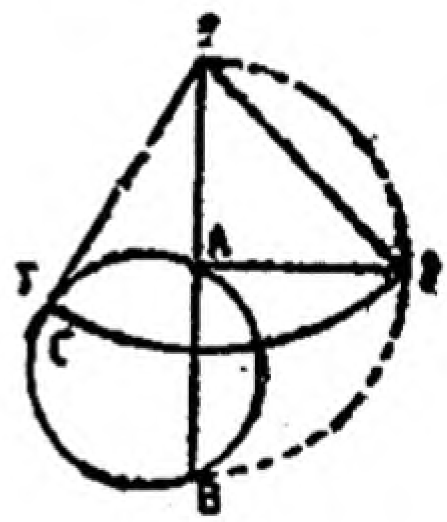


8. *Draw a tangent to a given ⊙ ABC from an external point P.*

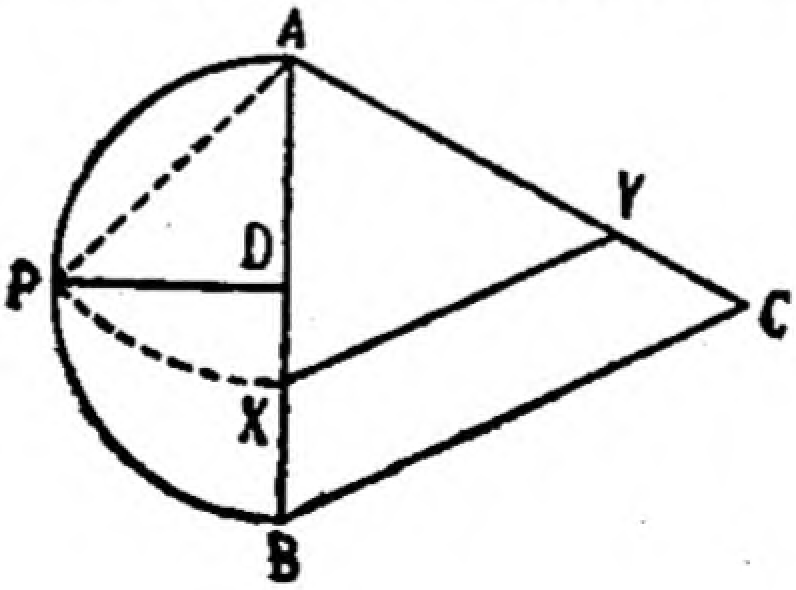
Hint.—From P draw any secant PAB.

Find PX, the mean proportional between PA and PB.

With P as centre and PX as radius describe an arc cutting the circle at T. Then PT is the required tangent. How?



9. *Bisect the* △ *ABC by a line* ∥ *to one of the sides.*



Hint :—Find D the middle point of the side AB.

Determine AP, the mean proportional between AD and AB. Cut off AX=AP.

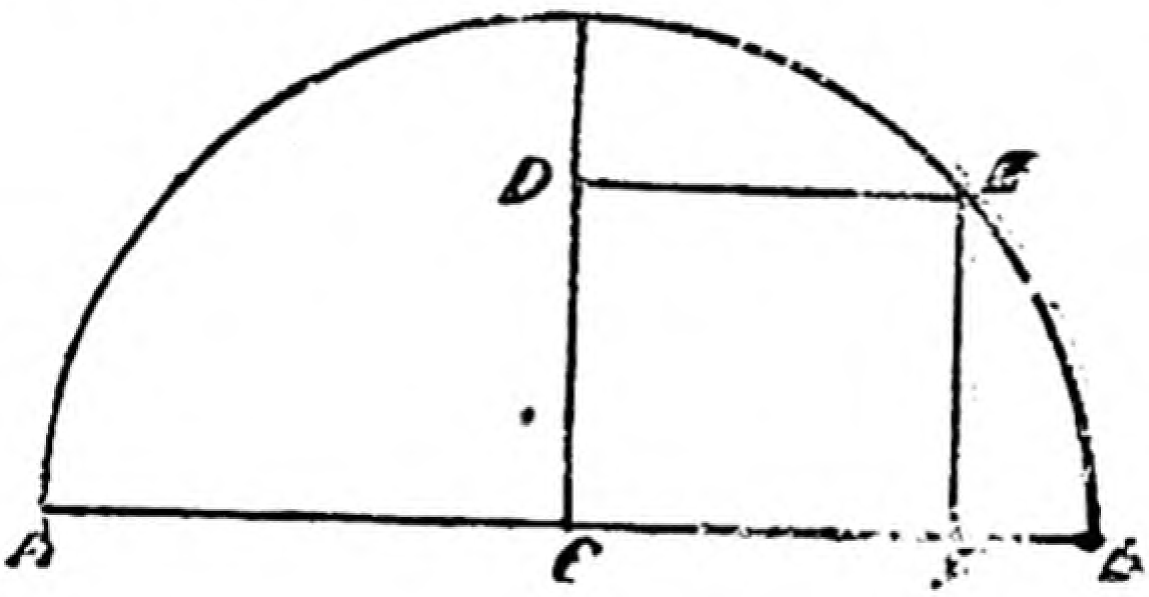
Through X draw XY ∥ BC meeting AC in Y. Then XY bisects the triangle ABC.

Proof :— $\frac{\triangle AXY}{\triangle ABC}=\frac{AX^2}{AB^2}=\frac{AD\times AB}{AB^2}=\frac{AD}{AB}=\frac{1}{2}$

10. *Draw a straight line 7·5 cm. long and divide it so that the rectangle contained by its segments may be equal to a square whose side is 2·6 cm. Measure the greater segment and verify it by calculation.*

(*Delhi*, 1929).

Hint :—Take AB=7·5 cm. Draw a semi ⊙ on this line. Draw CD the perpendicular bisector of AB. On this measure a distance CD=2·5 cm.



Through D draw DE ∥ AB cutting the ⊙ at E.

From E drop ⊥ EP to AB. Then point P divides AB into the two required segments.

By measurment AP=6·54 cm.

By calculation, AP+PB=7·5 cm.

and AP.PB=$(2·5)^2$.

$\therefore$ AP$-$PB= $\sqrt{(AP+PB)^2-4(AP.PB)}$

= $\sqrt{7·5^2-4(2·5)^2}$

=5·59 cm.

Whence 2AP=5·59+7·5=13·09

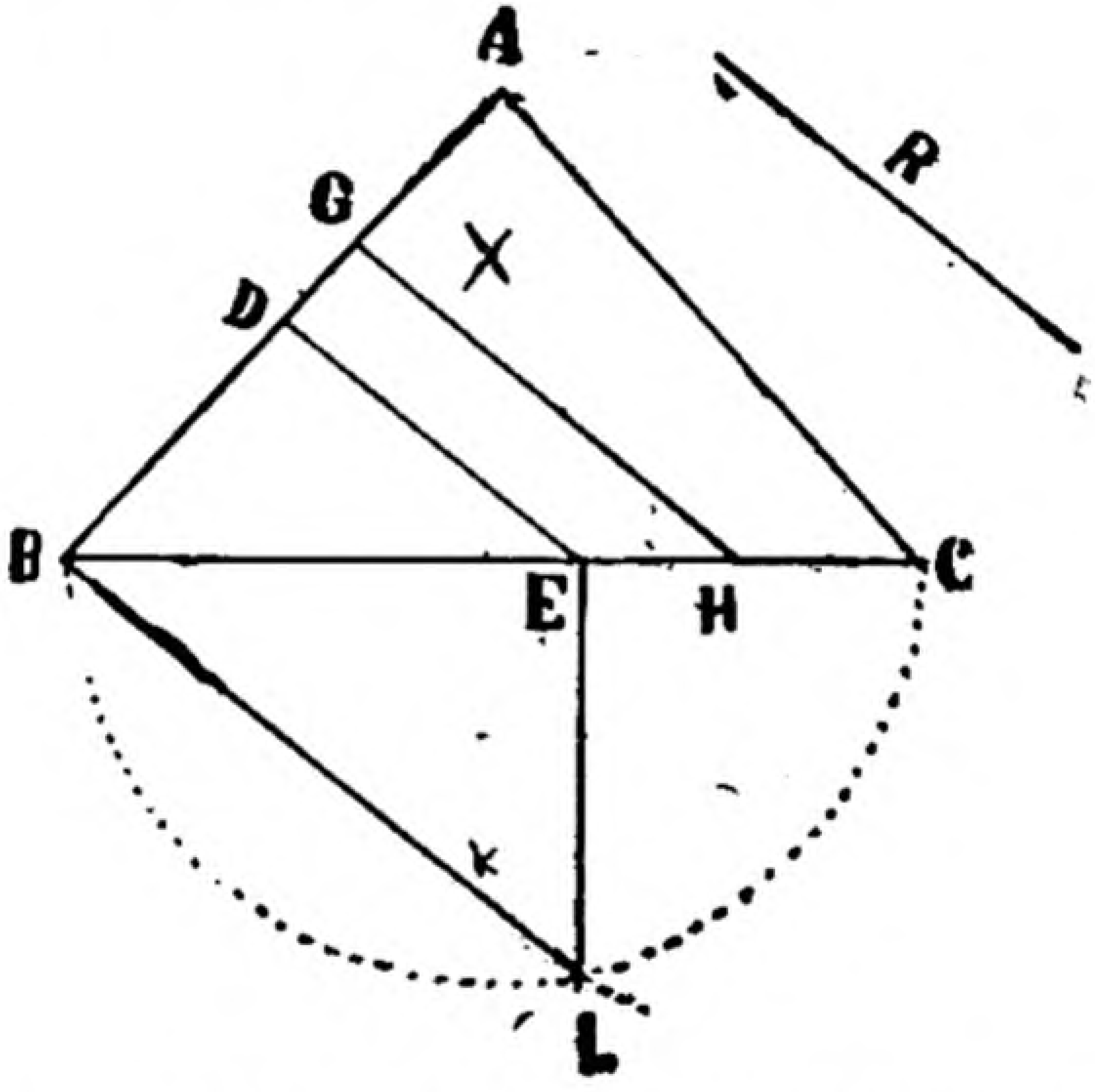
or AP=6·54 cm.

Note:—This construction can be used for finding x and y when $x+y$ and xy are given.

11. Taking 1" as the unit of length, find graphically the value of $\sqrt[4]{30}$ inches.

Hint :—Find $\sqrt{6}''$ and $\sqrt{5}''$ and then take a line equal to the sum of the two lengths and find their mean proportional.

12. Divide a △ ABC into two equal parts by a line parallel to a given st. line.



Hint: —We want to bisect △ ABC by a line ‖ the given line R.

Bisect AB in D. Draw DE ‖ R.

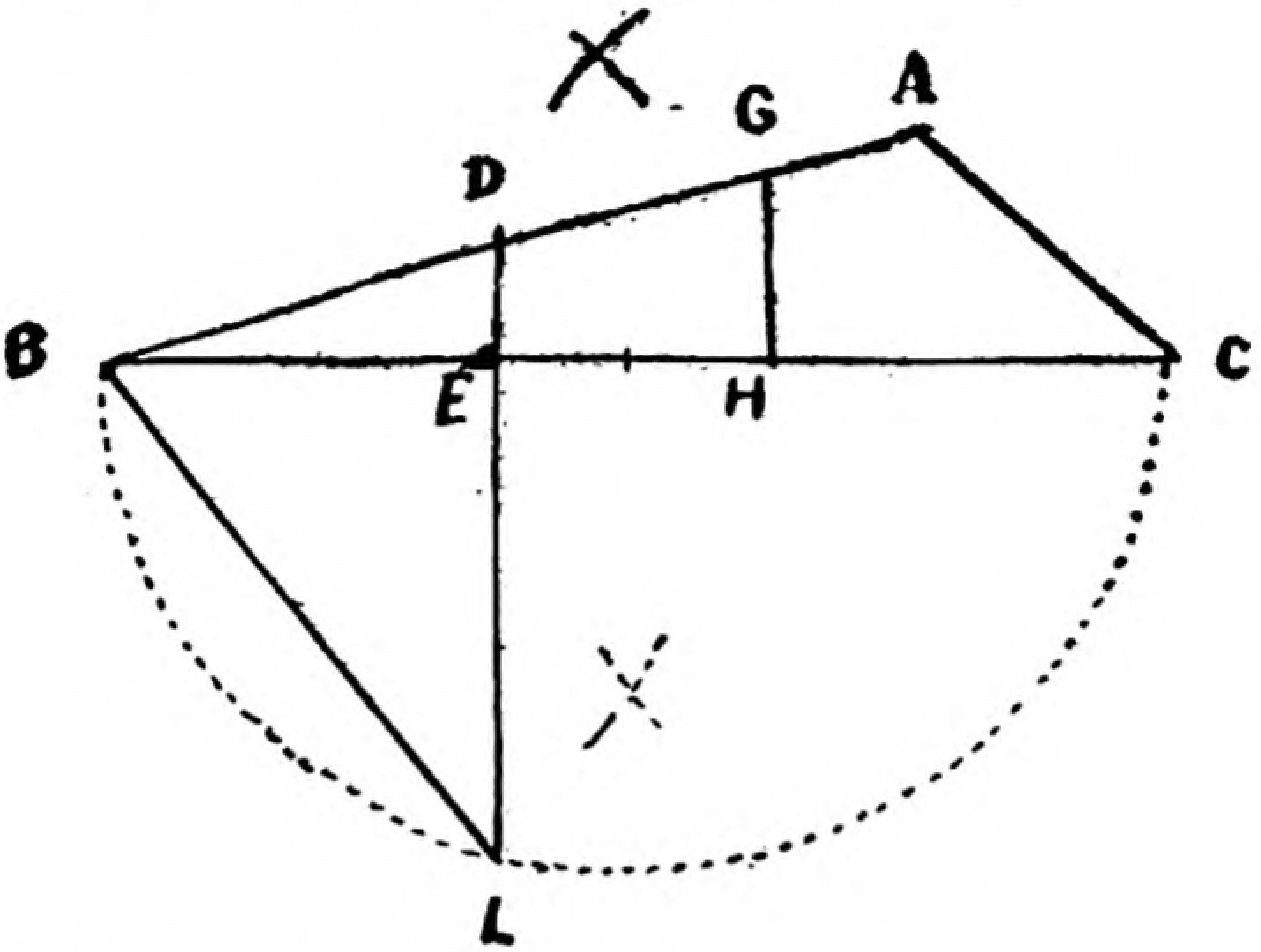
Draw BL mean proportional between BC and BE. off BH = BL.

Draw HG DE.

Then HG biscects the △ ABC and ‖ to DE ‖ R

Q.E.F.

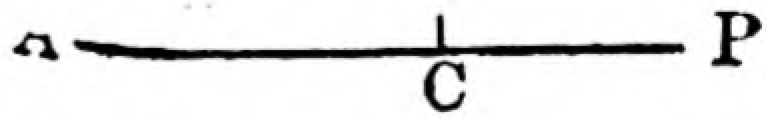
13. Divide a $\triangle$ ABC into two equal parts by a st, line drawn perpendicular to BC.



Hint:—Bisect AB in D.
Draw DE $\parallel$ BC
Draw BL mean proportional between BC and BE
Cut off BH=BL
Draw GH $\parallel$ DE.
Then GH is $\perp$ BC and bisects the $\triangle$ ABC.

Q.E F.

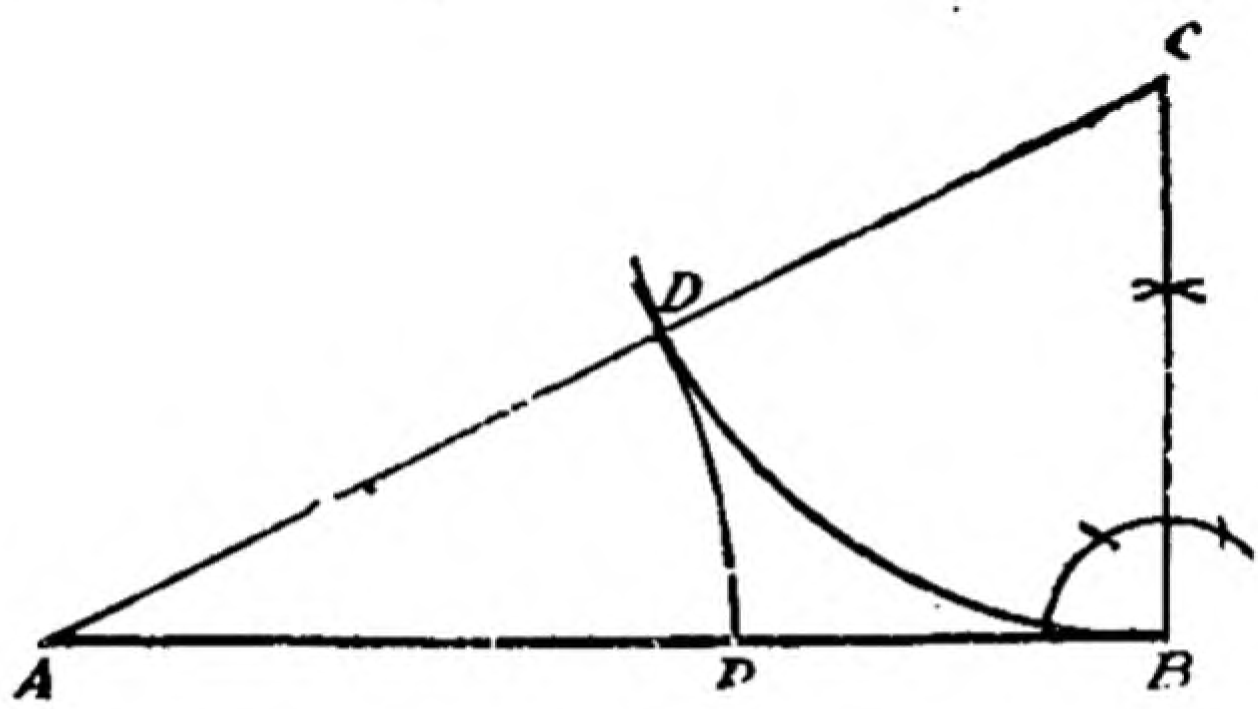
Def. A straight line is said to be divided in **extreme and mean ratio** when the square on one part is equal to the rectangle contained by the whole line and the other part. For instance, if $AC^2 = AB.BC$, the line AB is said to be divided at C in extreme and mean ratio



The division of a line in extreme and mean ratio is also called division in **Golden Section** or **Medial Section**.

Proposition 80 (*Problem*)

To divide a given straight line in extreme and mean ratio.



Given :—A straight line AB.

Required :—To divide it into two equal parts such that the square on one part be equal to the rectangle contained by the whole line and the second part.

Construction :—At B draw BC perpendicular to AB and cut off BC=$\frac{1}{2}$ AB. Join AC.

With C as centre and radius equal to CB describe an arc cutting AC at D.

With A as centre and radius equal AD draw another arc cutting AB at P.

Then P is the required point.

Proof :—Suppose $AB=a$ and $AP=x$,

$$\therefore\ BC=\frac{a}{2},\ CD=\frac{a}{2},\ \text{and } AD=x.$$

Now $\because\ AC^2=AB^2+BC^2$ ($\angle B=1$ rt. $\angle$).

$$\therefore \left(x+\frac{a}{2}\right)^2=a^2+\frac{a^2}{4}$$

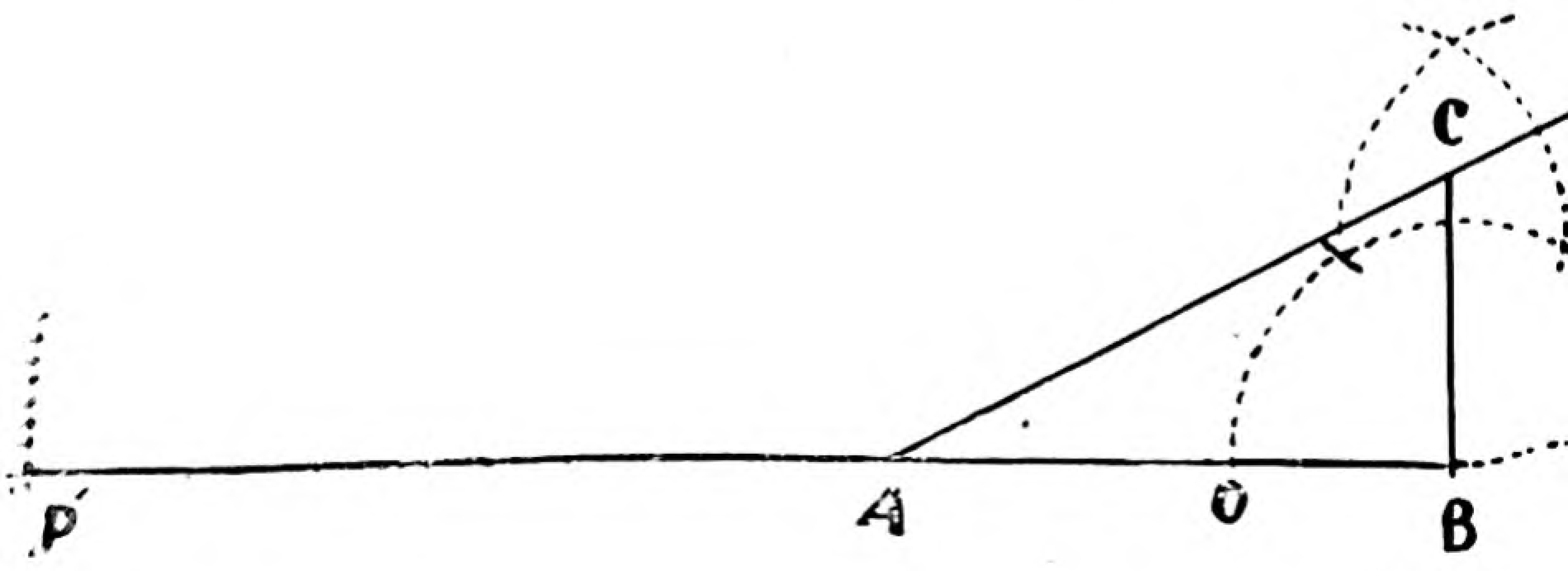
or $x^2+ax=a^2$.

or $x^2=a(a-x)$.

$\therefore\ AP^2=AB.PB$.

Q.E.F.

Note. 1. If from AC *produced* CD′ is cut off equal to CB and AP′=AD′ from BA *produced*, then the line AB shall be divided at P′ externally in golden section. *i.e.* $AP'^2=BP'.AB$.



Note. 2. It will be advantageous if students remember that in medial section the greater segment (as AP in the figure) is always equal to $\frac{\sqrt{5}-1}{2}$ of the whole line. $\frac{\sqrt{5}-1}{2}=$ ·6 nearly.

Exercises

1. Divide a line 3·4″ long in extreme and mean ratio. Measure the length of each part.

(*Punjab*, 1921).

2. Divide AB, 2·3″ long into two parts at C such that AC^2 is equal to AB. BC. Measure the length of each part.

3. *Divide a line 4·6 cm. long in golden section. With two parts as sides construct a rectangle. Draw a square equal in area to this rectangle. Measure the side of the square in inches.*

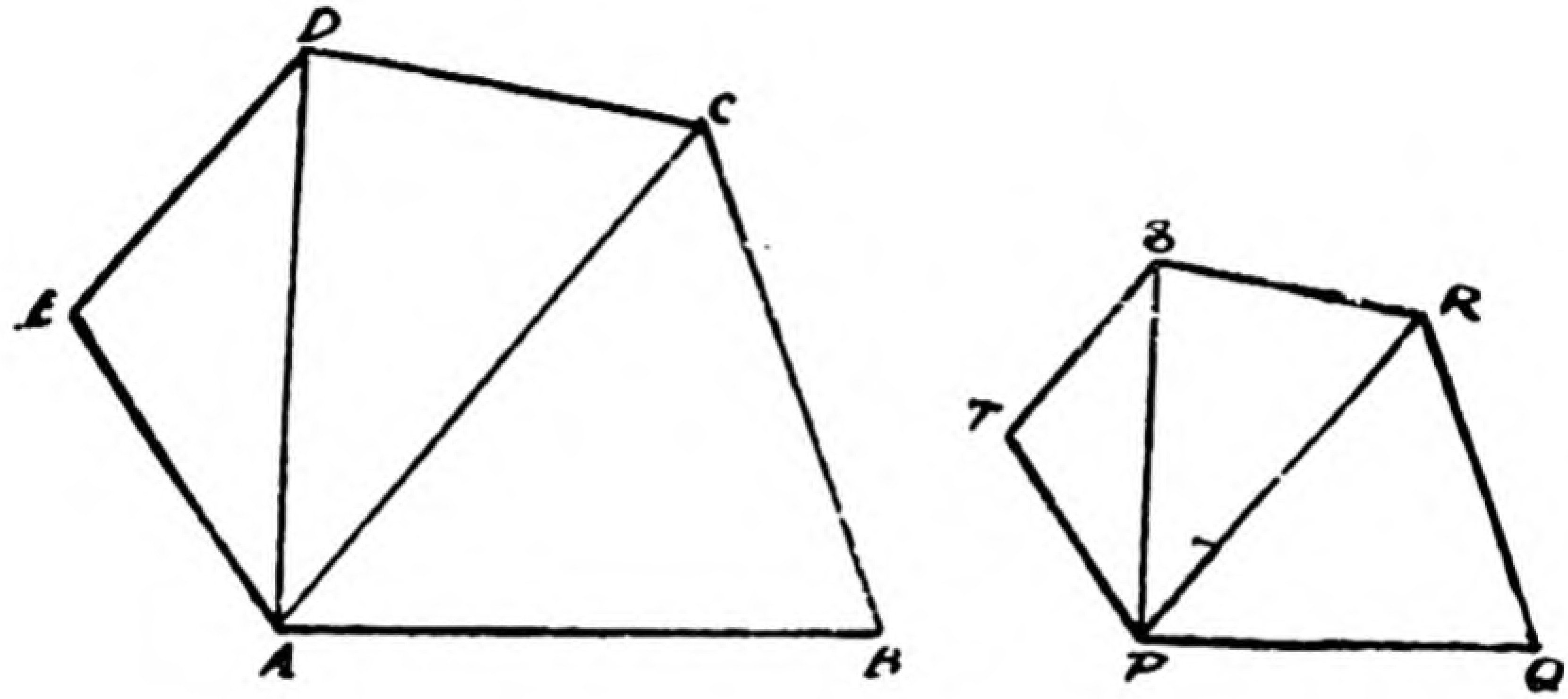
4. *Divide a line 3·1 cm. long externally in medial section.*

5. Draw geometrically st. lines (*a*) $\frac{1}{2}(\sqrt{5}+1)$ and (*b*) $\frac{1}{2}(\sqrt{5}-1)$ inches long.

Hint.—Divide a st. line AB=1″ in its medial section at M. AM is the required length in each case.

Proposition 81 (*Problem*)

On a given straight line to construct a polygon similar to a given polygon.



Given :—A polygon PQRST and a straight line AB.

Required :—To construct on AB a polygon similar to PQRST.

Construction :—Join PR, PS.

Make angles BAC and ABC equal to angles QPR and PQR respectively.

Make angles CAD and ACD equal to angles RPS and PRS respectively. Make angles DAE and ADE equal to $\angle$s SPT and TSP respectively. Then PQRST is the required polygon.

Proof :—The polygons ABCDE, PQRST are evidently equiangular by construction.

Now $\because$ $\triangle$s ABC, PQR are equiangular (Const).

$$\therefore \frac{AB}{PQ}=\frac{BC}{QR}=\frac{AC}{PR} \qquad ......(i)$$

$\therefore$ $\triangle$s ACD, PRS are equiangular (Const.)

$$\therefore \frac{AC}{PR}=\frac{CD}{RS}=\frac{AD}{PS} \qquad ......(ii)$$

and $\because$ $\triangle$s ADE, PST are equiangular (Const.)

$$\therefore \frac{AD}{PS}=\frac{DE}{ST}=\frac{AE}{PT} \qquad ......(iii)$$

Therefore $\frac{AB}{PQ}=\frac{BC}{QR}=\frac{CD}{RS}=\frac{DE}{ST}=\frac{AE}{PT}$ From (*i*), (*ii*) and (*iii*).

Hence polygon ABCDE is similar to the given polygon PQRST and has been constructed on the given line AB.

Q. E. F.

Exercises

1. Draw a quadrilateral ABCD having given AB=·3″, BC=1·9″, DA=1·8″, $\angle A=45°$ and $\angle B=60°$. On a straight line 3·8″ long construct a quadrilateral similar to it.

2. Inscribe a regular hexagon in a circle of 2″ radius. Construct another hexagon similar to it on a line 3·5 cm. long.

3. Draw a figure similar to a given figure ABCDE and having the sides in the ratio of 2 : 3. Compare their areas.

4. Draw a △ similar to a given △ but equal to thrice its area.

Hint:—Areas of two △s are proportional to the squares on their corresponding sides.

5. Make a quadrilateral similar to a given quadrilateral having one fourth its area.

Hint.—Construct an equiangular quad. on half the side of the given quad.

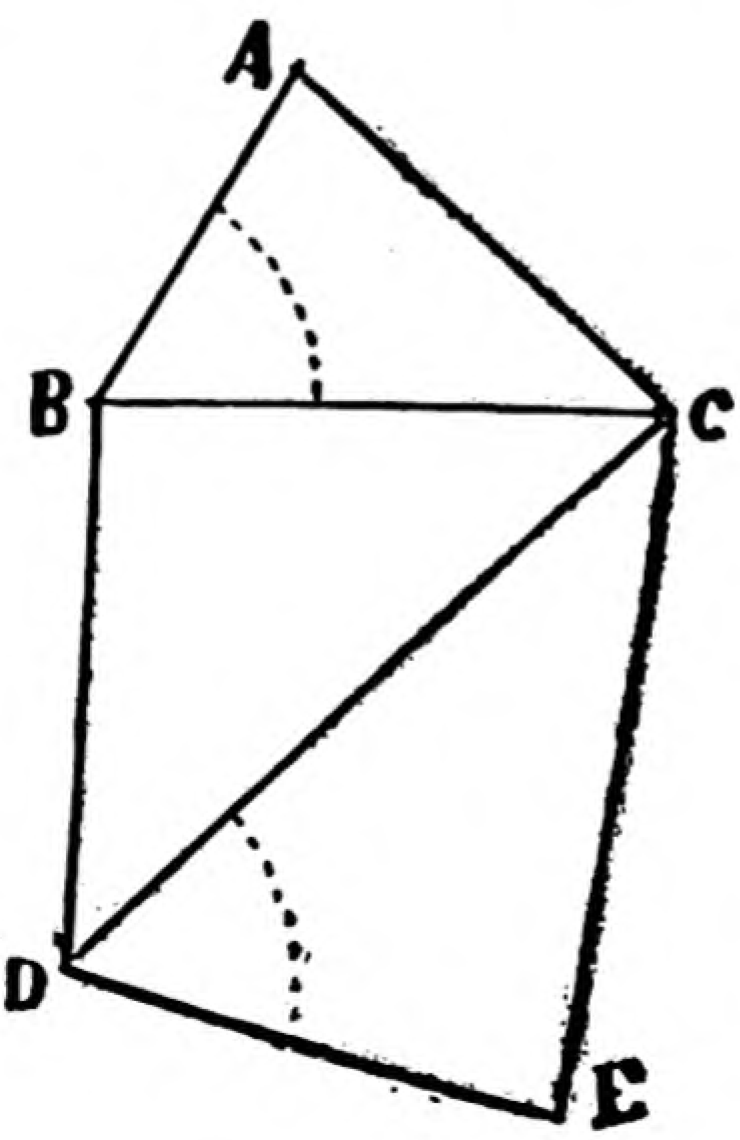
6. Construct a quad. similar to the quad. in the above exercise but four times its area.

Hint.—Take double sides.

7. Construct a △ similar to a given △, but double its area.

Hint.—ABC is given △. Draw $BD \perp BC$ and $= BC$. Join CD.

Make $\angle DCE = \angle ACB$ and $\angle CDE = \angle ABC$ then △DCE is the rqd. △.



8. Construct a polygon similar to a given polygon ABCDE and equal $\frac{3}{5}$ its area.

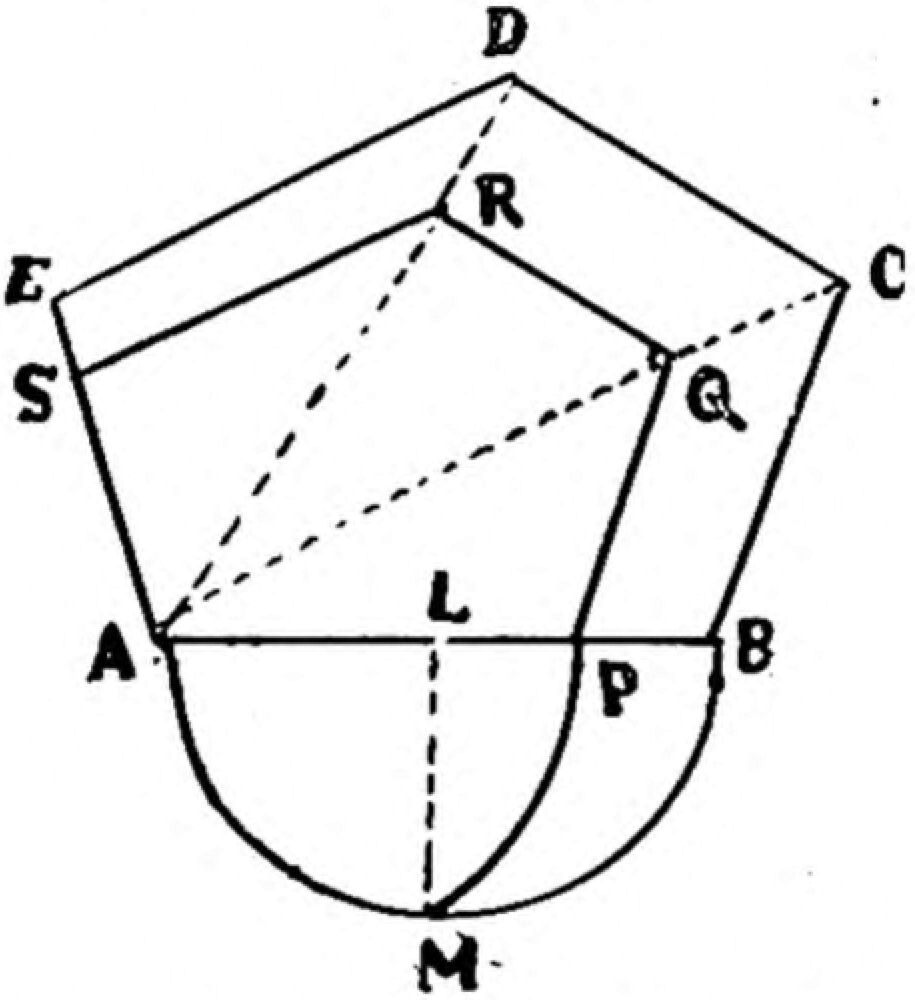
Hint.—Join AC, AD. Cut off $AL=\frac{3}{5}$ AB. Cut off $AP=AM=$mean proportional between AB and AL.

Draw $PQ \parallel BC$,

$QR \parallel CD$

and $SR \parallel DE$

Then APQRS is the rqd. polygon.



Miscellaneous Exercises on Ratio and Proportion and Similar Triangles

1. The bisector of the vertical angle of an isosceles △ bisects the base.

2. If the bisector of the vertical angle of a triangle bisects the base, the triangle is isosceles.

3. ABC, DBC are two triangles such that $\frac{BA}{AC}=\frac{BD}{DC}$. Show that the bisectors of the angles A and D meet on the base BC.

4. FE is drawn parallel to the side BC of a given triangle ABC and meets the sides AB, AC in F, E respectively; show that the bisectors of the angle A divide FE and BC in the same ratio.

5. AD, the bisector of the angle A of the triangle ABC meets BC in D; FE is any st. line parallel to BC meeting AB, AC, in F, E, respectively; show that $\frac{BD}{DC}=\frac{BF}{CE}$.

6. The sides of one triangle are respectively perpendicular to the sides of another, each to each; prove that the triangles are similar.

7. The ratio of the areas of two similar rectilineal figures is equal to the ratio of the squares on the corresponding sides.

8. Compare the areas of regular hexagons described in and about a given circle.

9. On the sides of a right-angled triangle three regular hexagons are described; prove that the sum of the areas of the hexagons on the sides containing the rt. angle is equal to the area of the hexagon on the hypotenuse.

10. Two triangles, which have their sides parallel each to each are similar.

11. The perimeters of similar polygons are proportional to their corresponding sides.

12. The bisector of the angle A of the triangle ABC meets BC in D; that of the angle B meets AD in I; show that CI bisects the angle C.

13. The line joining the mid-points of the parallel sides of a trapezium passes through the intersection of the diagonals.

14. Through O, the intersection of the diagonals of a trapezium, a line is drawn parallel to the parallel sides and cutting the oblique sides in A and B. Prove that AB is bisected at O.

15. *A common tangent to two circles cuts the line joining their centres, internally or externally, in the ratio of their radii.*

Hint.—The common tangents meet the line of centres in the same point; let the direct common tangent TT′ meet the line of centres AB produced in S. Then from similar $\triangle$s ATS and BT′S $\frac{AS}{BS}=\frac{AT}{BT'}$, thus the line AB is divided externally in S in the ratio of the radii of the circles. Similarly we can prove that the transverse common tangent divides AB internally at S′ in the ratio of the radii.

16. If two triangles are similar, their areas are proportional to the squares on (*i*) their in-radii, (*ii*) their corresponding medians, (*iii*) their circum-radii.

17. Bisect a triangle by a st. line parallel to one of the sides.

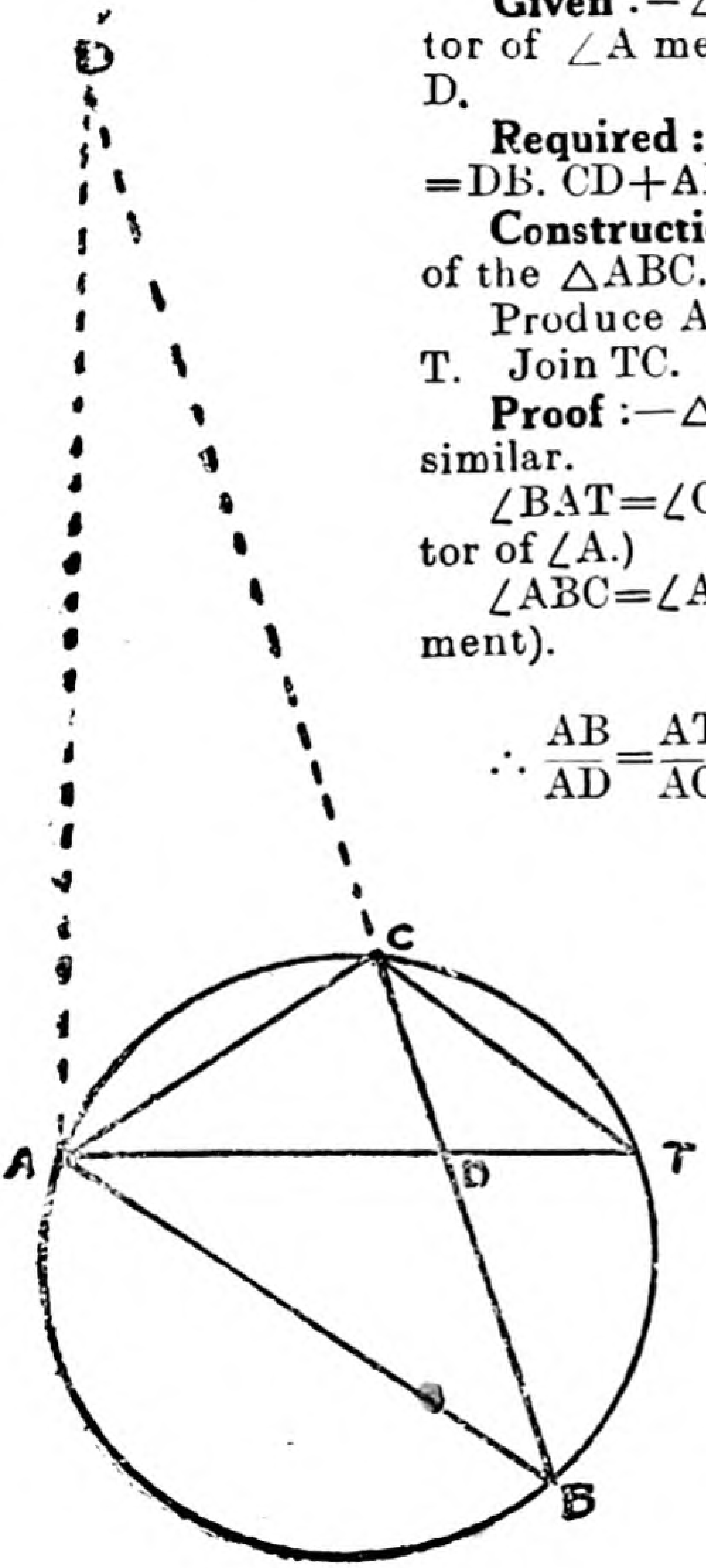
18. *Construct a triangle, having given the base, the ratio of the sides and the vertical angle.*

Hint.—On the base describe a segment of a circle capable of the vertical $\angle$ and divide the base in the ratio of the sides.

19. The locus of a point which moves so that the ratio of its distances from two fixed points is constant is a circle (Circle of Apollonius).

Hint.—See Prop. 85 Ex. 9.

20. *If the internal bisector of the vertical angle of a triangle meets the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base together with the square on the bisector.*



Given :—$\triangle$ABC and DA bisector of $\angle$A meeting the base BC at D.

Required :—To prove that AB. AC $=$DB. CD$+$AD^2.

Construction :—Draw circumcircle of the $\triangle$ABC.

Produce AD to meet the circle at T. Join TC.

Proof :—$\triangle$s ABD and ATC are similar.

$\angle$BAT$=\angle$CAT (AD being the bisector of $\angle$A.)

$\angle$ABC$=\angle$ATC ($\angle$s in the same segment).

$$\therefore \frac{AB}{AD}=\frac{AT}{AC}.$$

Or AB. AC$=$AT. AD

$=$(AD$+$DT) AD

$=$AD$^2+$DT. AD

But DT. AD

$=$BD. CD (Being products of the segments of two intersecting chords).

$\therefore$ AB. AC$=$BD. CD$+$AD.^{2}

21. *If from the vertical angle of a triangle a perpendicular is drawn to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circumcircle. (Theorem due to Brahma Gupta.)*

22. *The diagonal AC of a cyclic quad. ABCD is bisected by BD. Show that AB. AD=CB. CD.*

Hint —Let D′ be the circum-diameter and AE and CF ⊥ s to BD

Then AB. AD=D′.AE
and CB. CD=D′. CF

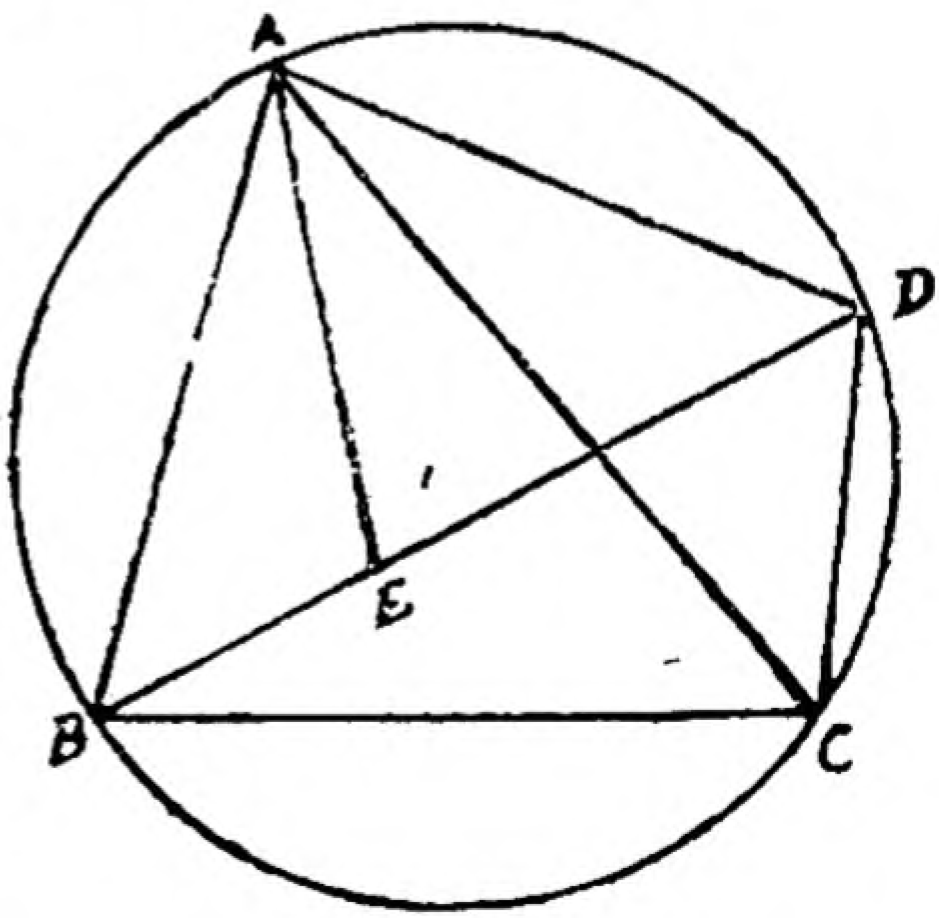
but AE and CF are equal ∵ △s AEO and COF are congruent ∴ AB. AD=CB. CD.

23. *The rectangle contained by the diagonals of a cyclic quadrilateral is equal to the sum of the two rectangles contained by the pairs of opposite sides.*

Given :—ABC a cyclic quadrilateral with disgonals AC and BD.

Required :—To prove that AC. BD=AB.C D+AD. BC.

Construction :—Make ∠BAE=∠DAC.



Proof :—In two △s ABE and ACD ∠BAE=∠CAD (Const).

∠ABE=∠ACD (∠s in the same segment).

∴ △s are similar

$$\therefore \frac{AB}{BE}=\frac{AC}{CD} \text{ or}$$

AB. CD=AC. BE... (*i*)

Again △s EAD and BAC are similar.

$\therefore \angle EAD = \angle BAC$

and $\angle C = \angle D$.

$\therefore \dfrac{AD}{DE} = \dfrac{AC}{CB}$ or $AD.CB = DE.AC$....................(*ii*)

Adding (*i*) and (*ii*) both sides

$$AB.CD + AD.CB = AC.BE + AC.DE.$$
$$= AC(BE + DE)$$
$$= AC.CD.$$

This theorem is known as *Ptolemy's theorem.*

24. *Enunciate and prove the converse of Exercise* 23.

Hint.—If the rectangle contained by the diagonals of a quadrilateral is equal to the sum of the rectangles contained by pairs of opposite sides, the quadrilateral can be inscribed in a circle.

(*ii*) Use the theorem : "The rectangle contained by the diagonals of *any* quadrilateral is *less* than the sum of the two rectangles contained by pairs of opposite sides." (*Reductio ad absurdum*).

25. *If the diagonals of a cyclic quadrilateral are at rt. $\angle$s, the sum of the rectangles contained by pairs of opposite sides is equal to twice the area of the quadrilateral.* (*Twice area = product of diagonals, use Ex.* 23).

26. The st. lines which join the vertex of a triangle to the points which divide the base internally and externally in the ratio of the sides are at rt. $\angle$s

Hint.—The internal and external bisectors are at rt. angles.

27. AD bisects the ext. $\angle A$ of the $\triangle ABC$ and meets BC produced in D ; CE bisects the angle ACD and meets AD in E ; show that BE bisects the $\angle B$.

Hint.—$\because$ AD is bisector of ext. $\angle A$ $\therefore \dfrac{BD}{CD} = \dfrac{AB}{AC}$

or $\dfrac{AB}{BD}=\dfrac{AC}{CD}$ (i)

$\therefore$ CE bisects $\angle ACD$ $\therefore \dfrac{AC}{CD}=\dfrac{AF}{DE}$ (ii)

From (i) and (ii) $\dfrac{AB}{BD}=\dfrac{AE}{DE}$ $\therefore$ BE bisects $\angle ABD$.

28. The bisector of $\angle A$ of the $\triangle ABC$ meets the opposite side in D ; DE, parallel to AC, meets AB in E ; show that $\dfrac{BE}{EA}=\dfrac{AB}{AC}$.

Hint.—AD bisects $\angle A$ $\therefore \dfrac{AB}{AC}=\dfrac{BD}{CD}$ $\because$ DE $\parallel$ AC.

$\therefore \dfrac{BD}{CD}=\dfrac{BE}{AE}$ hence $\dfrac{BE}{AE}=\dfrac{AB}{AC}$.

29. ABC is a quad. E, F the points of trisection of AB, BC nearest B, and G, H the points of trisection of CD, DA nearest D ; Show that EFGH is a $\|^m$.

Hint.—Join AC, EF is $\parallel$ AC and $\frac{1}{3}$ AC. GH is $\parallel$ AC and $\frac{1}{3}$ AC. $\therefore$ EF and GH are equal and $\parallel$. Hence EFGH is a $\|^m$

30. In two similar triangles (i) altitudes, (ii) medians, (iii) the bisectors of the angles are proportional to corresponding sides.

31. The lines drawn from the ends of the base of a triangle perpendicular to the bisector of the vertical angle are in the same ratio as the sides of the triangle.

Hint.– BE, CF are $\perp$s to the bisector AEDF of $\angle BAC$. $\triangle$s BDE and CDF are equiangular,

$$\therefore \frac{BD}{CD}=\frac{BE}{CF}$$

$\therefore$ AF bisects $\angle A \therefore \dfrac{BD}{CD}=\dfrac{AB}{AC}$ $\therefore \dfrac{BE}{CF}=\dfrac{AB}{AC}$.

32. ABC is a triangle; D, E, are points which divide BC internally and externally in the same ratio; O is the mid-point of DE; show that OA = OD = OE.

Hint.—AD, AE are internal and external bisectors of $\angle A \therefore \angle DAE$ is a rt. $\angle$ and O the mid-point of hypotenuse DE $\therefore$ AO = DO = EO.

33. The angle A of a triangle ABC is bisected by a line which cuts the base in D and the circum-circle in E; prove that $\dfrac{BA}{EA}=\dfrac{AB}{AC}$.

Hint—Join EC. $\triangle$s ABD, ACE are equiangular.
$\therefore \angle BAD=\angle CAE, \angle B=\angle E \therefore \dfrac{AB}{AE}=\dfrac{AD}{AC}$.

34. ABC is a triangle: E, F are points in CA, AB respectively, such that $\dfrac{EA}{FA}=\dfrac{BA}{AC}$ show that the quadrilateral BCEF is cyclic.

Hint.—$\therefore \dfrac{AE}{AF}=\dfrac{AB}{AC}$ and $\angle A$ is common,

$\therefore$ $\triangle$s AEF and ABC are equiangular or similar.
$\therefore \angle AEF=\angle B$ $\therefore$ BCEF is cyclic.

35. If BE, CF be altitudes of a triangle ABC, the triangles AEF, ABC are equiangular to each other and hence show that quadrilateral BCEF is cyclic.

36. $\|^{ms}$ of the same altitude are to one another as their bases.

37. Through the vertices A, B, C of an equilatera triangle st. lines are drawn perpendicular to AB,

BC, CA respectively, so as to form another equilateral triangle; compare the areas of the two triangles.

Hint.—Through A, FAD, through B, FBE; through C, DCE are perpendiculars drawn to meet at D, E and F and from equilateral $\triangle$ DEF. Let side AC or BC $=x$. In rt. $\angle d$ $\triangle$ACD, AC$=x$;

$$\therefore \quad CD=\frac{x}{\sqrt{3}}.$$

$$\therefore \quad \angle CAD=30^\circ, \text{ similarly } \because BC=x \therefore BE=\frac{x}{\sqrt{3}}.$$

$$\therefore \quad EC=\frac{2x}{\sqrt{3}} \therefore DE=\frac{x}{\sqrt{3}}+\frac{2x}{\sqrt{3}}=\frac{3x}{\sqrt{3}}=\sqrt{3}x.$$

Now equal $\triangle$s ABC and DEF are similar.

$$\therefore \quad \frac{\triangle ABC}{\triangle DEF}=\frac{BC^2}{DE^2}=\frac{x^2}{3x^2}=\tfrac{1}{3}.$$

38. The area of the square formed by joining the mid-points of the sides of a given square is one-half the area of the latter.

39. The sides AB, BC, CD, DA of a square are bisected in E, F, G, H: show that the sqaure enclosed by AF, BG, CH, DE is one-fifth of the given square.

Hint.—DE cuts CH at P and AF at Q; BG cuts CH at S and AF at R. Then PSRQ is a a square$=\frac{1}{5}$ sq. ABCD. Prove CH and AF$=$and parallel also to BG and DE$=$and parallel $\triangle$s ADE, ABF are$\equiv$hence $\angle$EAQ$=\angle$ADP and $\angle$AED$=\angle$AFB hence prove $\angle$EAQ$+\angle$AEQ$=$1rt. $\angle$. $\therefore$ $\angle$AQE$=$1 rt. $\angle=\angle$PQR. Similarly $\angle$s at R, S, P are rt. $\angle$s. Again $\triangle$s ADQ, ABR are congruent $\therefore$ DQ$=$AR and DP$=$PQ. Similarly AQ$=$QR $\therefore$P Q$=$QR and PQRS' is a rect. $\therefore$ PQRS is a square. Join AP, PR $\therefore$ AQ$=$QR $\therefore$ $\triangle$APQ$=\triangle$PQR. But $\triangle$APQ$=\frac{1}{2}$ $\triangle$ADQ and $\triangle$PQR$=\frac{1}{2}$ sq. PQRS. $\therefore$ sq. QS'$=\triangle$ADQ. Similarly

sq. QS $=\triangle ABR=\triangle BCS=\triangle CPD$ $\therefore$ sq QS$=\frac{1}{5}$th sq. ABCD.

40. *Cut off the third part of a triangle by a st. line parallel to one of the sides.*

Hint. – Trisect the side AB of a $\triangle ABC$ at P and Q. P nearest to A ; on AB as diameter describe a semi$\odot$ and from P draw PR $\perp$ AB cutting semi @ at R, with centre A and radius AR describe an arc cutting AB at X, through X draw XY $\parallel$ BC. Then $\triangle AXY=\frac{1}{2}\triangle ABC$.

41. In a cyclic quadrilateral ABCD, AB : CD$=$ AD : BC ; prove that the diagonal AC bisects the diagonal BD.

Hint .—(Converse of Ex. 22) $\therefore\ \dfrac{AB}{CD}=\dfrac{AD}{BC}$

$\therefore$ AB, BC$=$AD.CD. Let AC and BD cut at O, let BE, DF be $\perp$s to AC $\because$ the quadrilateral ABED is cyclic,

$\therefore$ AB.BC$=$BE circum-diameter
and AD.CD$=$DF, circum-diameter.

$\therefore$ BE$=$DF $\therefore$ $\triangle$s BEO and DOF are $\equiv$, hence BO$=$DO.

42. The rectangle contained by two sides of a triangle is never less than twice the triangle.

43. If the bisectors of two angles of a triangle are equal, the triangle is isosceles.

Hint:—ABC is a $\triangle$, bisectors (BD, CE) of $\angle$s B and C cut AC, AB respectively in D, E ; BD$=$CE then AB$=$AC.

Or $\angle ABC=\angle ACB$ if not, $\angle ABC>\angle ACB$.

$\therefore$ $\angle DBC>\angle ECB$. Then in $\triangle$s DBC, ECB,

BC$=$BC, BD$=$CE, $\angle DBC>\angle ECB$ $\therefore$ CD$>$BE. (*i*)

Let EC, BD cut in P. On the other side of BC from construct $\triangle$ BQC such that BQ$=$EC, QC$=$EB. Join DQ. Then $\therefore$ BQ$=$BD $\therefore$ $\angle BDQ=\angle BQD$. Also

in $\triangle$s EPB, DPC $\angle EPB=\angle DPC$, $\angle EBP>\angle DCP$ (hyp.)
$\therefore$ $\angle PDC>\angle PEB=\angle BQC$.

$\therefore \angle BDC>\angle BQC$ and $\angle BDQ=\angle BQD$.

$\because \angle BDC>\angle BQC$ and $\angle BDQ=\angle BQD$ (proved)

$\therefore$ ADC $>$ DQC $\therefore$ CQ$>$QO $\therefore$ BE$>$CD (*ii*)

(i) and (ii) are contradictory $\therefore$ $\angle ABC>\angle ACB$ is false. Similarly $\angle ACB>\angle ABC$ is false. $\therefore$ $\angle ABC=\angle ACB$ $\therefore$ AB$=$AC.

44. Show how to draw a st. line XY $\parallel$ BC the base of a triangle ABC, so that the area of the $\triangle$ may be $\frac{9}{7}$ of the figure BCYX.

Hint :—If area of BCYX$=1$, area of $\triangle$ AXY$=\frac{9}{7}$.

$\therefore$ Area of $\triangle ABC=1+\frac{9}{7}=\frac{16}{7}$.

$\therefore$ Ratio of the area $\triangle$s AXY and ABC$=\frac{9}{7}:\frac{16}{7}=9:16$

Ratio of sides$=3:4$

$\therefore$ If the side AB of the $\triangle$ABC be divided into 4 equal parts, AX$=\frac{3}{4}$ of AB ; hence the construction.

45. *Bisect a $\triangle$ by a st. ine $\perp$ to the base.*

Hint :—Draw AD $\perp$ BC, the base of the $\triangle$ABC. Bisect BC at E. Draw EF $\perp$ BC. On BA find a point X so that BX is a mean proportional to BA and BE. Draw XY $\perp$ BC.

46. *Draw a st line in the ratio $\sqrt{2}:\sqrt{3}$.*

47. *Construct a figure similar to a given rectileal figure and equal to $\frac{3}{4}$ of its area.*

———

Proposition. 82. (*Problem*)

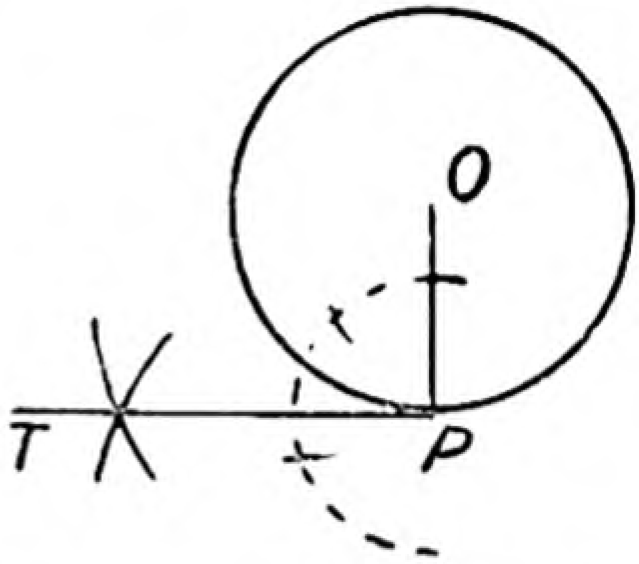
To draw a tangent to a given circle at a given point on its circumference.

Given :—A circle with centre O and a point P in the circumference.

Required :—To draw a tangent at P.

Construction :— Join OP. Draw PL ⊥ OP.

Then PT is the required tangent.



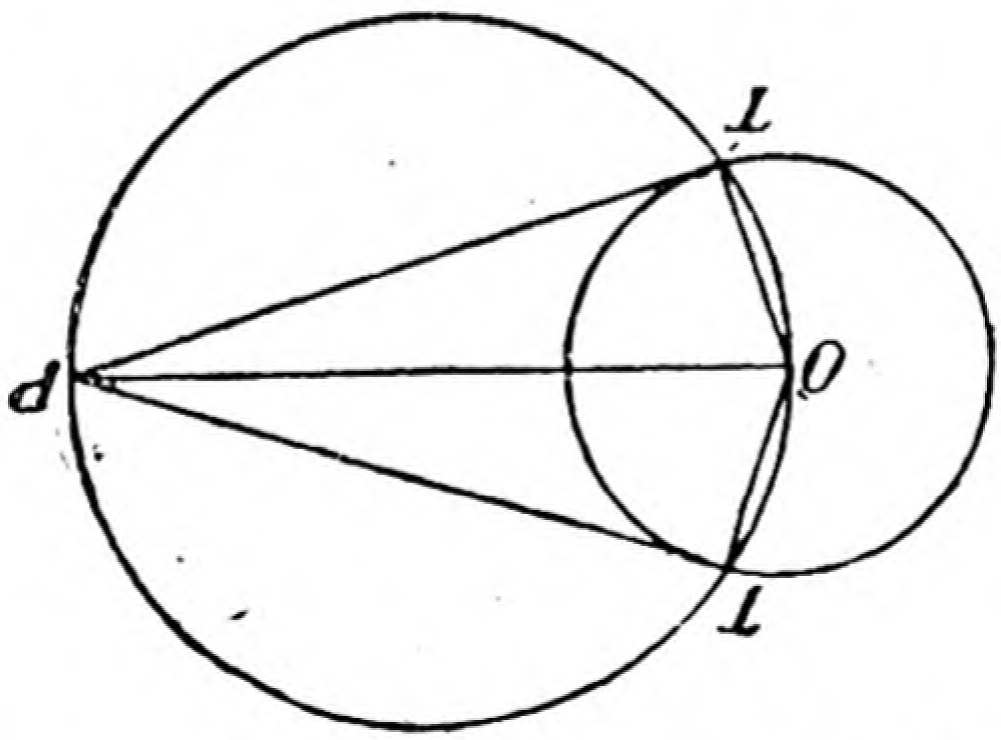
Proof: —∠OPT is a right angle, and OP is a radius.

∴ PT is the tangent.

Q. E. D.

Proposition 83. (*Problem*).

To draw a tangent to a given circle from a point outside it.



Given :—A circle with centre O, and a point P outside the circle.

Required :—To draw a tangent to the circle from P.

Construction :—Join OP.

On OP as diameter describe a circle OTPT′ cutting the given circle at T and T′

Join PT, PT′. Then PT, PT′ are the required tangents.

Proof :—Join OT, OT′, ∠OTP is a right angle being an angle in the semi-circle, and OT is the radius.

∴ PT is the tangent from P.

Similarly PT′ is also a tangent from the same point P.

Q. E. F.

Note.—Students should note carefully that from any point outside the circle two tangents can always be drawn to the circle.

Exercises

1. *If from a point distance d from the centre of a ⊙ of radius r a pair of tangents is drawn, prove that*

(*i*) *The length of each tangent* $=\sqrt{d^2-r^2}$.

(*ii*) *The distance of the chord of contact from the centre is* $\dfrac{r^2}{d}$.

(*iii*) *The length of the chord of contact is*

$$\frac{2r\sqrt{d^2-r^2}}{d}$$

2. Draw a ⊙ of radius 4 cm. and from a pt. which is at a distance of 8·5 cm., from the centre draw a pair of tangents to the ⊙. Measure the lengths of the tangents and the ∠ included between the tangents.

(Alig., 1922)

3. Draw a ⊙ of radius 1″ and from a pt. 2″ distant from the centre, draw two tangents to the ⊙.

Measure the length of any tangent. Verify it by calculation. (Alig. 1923).

4. *Draw a ⊙ of radius 3·7″. Draw a line through the centre. On it take two pts. each 2·3″ from the centre and draw tangents to the ⊙ from these points.* (Punjab, 1919)

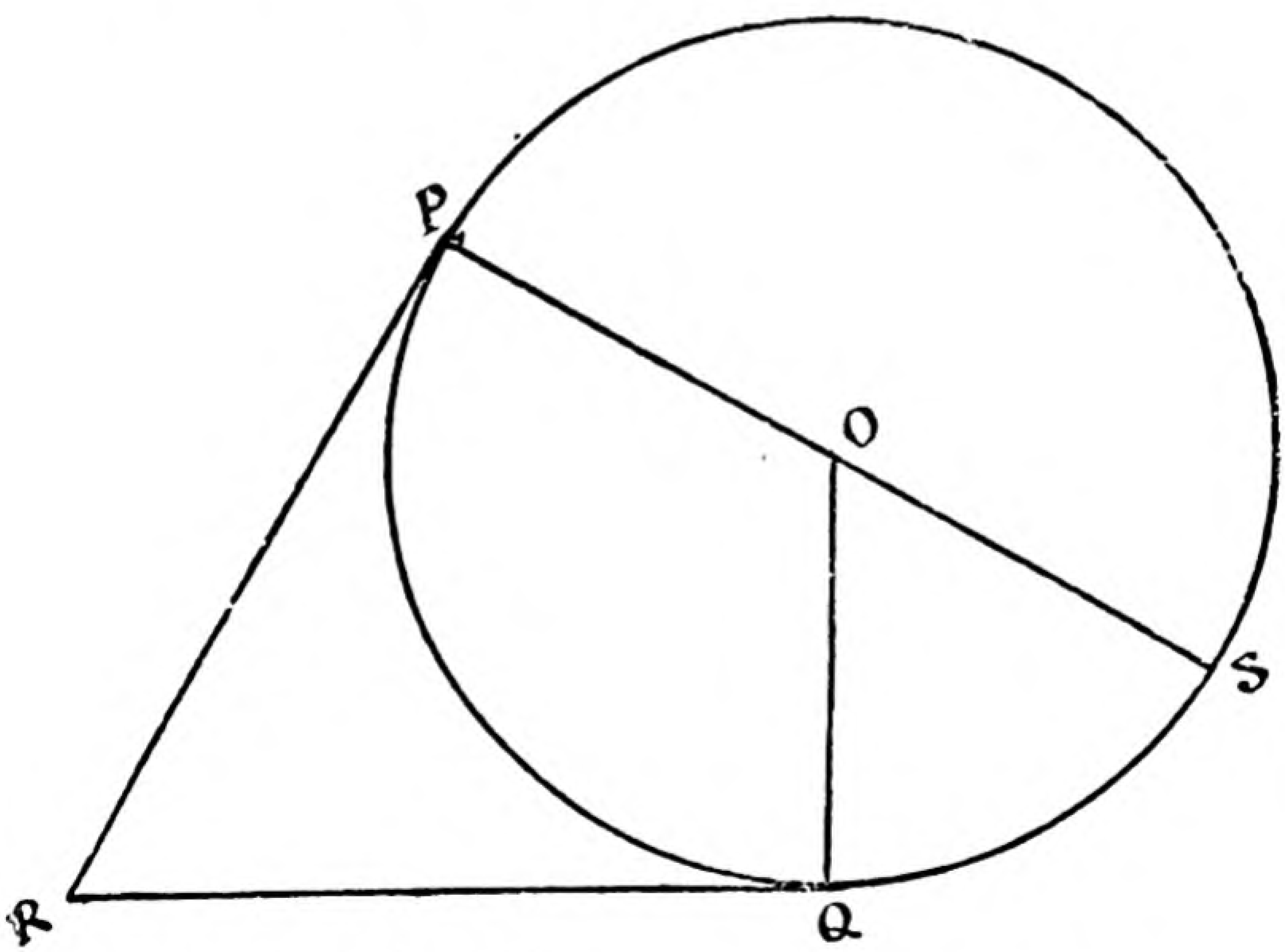
5. Take a st. line AB 3·3″ long and divide it at C so that AC : CB = 3 : 8 ; on CB as diameter describe a ⊙ and draw a pair of tangents to this ⊙ from A.

(Punjab, 1914).

6. *Draw a ⊙ of radius 1·2″ ; draw two tangents to it inclined at an ∠ of 60° to each other.*

(Punjab 1929 & C. P. 1929)

Hint.—Draw a ⊙ of radius 1·2″.



Make ∠SOQ = 60°,

At P and Q draw PR and QR ⊥ to OP and OQ respectively and meeting in R.

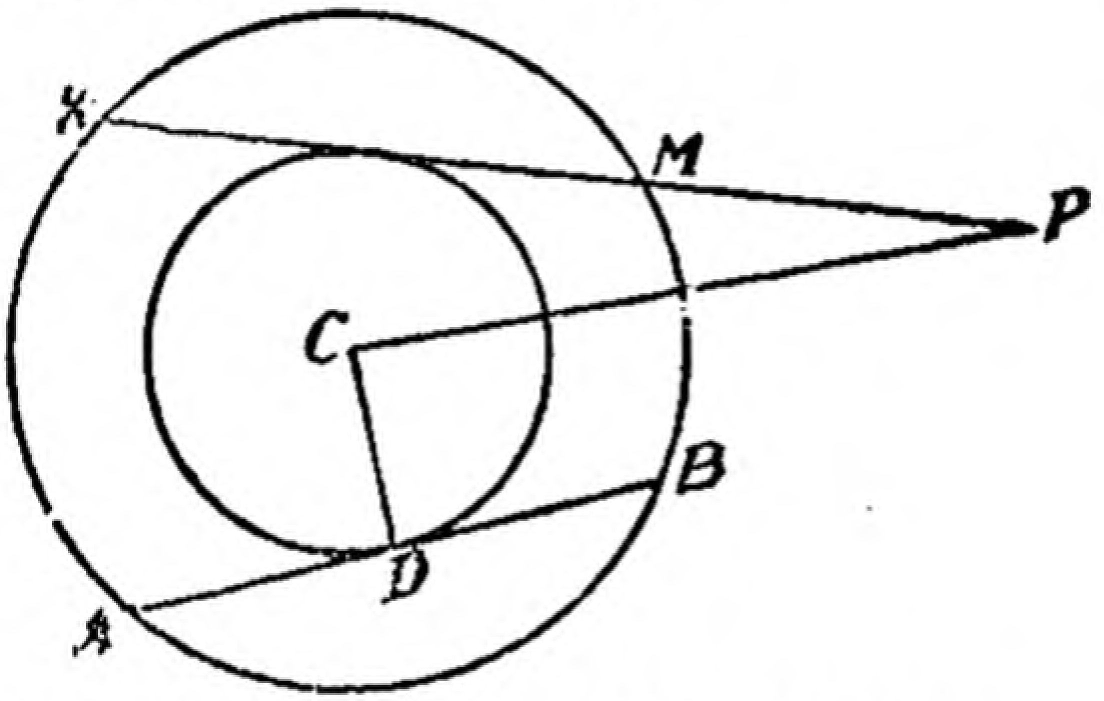
Then RQ and PR are the required tangents.

7. Describe a ⊙ of 3·6″ diameter and in its place a chord 2″ long. Bisect the angle subtended by this chord at the centre. (Punjab, 1912).

8. In a ⊙ of 4 cm., radius draw a chord whose distance from the centre is 3 cm. and measure the length of this chord in millimetres. (Punjab, 1915)

9. *From a point outside a circle draw a straight line such that its portion intercepted within the circle may be equal to a given length.*

ABK is the given circle with centre C and P a point outside it and l a given length.



In ⊙ ABK place AB a chord equal to l.

Draw CD ⊥ to AB ; with centre C and radius CD describe a circle. From P draw a tangent to this circle cutting the given circle at M and K. Then PMK is the required line with the portion $KM = l$.

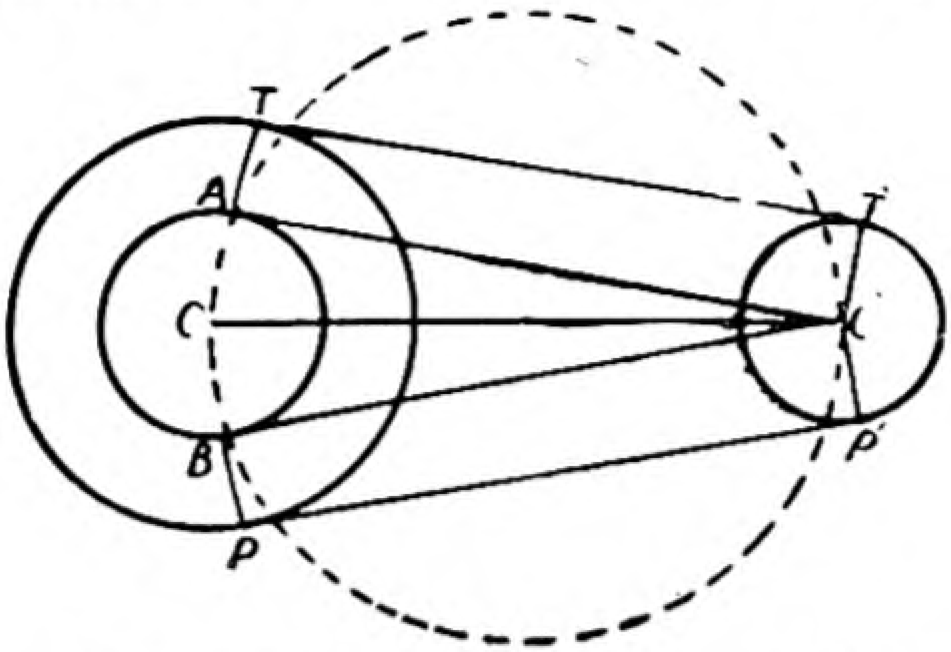
KM and AB are equidistant chords of the given ⊙.

10. Describe a ⊙ having a diameter 5 cm. long and place in it two chords each 4 cm. long such that when they are produced, they meet outside the circle at an angle of 60°.

11. In a ⊙ of ·7″ radius place a chord equal in length to twice its distance from the centre.

Proposition 84 (*Problem*).

To draw direct common tangents to two given circles.



Given :—Two circles with centres C and C′.

Required :—To draw direct common tangents to these two circles.

Construction :—Join CC′. With CC′ as diameter describe a circle. With centre C and radius equal to the difference of the two radii describe a circle cutting the first circle in A and B. Join CA and CB and produce them to meet the original circle in T and P. Draw radii C′T′ and C′P′ parallel to CT and CP respectively.

Join TT′ and PP′.

Then TT′ and PP′ are required direct common tangents.

Proof :—Join AC′ and BC′, CAC′ is a right angle and C′T′ is parallel and equal to AT. ∴ ATT′C′ is a rectangle CT and C′T′ being radii, TT′ is a tangent to both the circles. Similarly PP′ is a tangent to both the circles.

Hence TT′ and PP′ are the required direct common tangents.

Q. E. D.

Exercises.

1. *If R and r are the radii of the two circles and the d distance between the centres, prove that the length of the direct common tangent* $= d^2-(R-r)^2$. *Using this formula calculate the length of the direct common tangent to two* ⊙*s whose radii are* 4″ *and* 1″ *and centres* 5″ *apart.*

2. *Draw two* ⊙*s of* ·8″ *and* ·6″ *radius with centres* 2·8″ *apart. Draw the two external common tangents. Measure their lengths and verify the result by actual calculation.* (*Punjab*, 1918, 1926).

3. Draw direct common tangents of two ⊙s whose radii are 4 cm. and 2 cm. and centres 5 cm. apart. Measure the length of their common chord.

4. *Draw direct common tangents to two* ⊙*s each of* .8″ *radius, having their centres* 2·1″ *apart. Measure one of them.*

5. If two external common tangents being produced, intersect, the point of intersection shall lie on the line of centres.

6. In Example 5 above, show that the tangents are equally inclined to the line of centres.

7. Prove that the direct common tangents of two circles which touch externally is a mean proportional between their diameters.

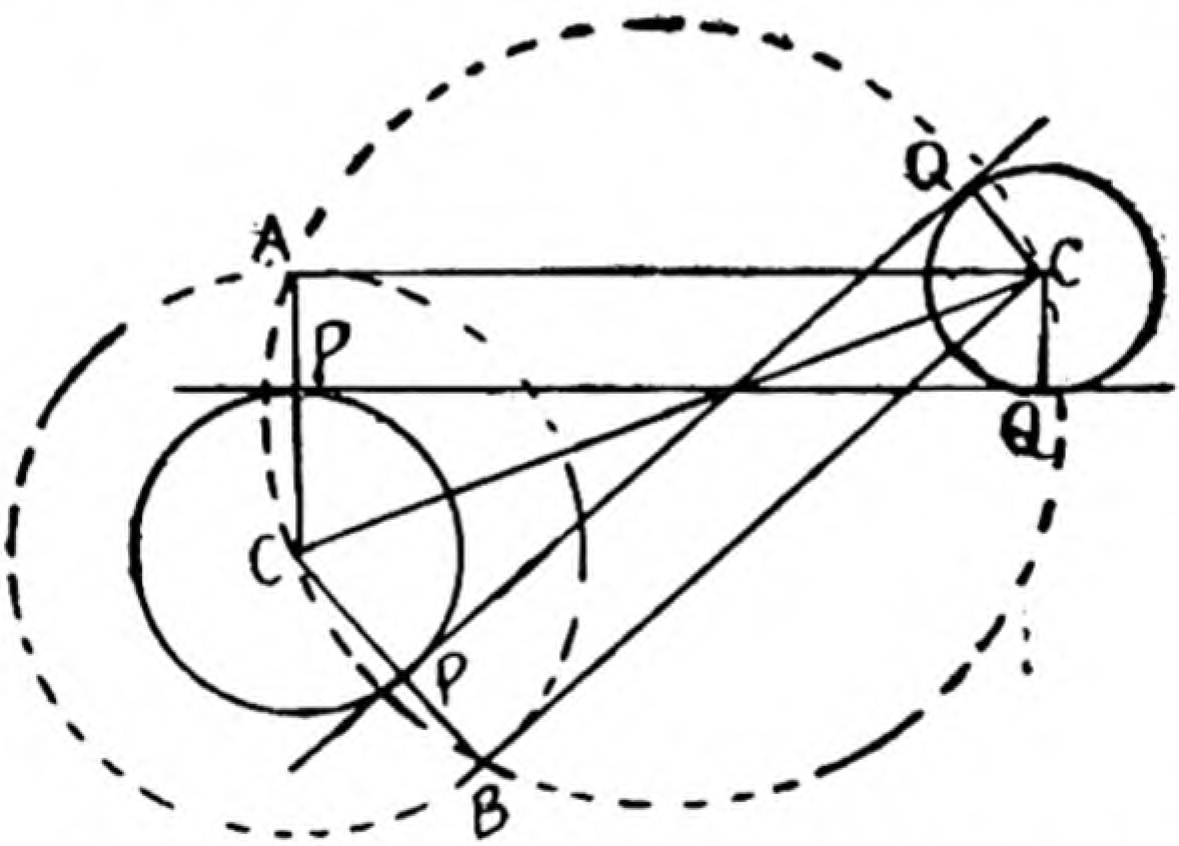
Hint :—In the case of external contact $d=R+r$.

8. Show how to draw a st. line ABCD such that the chords AB, CD, intercepted on it by two given circles may be of given lengths.

9. Two circles whose centres are at P and Q, touch externally at R. The common tangent at R cuts another common tangent TT′ at S. Prove that (*i*) TT′ is bisected at S. (*ii*) ∠TRT′=90°.
(*iii*) the perpendicular to T′T′ through S bisects **PQ**, (*iv*) ∠PSQ=90°. (*v*) The circle on PQ as diameter touches TT′ at S.

Proposition 85 (*Problem*).

To draw direct common tangents to two given circles.



Given :—Two circles with centres C and C′.

Required :—To draw transverse common tangents to these circles.

Construction :—Join CC′. On CC′ as diameter draw a circle. With C as centre and radius equal to the sum of the two radii draw a circle meeting the first one in A and B. Join CA and CB meeting the original circle in P and P′. Draw radii C′Q and C′Q′ parallel to C′P′ and C″P′ respectively.

Join PQ and P′Q′. Then PQ, P′Q′ are the required transverse common tangents

Proof :—Join AC′, BC′. CAC′ is a right angle and CQ is parallel and equal to PA. ∴ PAC′ is a rectangle.

Hence PQ is at right-angles to CP and C′Q. ∴ it is a transverse common tangent.

Similarly P′Q′ is a transverse common tangent.

Q. E. F.

Exercises.

1. *If R, r the radii of the two ⊙s and d the distance between the ecntres, prove that the length of the transverse common tangent.* $=\sqrt{d^2-(R+r)^2}$.

Using this formula find the length of the Transverse Common Tangent to two △s whose radii are 9 mm., 7 mm. and centre 20 mm. apart.

2. *Draw two ⊙s with radii 2·0″ and 0·8″, placing their centres 3·8″ apart. Draw transverse common tangents and find the lengths between the point of contact both by calculation and by measurement.*

3. Prove that the transverse common tangents meet the line of centres in the same pt. and are equally inclined to it.

4. Take two pts. P, Q 2·1″ apart. Draw a st. line AB such that the ⊥s P, Q on it are ·8″ and ·5″ respectively. Show that there are 4 different positions.

Hint:—Draw two ⊙s of radii ·8″, ·5″ and centres 2·1″ apart. Draw direct end transverse common tangents. As there are 4 tangents, there are 4 different positions of AB.]

5. How many common tangents can be drawn in each of the following cases ?

(*i*) When the given circles do not intersect and lie wholly outside each other;

(*ii*) when two circles touch each other ;

(*iii*) when two circles intersect each other ;

(*iv*) when two circles have internal contact ;

(*v*) when one circle lies wholly within the other?

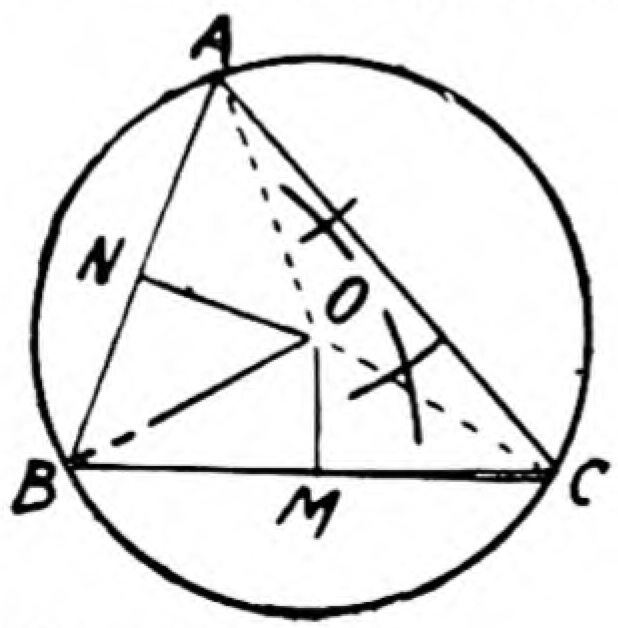
Illustrate your answers in the above cases by drawing two circles of radii 1·3″ and 1″ with distance between the centres in (*i*) 3·1″ (*ii*) 2·3″ (*iii*) 1·3″ (*iv*) ·3″ anb (*v*) ·1″.

6. Draw all the common tangents to two ⨀s whose centres are 1·8″ apart and whose radii are ·6″ and 1·2″. Calculate and measure the length of the direct common tangents. (Delhi)

7. The distance between the centres of two ⨀s is 2·5″ and their radii are 1·3″ and 1·7″. Draw all the common tangents to these circles. (*Punjab*, 1912)

Proposition. 16. (*Problem*).

To describe a circle about a given triangle.



Given :—A triangle ABC.

Required :—To describe a circle about it.

Construction :—Draw OM and ON $\perp$ bisectors of the sides BC and BA respectively meeting at the point O. With O as centre and OB or OC or OA as radius describe a circle. Then ABC is the required $\odot$.

Proof :—Join OB, OC and OA.

Since O lies on the $\perp$ bisector of BC.

$\therefore$ OB=OC.

Also since O lies on the $\perp$ bisector of AB.

$\therefore$ OB=OA.

Hence OA=OB=OC, or O is equidistant from A, B and C.

Hence ABC is the required circle.

Q.E.D.

Exercises.

1. *Construct a $\triangle$ whose sides are* 3″, 4″, 5″ *and describe a $\odot$ about it. Measure its radius and verify it by using the formula* $R=\frac{abc}{4\triangle}$ *where the letters, a, b, c, have their usual meanings and $\triangle$ stands for the area of the triangle.*

2. Construct a $\triangle$ whose sides AB, BC are respectively 2″ and 2·5″ and $\angle$B=120°. Draw the circum $\odot$ of $\triangle$ABC and write down its radius.

(Bombay, 1913).

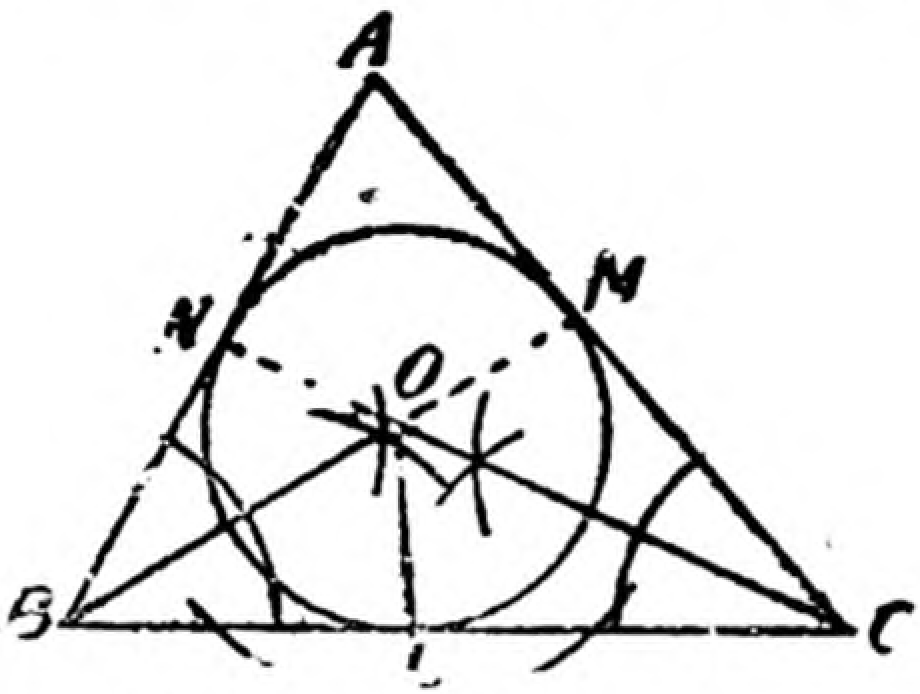
3. Describe a $\odot$ which shall pass through three given pts. A,B,C, not in a st. line the pt. B being 6·4 cm. distant from A, the point C being 4 cm. distant from A and also 5·6 from B.

(Allahabad, 1917)

4. Construct a $\triangle$ whose sides are 10, 4, 11, 2, 12 cm. Draw the $\odot$ passing through the feet of the $\perp$s drawn from the vertices of the $\triangle$ on the opposite sides and measure its radius. (Allahabad, 1928).

Proposition 87 (*Problem*)

To inscribe a circle in a given triangle.



Given:—A triangle ABC.

Required:—To inscribe a circle in it.

Construction:—Draw BO nd CO bisectors of $\angle$s B and C meeting each other in point O. Draw OM, ON perpendiculars to BC, CA and AB respectively. Now with centre O and radius OL, OM or ON describe a circle. Then LMN is the required $\odot$.

Proof:—Since O lies on the bisector of angle B.

$\therefore$ OL=ON.

Since O lies on the bisector of angle C,

$\therefore$ OL=OM.

Hence OL=OM=ON.

or O is equidistant from the sides BC, CA and AB.

Hence LMN is the required circle.

Q. F. F.

Exercises.

1. *In the $\triangle ABC$, if I is the in-centre, and r the length of the radius of the in-⊙, show that:—*

$\triangle IBC = \frac{1}{2}ar$; $\triangle ICA = \frac{1}{2}br$; $\triangle IAB = \frac{1}{2}cr$.

Hence prove that $\triangle ABC = \frac{1}{2}(a+b+c)r$.

Deduce, $r = \frac{\Delta}{s}$, *where* $2s = a+b+c$.

Verify the formula by measurement when $a = 9$ *cm.* $b = 8$ *cm. and* $c = 7$ *cm.*

2. Construct a $\triangle$ whose sides are 2·6″, 2·8″ and 3″. Inscribe a ⊙ in it and measure its rudius. Verify by calculation. (Alig., 1922)

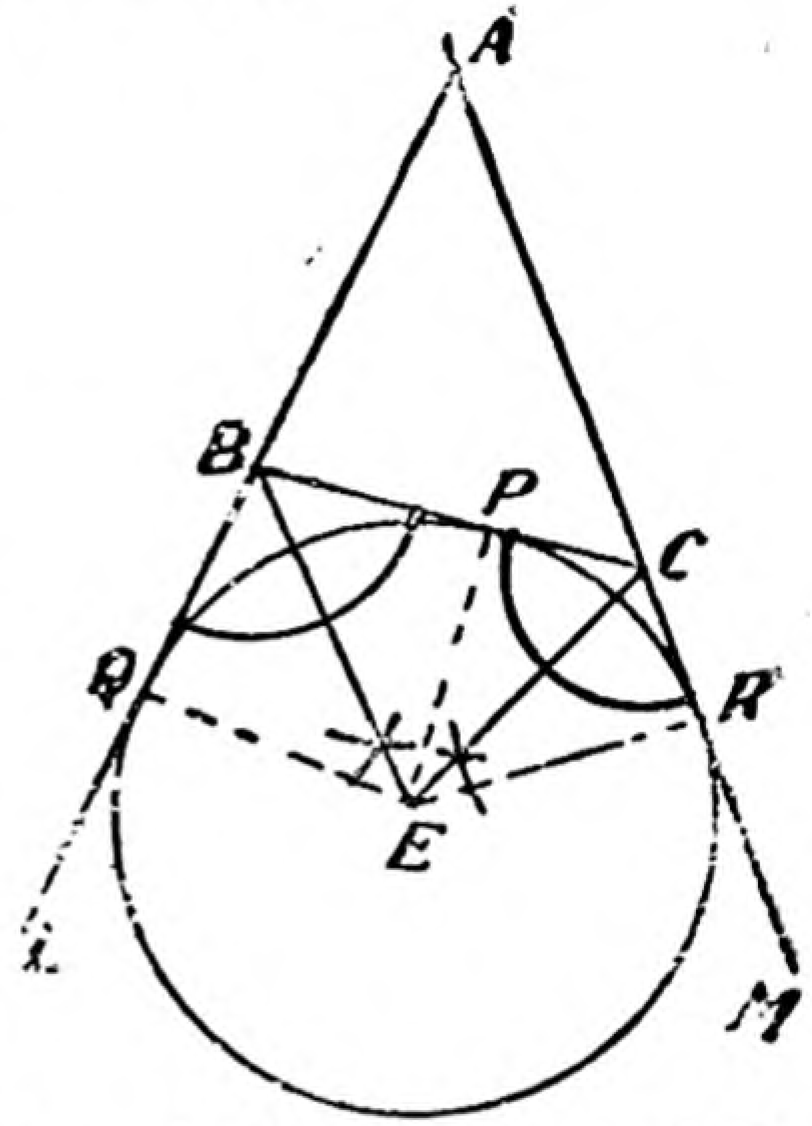
3. Draw two ∥ st. lines 2″ apart and draw a transversal cutting each of them at 45°. Draw a ⊙ to touch each of the two ∥s and the transversal. Measure the length of the smallest side of the $\triangle$ formed by joining the points of contact.

4. Draw 3 ⊙s each touching the other two externally and having their centre at the vertices of the given $\triangle$.

(Hint: —Inscribe a ⊙.)

Proposition 88. (*Problem*)

To escribe a cicle to a given triangle.



Given:—A triangle ABC.

Required:—To draw an escribed circle.

Construction:—Produce AB, AC to L and M respectively.

Bisect the exterior angles CBL and BCM and let these bisectors meet at E.

Draw perpendiculars EP, EQ and ER on the lines BC, BL and CM respectively. With centres E and radius equal to EP, EQ or ER, describe a circle. This will touch the lines AL, AM and BC at Q, R and P respectively and is the required circle.

Proof:—$\because$ E lies on the bisector of $\angle$CBL.

$\therefore$ QE=PE.

$\therefore$ E lies on the bisector of the $\angle$BCM,

$\therefore$ ER=EP

Hence QE=PE=ER.

or E is equidistant from AL, BC, and CM.

Hence QPR is the required $\odot$

Q.E.F.

Note:—This circle touches the side BC and the other two sides produced. In the same way two more circles can be drawn so as to touch sides AC and AB respectively and the other two sides produced. Thus *a triangle has three escribed circles which are also called E-circles or Ex-circles.*

Exercises.

1. *In a triangle ABC if I is the centre and r_1 the radius of the escribed $\odot$ touching the side a and the other two sides produced, prove that*

(i) $\triangle IBC=\frac{1}{2}ar_1$; $\triangle ICA=\frac{1}{2}br_1$; $\triangle IAB=\frac{1}{2}cr_1$.

(ii) $\triangle ABC=\frac{1}{2}(b+c-a)r_1$,

(iii) $r_1=\dfrac{\triangle}{s-a}$ *where* $2s=a+b+c$.

similarly $r_2=\dfrac{\triangle}{s-b}$ *and* $r_3=\dfrac{\triangle}{s-c}$.

1. *Draw a $\triangle$ whose sides are 3, 3·5 and 5 cm. Draw the escribed circle touching the shortest side and the other two produced. Find by measurement and by calculation the radius of this $\odot$.*

3. In an equilateral triangle $r : R : r_1 :: 1 : 2 : 3$.

4. Prove that $\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}=\dfrac{1}{r}$, if r_1, r_2, r_3 are used in the usual sense.

5. D, E, F are the points of contact of the in-$\odot$ of the $\triangle$ABC and D_1, E_1, F_1, the points of contact of the e-$\odot$ opposite to A, also a, b, c, s have their usual meanings, prove that :—

(i) AF+BD+CE=AE+BF+CD=s.

(ii) AE=AF=$s-a$.
BD=BF=$s-b$.
CD=CE=$s-c$

(iii) $AE_1=AF_1=s$.
$CD_1=CE_1=s-b$.
$BD_1=EF_1=a-c$.

(iv) $CD=BD_1, BD=CD_1$.

(v) $EE_1=FF_1=a$.

(vi) $\angle A$ is bisected by AI_1. (I_1 being the centre of e=⊙ opposite to A.)

(vii) Figure IBI_1C is cyclic.

6. Show that the orthocentre and vertices of a △ are the centres of the inscribed and escribed ⊙s, of the △ formed by joining the feet of the altitudes.

7. I is the incentre and I_1, I_2, I_3, the ex-centres opposite to A, B, C respectively of a △ABC, prove that :

(*i*) The points A, I, and I_1, are collinear ; so are B, I, I_2, and C, I, I_3.

(*ii*) The points I_2, A, I_3 are collinear ; so are I_3, B, I_2 and I_1, C, I_2.

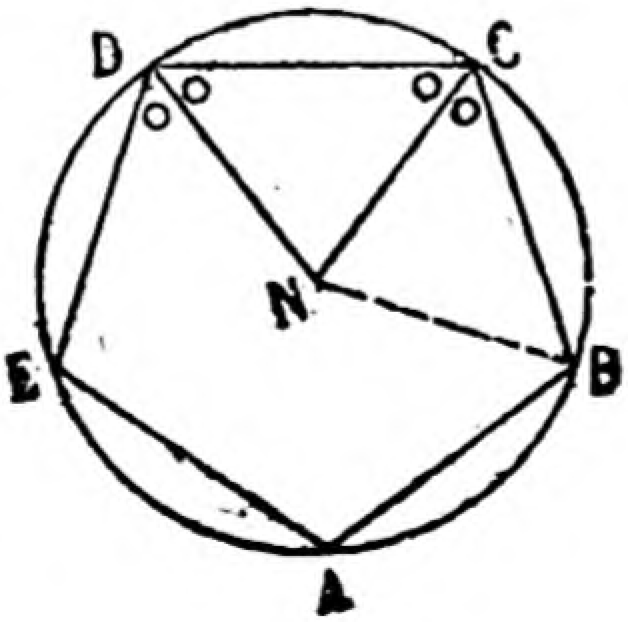
(*iii*) The triangle BI_1, C, CI_2, A, AI_3, B are equiangular to one another.

(*iv*) The $\triangle I_1 I_2 I_3$ is equiangular to the △DEF.

(*v*) Each of the four points I, I_1, I_2, I_3, is the orthocentre of the triangle whose vertices are the other three.

Proposition 89. (*Problem*)

(*a*) *To describe a ⊙ about a regular polygon of any number of sides.*



Given :—ABCDE a regular polygon.

Required :—To describe a circle about it.

Construction :—Draw CN and DN the bisectors of any two consecutive angles C and D of the polygon, meeting in N. With N as centre and NC as radius draw a ⊙. This is the required ⊙.

Proof.—Join NB. Now $\angle NCD=\frac{1}{2}$ the angle of the polygon.

Also $\angle NDC=\frac{1}{2}$ the angle of the polygon.

$\because \quad \angle NCD=\angle NDC.$

$\therefore \quad NC=ND.$

Now in the $\triangle$s NCD and NCB.

$CD=CB$ (sides of the regular polygon)

$NC=NC$

$\angle NCD=\angle NCB$ (CN being the bisector of $\angle C$).

$\therefore$ The triangles are congruent.

$\therefore \quad NB=ND$ but $NC=ND$.

$\therefore \quad ND=NC=NB.$

Similarly we can show that all the lines joining N with other angular points of the polygon are equal and each is equal to NC.

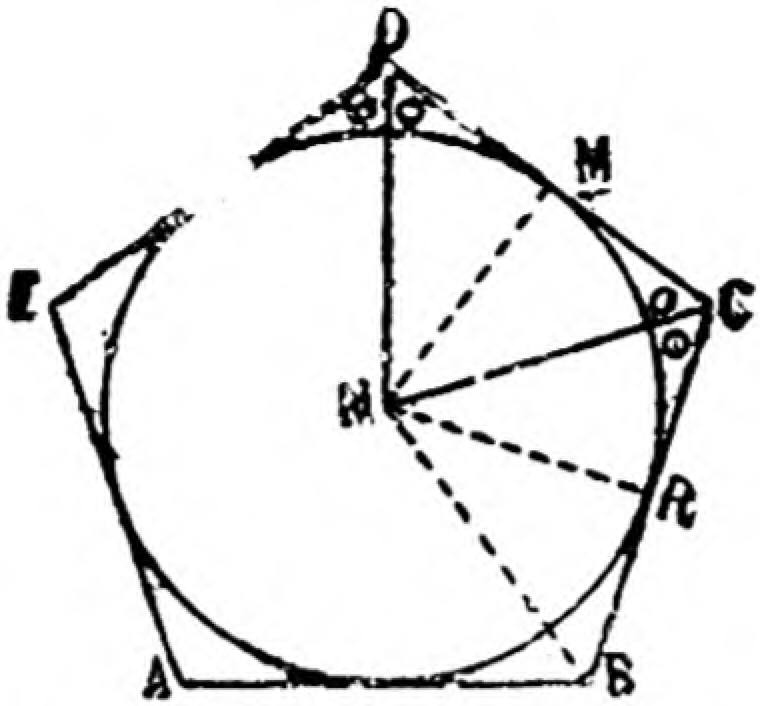
Thus the circle described with N as centre and with NC as radius will pass through A, B, C, D, E and is therefore the circumscribed circle of the polygon.

(*a*) *To inscribe a* ⊙ *in a regular polygon of any number of sides.*

Given :—ABCDE a regular polygon.

Required :—To inscribe a circle in it.

Construction :—Draw CN and DN, the bisector of any two consecutive $\angle$s C and D of the polygon, meeting in N.



Draw perpendiculars NM and NR on CD and BC respectively. With N as centre and NM as radius draw a ⊙. This is the required ⊙.

Proof :—As NCD and BNC are isosceles △s proved in (*a*) above and NM and NR are the perpendicular bisectors of CD and CB.

∴ In △s NCM and NCR

CM=CR

CN=CN

∠NMC=∠NRC (being rt. angles.)

∴ the △s≡and

NR=NM.

Similarly by dropping ⊥s from N on AB and DE we can show that these lines are also equal to NM.

Thus a circle drawn with N as centre and radius equal to NM will touch the sides of the polygon at the middle points of the sides. This circle is the inscribed circle.

Q. E. F.

Exercises.

1. Draw a ⊙ about a given square.
2. Draw a ⊙ in a given square.

3. ABCD is a quadrilateral having $\angle A + \angle C = 180°$. Circumscribe a ⊙ about this figure.

4. Draw a rectangle whose adjacent sides are 1·4″ and 2·5″. Draw a circle passing through its angular points.

5. Draw a ⊙ passing through the angular points of an isosceles trapezium ABCD, having given ∥ sides equal to 3·2 cm., and 2·3 cm., and the non-parallel sides equal to 1·3 cm. each.

6. Construct a rhombus each of whose sides is 7 cm. and one of whose diagonals is 5 cm. Inscribe a ⊙ in the rhombus and measure its radius.

(Punjab, 1918)

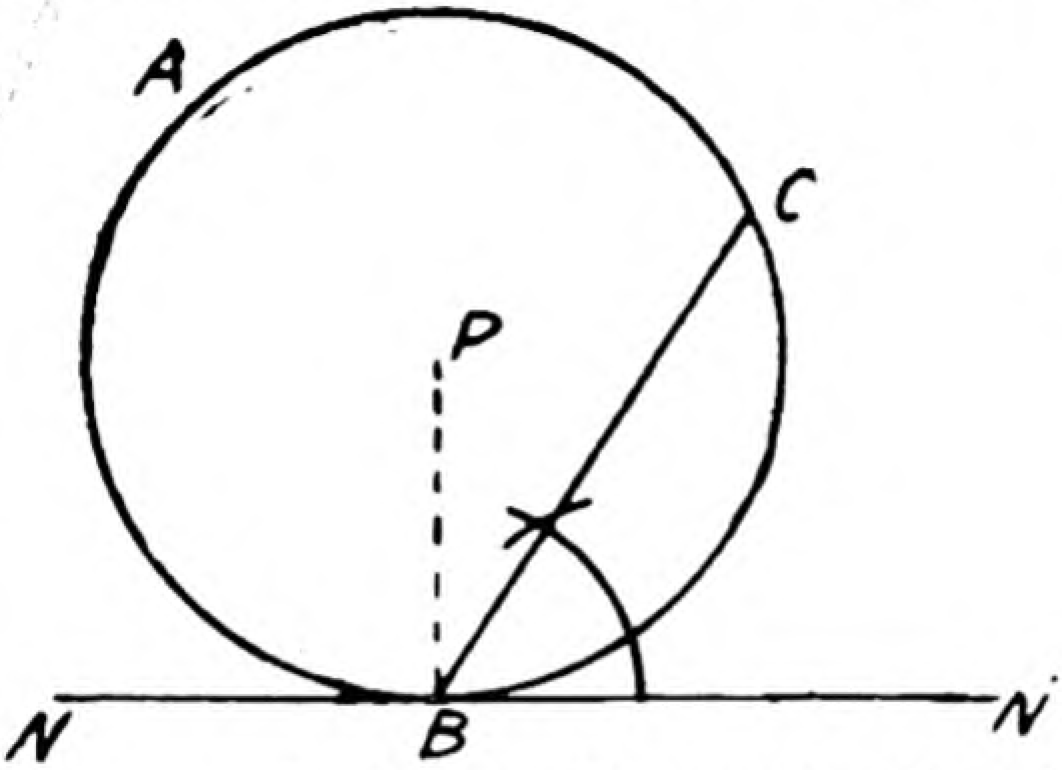
Hint.—The diagonals of a rhombus bisect the angles through which they pass.

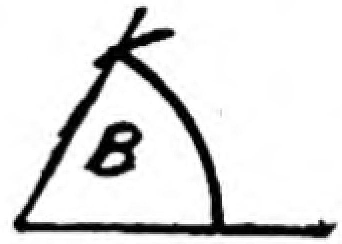
7. *Construct a rhombus whose diagonals are 9 cm., and 6 cm. Inscribe a ⊙ in the rhombus and measure its radius.* (Punjab, 1930)

8. *Construct a regular hexagon on a side of 2″. Draw the inscribed and the circumscribed ⊙s. Calculate and measure their diameter to the nearest hundredths of an inch.* (Punjab, 1909 ; Delhi 1930)

Proposition 90 (*Problem*)

To cut off from a given circle a segment capable of containing an angle equal to a given angle.





Given :—A circle with centre P and an angle B.

Required :—To cut off from the circle a segment capable of an angle equal to $\angle$B.

Construction :—Take any point B on the circumference and draw NBN' tangent to the circle. Make angle N'BC equal to angle B. Then BAC is the required segment.

Proof :—Since NBN' is a tangent and BC is any chord, the angle CBN' which the chord makes with the tangent must be equal to the angle in the alternate segment. But $\angle$CBN$=\angle$B.

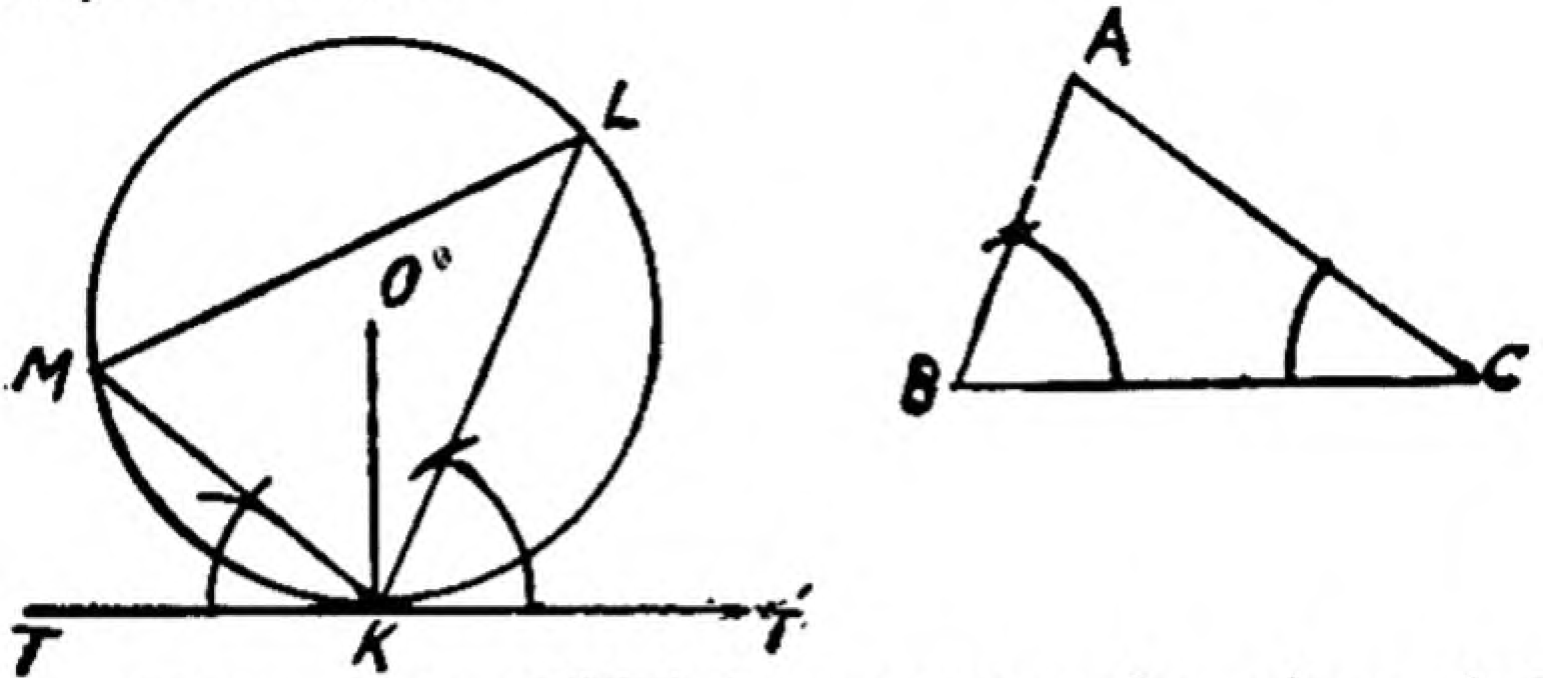
$\therefore$ $\angle$B=the angle in the segment CAB.

Hence BAC is the segment of the circle capable of containing an angle equal to angle B.

Exercises.

Draw a $\odot$ of radius 1" and from it cut off a segment capable of holding an angle of (*i*) 60°, (*ii*) 90° (*iii*) 120°.

2. *In a given circle, inscribe a triangle equiangular to a given triangle.*



Hint :—Draw TKT′ the tangent to the given circle at a point K.

Make $\angle$LKT′ ≡ $\angle$B of the triangle ABC.

Also make $\angle$MKT ≡ $\angle$C of triangle ABC. Join ML.

Then MLK is the required triangle which is equiangular to the given triangle ABC.

3. In a ⊙ of radius 5 cm. inscribe a △ two of whose angles are 40° and 80° and measure the length of the longest side. (Allahabad, 1919)

4. Describe a circle of radius 2″. Inscribe in it a triangle the angles of which are 30°, 80°, 70°. Measure its sides.

5. Draw a triangle with sides 1·5″, 1·6″, 1·7″ and inscribe a △ equiangular to it in a ⊙ of radius 1·3″.

6. Construct a △ ABC being given two $\angle$s B, C and the radius of the circumscribed ⊙.

7. *About a ⊙ circumscribe a △ equiangular to a given △.*

(**Hint.**—Draw three radii of the circle making at the centre two angles equal to the supplements of any two

angles of the given triangle. At their extremities terminating at the circumference draw three tangents. The △ thus formed is the required triangle).

8. About a ⊙ of radius 2 cm. circumscribe a △ with ∠s 30°, 60° and 90°. Measure its longest side.

(Bombay, 1925).

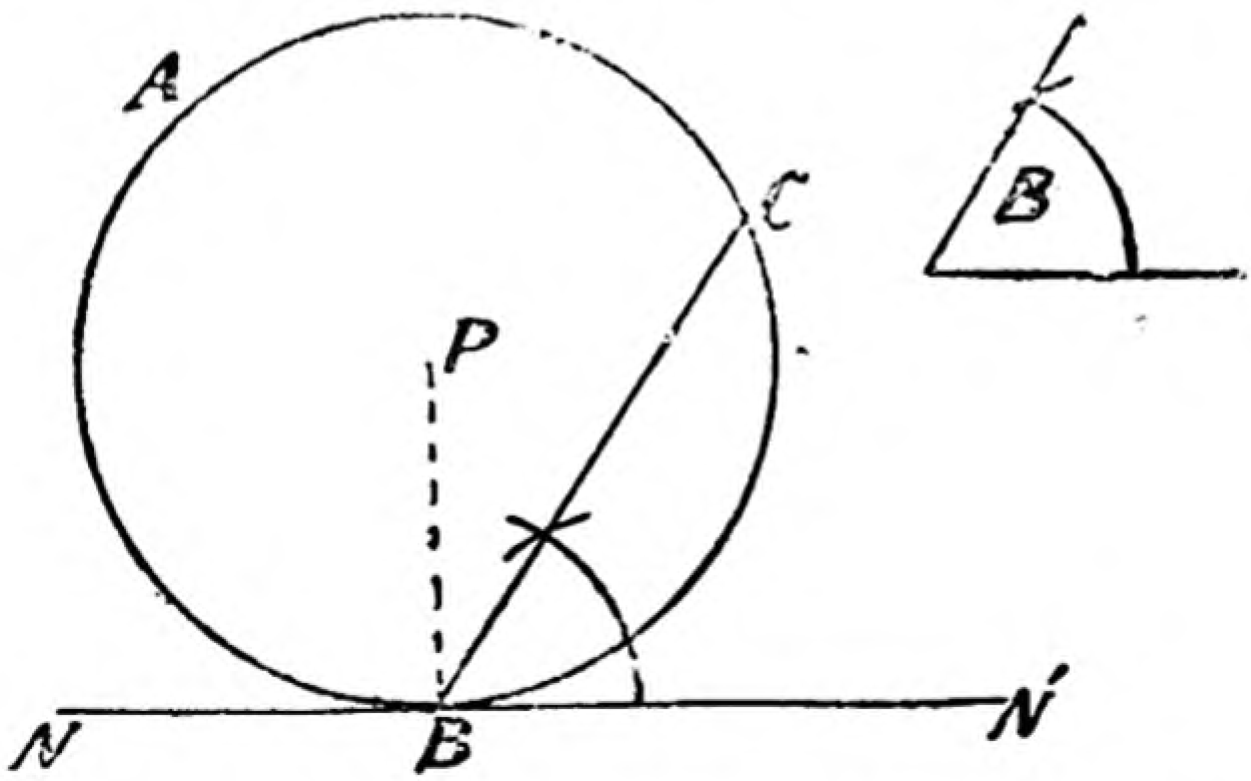
9. About a ⊙ of the ·6″ radius describe a rt. angled isosceles △.

10. Construct a triangle whose sides are in the ratio of 3 : 4 : 5 and the radius of whose inscribed ⊙ is 1″. Measure the longest side.

11. Construct a △, given two angles B, C and the radius of the inscribed ⊙.

Proposition 91 (*Problem*)

On a given straight line to describe a segment of a circle capable of holding an angle equal to a given angle.



Given :—A straight line AB and an angle B.

Required :—To construct on AB a segment of a circle capable of holding an angle equal to the given angle ∠B.

Construction :—At A, make $\angle$BAT equal to angle β. Erect AO perpendicular to TAT′. Draw NO perpendicular bisector of AB meeting AO in O. With O as centre and radius OA describe a circle.

Proof :—Since OA is a radius and TAT′ is at rt. angles to it at the point A. $\therefore$ TAT′ is a tangent to the circle. AB being the chord through A, the angle which it makes with the tangent (*i.e.*, angle BAT) is equal to the angle in the alternate segment. But $\angle BAT = \angle\beta$.

$\therefore$ ACB is the segment of the circle capable of holding an angle equal to $\angle\beta$.

Q. E. F.

Exercises.

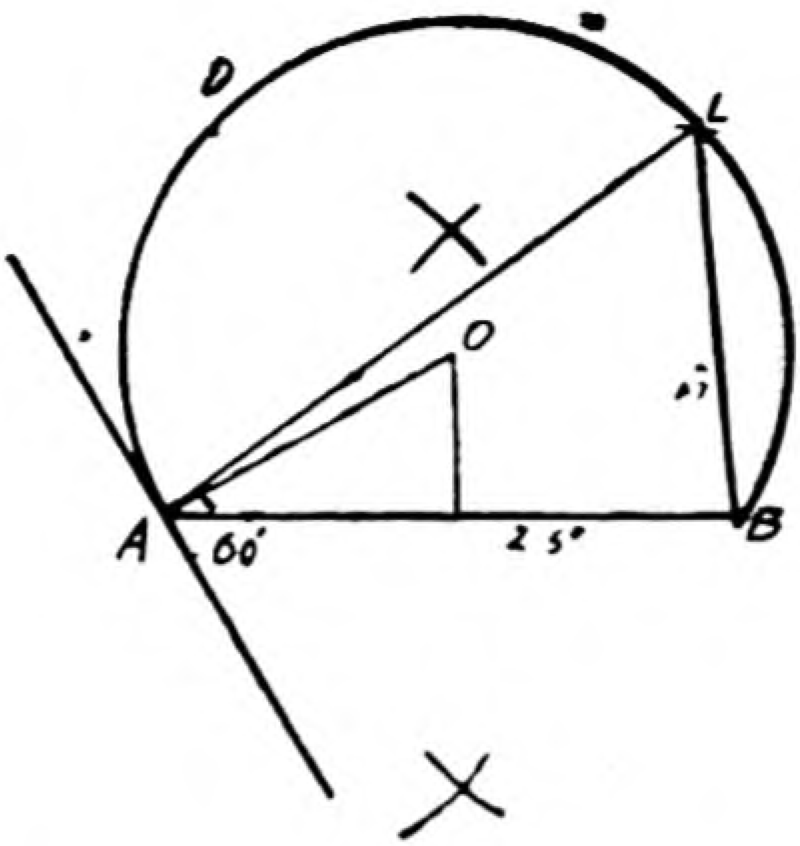
1. On a straight line 2·1″ long construct a segment of a $\odot$ capable of holding (*i*) 60°, (*ii*) 90°, (*iii*) 135°.

2. *Draw a* $\triangle$ *ABC with AB=1·1″. BC=2·4″ and CA=1·6″. Find a point in it at which the sides subtend equal angles.*

(**Hint.**—Each of the angles at the point will be=120°; so construct on two sides segments capable of holding 120°.

3. Construct a triangle having given the base= 2·5″, the vertical angle=60° and one of the sides=1·7″.

Hint.—Take a st. line AB=2 5″. On AB construct a segment of a $\odot$ capable of holding an $\angle$ of 60°.



With centre B and radius=1·7" describe an arc cutting ADB at L. Join AL and LB, ALB is the required triangle.

4. Construct a triangle ABC given that BC= 6 cm. ∠BAC=60° and AD=4·5 cm., where D is the mid-point of BC.

5. *Construct a △ having given the base, the vertical angle, and,*

(*i*) one other side.

(*ii*) the altitude.

(*iii*) the length of the median bisecting the given base.

(*iv*) the foot of the perpendicular from the vertex to the base.

(*v*) the straight line on which the vertex lies.

(*vi*) the point at which the base is cut by the bisector of the vertical angle.

(*vii*) length of the perpendicular from one end of the base on the opposite side. (*Punjab*, 1908)

(*viii*) the sum of the remaining sides.

(*ix*) the difference of the remaining sides.

Hint.—(*i*) On the given base construct a segment of a ⊙ capable of the given vertical ∠ and with one end of the base as centre and the other side as radius draw an arc cutting the segment.

(*ii*) On the given base construct a segment of a ⊙ capable of the given vertical angle and from one end of the base erect a ⊥ equal to the altitude and draw a parallel to the base, etc.

(*iii*) Draw the segment as before and with the mid-point of base as centre and radius equal to the median draw an arc cutting the segment, etc.

(*iv*) Let F be the foot of the perpendicular. Draw the segment as in (*i*) and from F erect perpendicular to cut the segment, etc.

(*v*) Draw the segment as before cutting the given st. line, etc.

(*vi*) Let P be the point at which the base is cut by the bisector of vertical angle, construct the segment as before and complete the circle ; bisect the conjugate arc and join the middle point of the arc to P, and produce the line to meet the segment, etc.

(*vii*) Construct the segment as before and on the base also describe a semi-circle. With one end of the base as centre and the given perpendicular as radius draw an arc cutting the semi-circle. Join this point to the other extremity of the base and produce it to meet the segment, etc.

(*viii*) Construct a segment on the base AB capable of half the vertical angle and with A as centre and sum of sides as radius mark an arc

cutting the $\odot$ at R. Draw the rt. bisector of RB cutting AR in C. Then ABC is the required $\triangle$.

(*ix*) On the base AB construct a segment ADB capable of an angle$=90^\circ+\frac{1}{2}$ vertical angle.

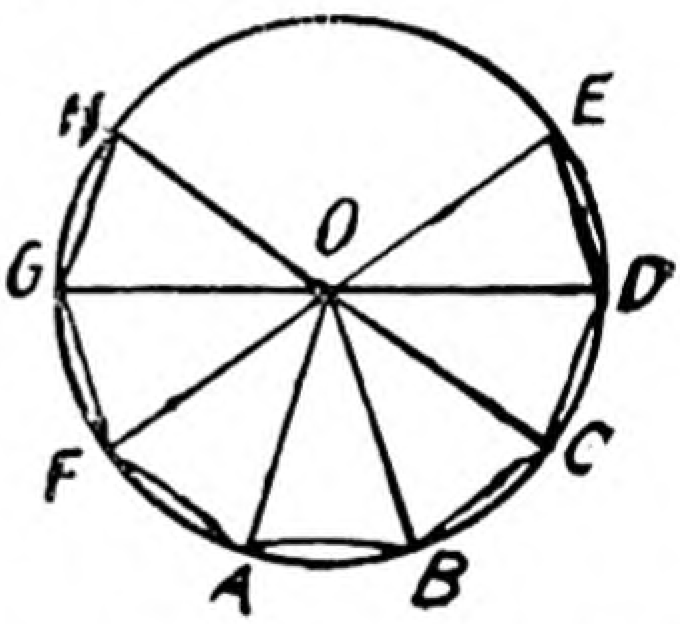
Place AD a chord$=$difference and join DB, draw the rt. bisector of BD meeting AD produced at C, then ACB is the reqd. $\triangle$.

Proposition 92. (*Problem*).

(*a*) *In a given circle, to inscribe a regular polygon of n-sides.*

Given :—A circle with centre O.

Required :—To inscribe a regular polygon of n sides in it.



Construction :—From O draw two radii OA and OB making an angle equal to $\frac{360^\circ}{n}$.

Then AB shall be a side of the required polygon of n sides.

From the circumference cut off arcs BC, CD, DE ....AF, FG, GH each equal to arc AB.

Join OC, OD, OE, OF, OG.........

Then ABCDE......HGFA shall be the required polygon of n sides.

Proof :—Arc AB$=$arc BC$=$arc CD$=$.......

$\therefore$ the chord AB$=$BC$=$CD$=$...............

$\therefore$ the figure ABCD......is equilateral.

Moreover arc AB$=$arc CD.

Add arc BC to both.

$\therefore$ Arc ABC=arcBCD.

$\therefore$ $\angle$ABC=$\angle$BCD (in equal segments.)

Similarly it can be shown that other angles such as CDE, BAF, AFG......are also each equal to $\angle$ABC.

$\therefore$ The figure formed is equiangular.

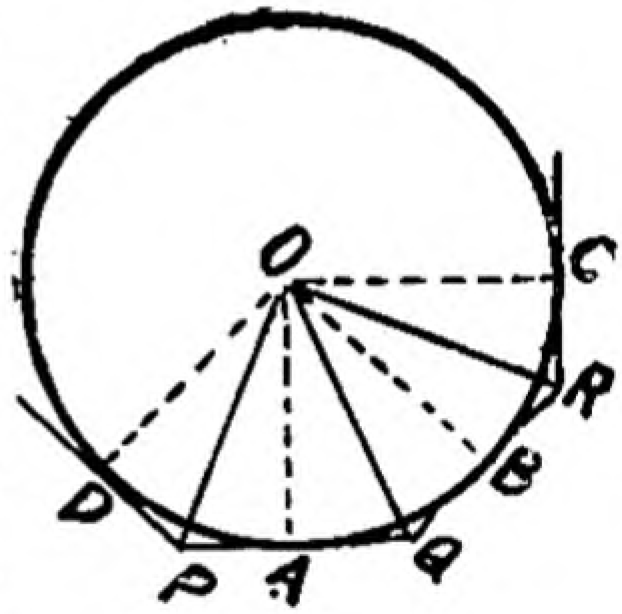
Thus the figure ABCDE......formed is a regular polygon.

Q. E. F.

(b) *About a given circle to circumscribe a regular polygon of n-sides.*

Given :— A circle with centre O.

Required :—To construct a regular polygon of n sides about it.



Construction :—Draw two radii OA, OB through O making an angle of $\frac{360°}{n}$ with each other.

Cut off from the circumference arcs BC, AD...each equal to AB.

Join OC, OD......

At D, A, B, C......draw $\perp$s to the radii intersecting at P, Q, R.....

Then P Q R......shall be the required polygon.

Proof :—As each of the angles DOA, AOB, BOC= $\frac{360°}{n}$, they are equal.

$\therefore$ their supplements $\angle$DPA, $\angle$AQB, $\angle$BRC are also equal;

$\therefore$ the figure is equiangular.

Now $\triangle$s QAO and QBO are congruent.

$\therefore$ OA=OB.

AQ=QB.

OQ=OQ. (tangents from the same point).

$$\therefore \quad BOQ = \angle AOQ = \tfrac{1}{2}\frac{360^\circ}{n} = \frac{180^\circ}{n}.$$

$\therefore$ OQ bisects the angle AOB.

Similarly OR and OP are also the bisectors of $\angle$BOC and $\angle$DOA.

Now $\triangle$s OAP and OBR are congruent.

$\therefore$ OA=OB.

$\angle$OAP=$\angle$OBR (each a rt. angle).

$\angle$AOP=$\angle$OBR $\left(\text{each} = \dfrac{180^\circ}{n}\right)$

$\therefore$ AP=BR.

But QA=QB.

$\therefore$ AP+AQ=BR+QB.

or PQ=RQ.

Similarly it can be shown that other sides of the figure PQRS......are each equal to PQ.

$\therefore$ the figure is equilateral.

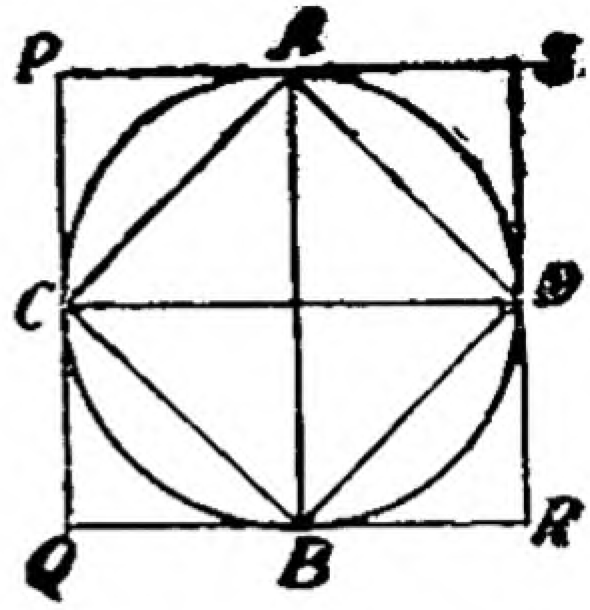
Thus it is a regular figure of n-sides described about the circle.

Q. E. F.

Exercises.

1. *Inscribe a square in a given circle.*

Draw two diameters AB and CD of the $\odot$ at right angles to each other.



Join AC, CB, BD and DA. Then ABCD is the required square.

Prove that its sides are equal and angles right angles.

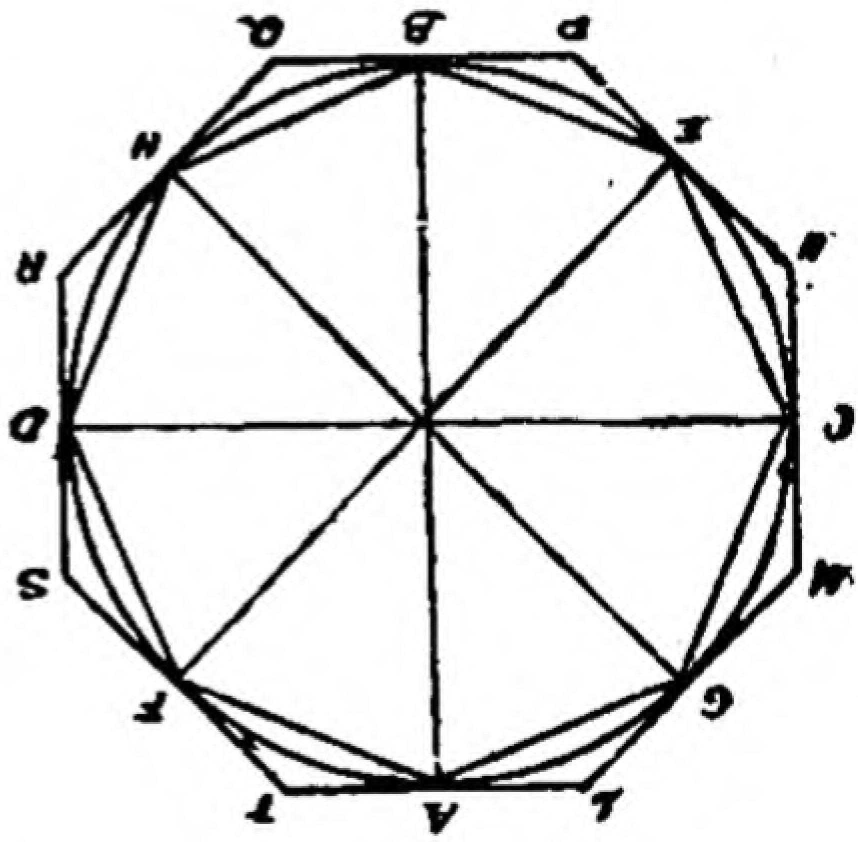
2. Circumscribe a sqaure about a given circle.

In the figure of Exercise 1 draw tangents to the circle at A, C, B and D intersecting at P, Q, R and S.

The figure PQRS is the required sqaure. Prove that it is square.

3. *Inscribe a regular octagon in a given circle.*

Hint.—Draw two diameters AB and CD at right angles.



Draw other diameters EF and GH bisecting the angles between the first two.

Join the extremities of these four diameters.

The figure AGCEBHDF is the required octagon.

Prove that it is a regular octagon.

4. *Circumscribe a regular octagon about a given circle.*

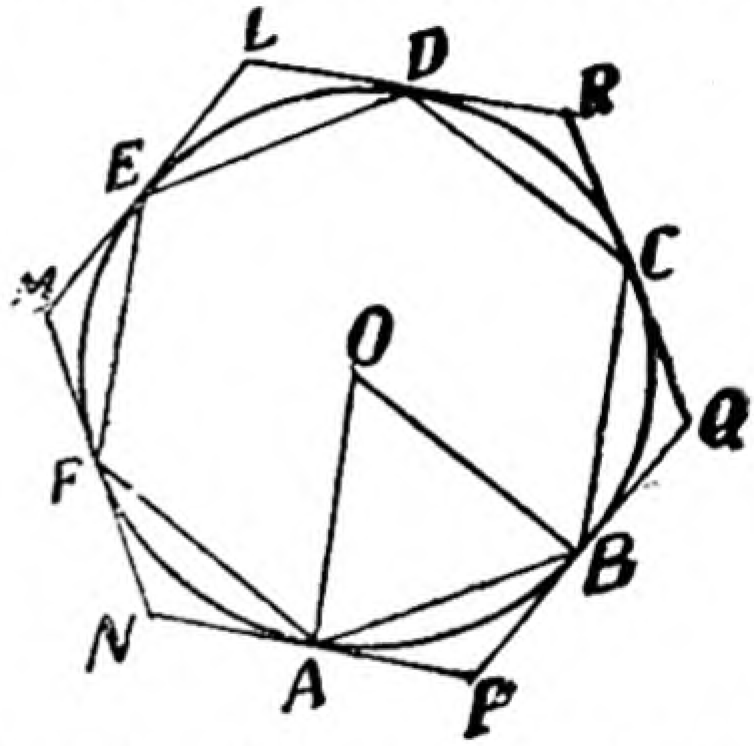
Hint.—In the figure of Exercise 3 draw lines at rt. angles to the radii through the points A, G, C, E, inter-

secting at points L, M, N, P, Q, R, S and T. The figure LMNPQRST is the required octagon.

Prove that it is a regular octagon.

5. *Inscribe a regular haxagon in a given ⊙.*

Hint – Place in the circle a chord AB=radius of the circle and join its extremities A, B with centre O. The △AOB thus form is an equilateral one. Therefore ∠AOB=60° or $\frac{1}{6}$ of 360°.



It follows from this that five more chords BC, CD, DE, EF and FA each equal to the radius can be placed in the circle end to end and each will be subtending at the centre an angle=60°.

Join BC, CD, DE, EF and FA.

The figure ABCDEF is the hexagon required.

Prove that it is a regular hexagon.

6. *Circumscribe a regular hexagon about a given circle.*

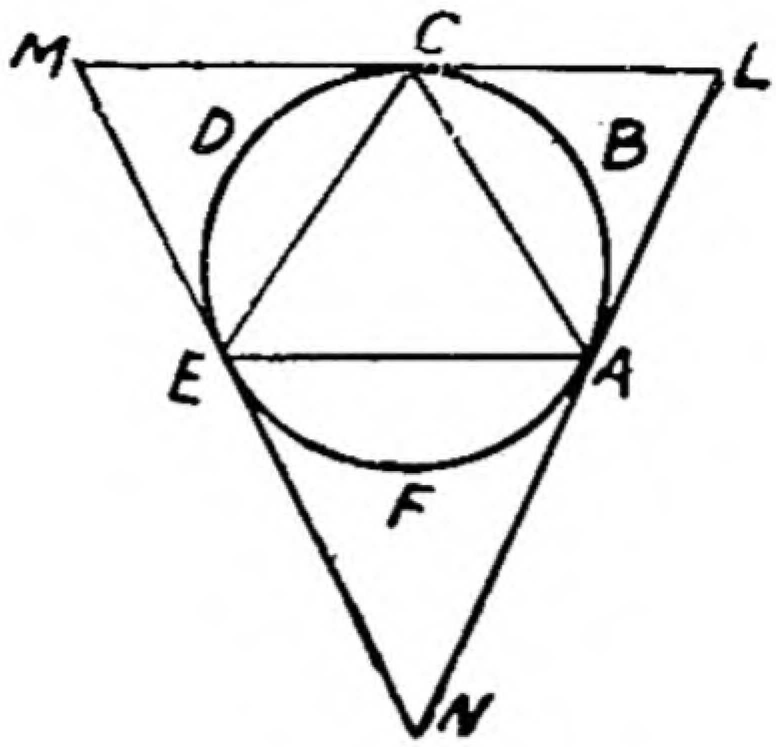
Hint.—In the figure of Exercise 5 draw tangents to the circle at D, E, F, A, B, C intersecting at L, M. N, P, Q, R.

The figure LMNPQR is the required hexagon.

Prove that it is regular hexagon.

7. *Inscribe an equilateral triangle in a given circle.*

Hint.—As in Exercises 5 divide the circumference into six equal parts.



Join the alternate points of division A, C and E.

The △ ACE is the required △.

Prove that it is equilateral.

8 *Circumscribe an equilateral triangle, about a given circle.*

In the figure of Exercise 7 draw tangents to the ⊙ at A, C and E, intersecting at L, M and N.

Then △ LMN is the required △.

Prove that LMN is an equilateral △.

9. ***In and about a given circle construct a regular Do-decagon.***

10. About a circle of radius 2·5″ describe a square. Measure its sides.

11. Construct a regular polygon of 8 sides in a ⊙ of radius 2·3″. Measure its angles.

(Punjab, 1922).

12. Draw a circle of radius 3 cm. and construct a regular hexagon about it. Measure the sides of the figure. (Allahabad, 1910).

13. Describe an equilateral $\triangle$ in a circle whose diameter is $2''$. Measure its angles.

(Punjab, 1915).

14. Circumscribe an equilateral $\triangle$ about a $\odot$ of $1{\cdot}2''$ radius. Measure the sides.

15. Draw a $\odot$ of radius $2''$, and

(*i*) Inscribe in it a regular do-decagon.

(*ii*) Circumscribe about it a regular dodecagon.

Hint:—Draw two radii making at the centre of the circle an angle$=30^\circ$. The chord joining the extremities of these radii is one side of the do-decagon in case (*i*).

16 Show that the area of the inscribed square is half of that of the circumscribed square.

17. If a be the side of a regular octagon and b that of a square inscribed in a $\odot$ of radius r, show that $b^2=a^2\,(2+\sqrt{2})=2r^2$.

(Bombay, 1921).

18. An equilateral $\triangle$ and a regular hexagon are inscribed in a given circle and a, b, denote the lengths of their sides respectively, show that (*i*) $\triangle=\frac{1}{2}$ (area of hexagon), (*ii*) $a=3b^2$. (Bombay, 1922).

19. Show that 4 times the area of the inscribed regular hexagon is equal to 3 times the area of the circumscribed hexagon.

20. The radius of a $\odot$ is r, show that the side of an equilateral $\triangle$ (*i*) inscribed in it $=r\sqrt{3}$ (*ii*) circumscribed about it$=2\,r\sqrt{3}$.

———

Hints for solving Geometrical Exercises.

The solution of geometrical exercises always presents some difficulty to beginners, but a student's progress in geometry can be measured more by his skill and ability to apply the knowledge of propositions read to solving successfully the exercises which form a new and original subject for him, rather than his capacity to memorize and his ability to repeat well the known facts. It is, therefore, desirable that he may be induced to do and be encouraged in this side of the work by all possible means. This ability of doing exercises is a gift which can only be acquired by a good deal of thinking and constant practice. We propose, therefore, to give below a few hints that may be helpful to him in learning how to attempt such exercises.

Theorems, generally, do not present much difficulty, because in proving them we proceed from the given hypothesis step by step to the conclusion, with the help of previous knowledge acquired and the figure assists our reasoning to a very great extent But this is not true of a problem, the solution of which becomes a sort of original work for a student for which he has no previous clue ; nor do the data ordinarily help him, except in very simple cases, to hit upon the construction. In either case he will, however, be well advised to :—

(1) *Draw always a good, large and clear figure.*

(2) *State clearly and mark on the figure, thus formed the data given.*

(3) *Understand clearly and state concisely what has to be found or done.*

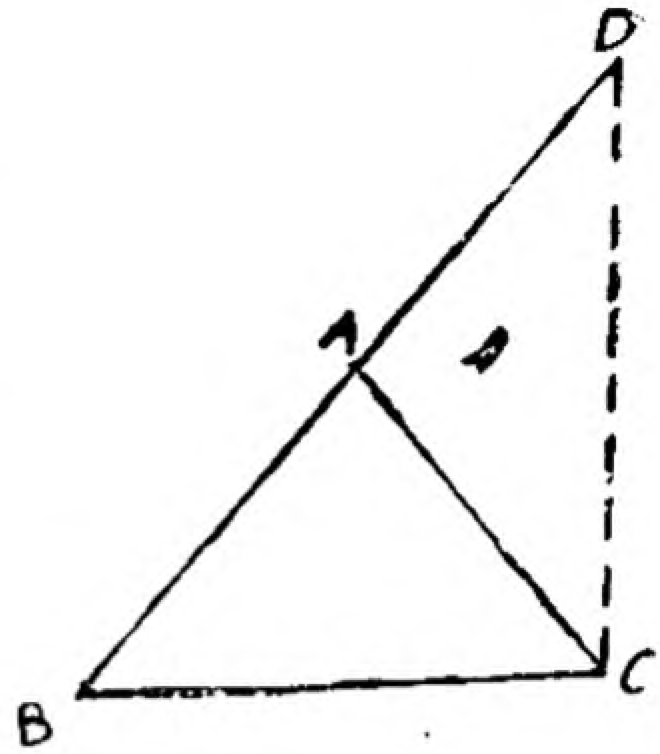
Next he should look at the figure and see what given facts have been indicated there and what results, if any, can be drawn from these facts. Noting down all the inferences that he may have been able to draw, he should read the *data* again in order to see if they can help him any further. Most likely the proof or the required construction will become obvious but if not, he should begin tackling the question from the other end. Let him assume the results to be proved correct or the required construction made. Then, examining very carefully the relations between the various parts of the figure, he may be able to deduce consequences from the assumption made, combined with results already established until some relation is discovered, which, with the help of some facts, already proved, lead him to a clue to the required construction or the desired result.

This method of tackling an exercise in geometry is called **Analysis**.

Now we proceed to explain how to apply this method of Analysis.

Suppose it is required to prove :—

1. *Any two sides of a triangle are greater than the third.*



(*a*) Make a triangle ABC. In this we want to prove that AB+AC>BC.

(*b*) Our previous knowledge tells us that in a triangle one side is longer than the other, if the angle opposite the former is greater than the angle opposite the latter.

(*c*) Here we have two lengths AB and AC, the sum of which is being compared with the third line BC. Then naturally we should think of some means of putting the first two lengths into one.

If we produce BA and cut off AD=AC then we have BD=BA+AC.

Now if we join CD we have a triangle having two sides BD and BC to be compared.

Can we show that BD is greater than BC ? If so, how ?

BD will be greater than BC if ∠BCD were proved to be greater than ∠BDC. Let us see if this is true.

In the figure we made AD=AC ∴ CAD is an isosceles triangle ; ∴ ∠ADC=∠ACD.

But ∠ACD is obviously less than ∠BCD.

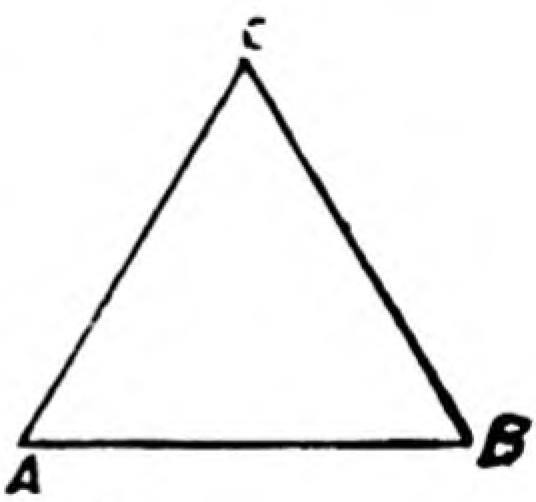
∴ ∠ADC is also less than ∠BCD, which makes BD>BC.

(*d*) Now the construction is obvious and retracing our steps we put as follows : Produce BA, one of the two sides and make the part produced AD equal to the other side AC. Join CD.

Now proving $\angle BCD > \angle BDC$ we show that BD or $BA+AC>BC$.

Now all that is given in (*a*), (*b*) and (*c*) is what we have called analysis. In this we went back from the conclusion to the hypothesis to discover the unknown links of the proof. In (*d*) we recapitulated all our steps and put them in the proper form leading us from the hypothesis to the conclusion. So the process explained in (*d*) is what we call **'synthesis'** (putting together).

2. *Construct an equilateral triangle on a given straight line.*



(*a*) Suppose ABC is the equilateral $\triangle$ made on the line AB.

(*b*) A little study of the figure shows that $AB=CB$, and also $AC=AB$; AB, BC and AC being the sides of an equilateral triangle.

(*c*) $\therefore$ A and C are at the same distance from B, and will both lie on a circle described with B as centre and BA as radius. Similarly B and C being equidistant from A will lie on a $\odot$ described with A as centre and AB as radius. Thus C will lie on both the circles.

Now the construction is obvious.

(*d*) Draw line AB equal to the given line ; with centre B and radius= BA draw a ⊙ and with centre A and radius=AB draw another ⊙. Join C, the point of intersection of these circles, to A and B.

Then ABC is the required △.

Here too, steps explained in (*a*), (*b*), (*c*) form analysis, whereas (*d*) gives the synthesis.

Students will find it very useful to remember easy and important exercises which though not difficult in themselves, yet are used extensively in establishing results of immense importance. In questions on construction or problems loci play a very great part. Very often intersection of two loci leads to the discovery of an unknown construction. We, therefore, give below some loci which have already been dealt with by the student in the body of this book, but will be found very useful by him in doing constructions on circles.

1. The locus of the centre of a ⊙ of given radius which passes through a given point is a circle whose centre is the given point and radius equal to the given radius.

2. The locus of the centre of a circle of given radius which touches a given circle is a concentric circle whose radius is equal to the sum of the radii.

3. The locus of the centre of a circle of given radius which touches a given straight line is a pair of lines parallel to the given line on either side of it and at a distance equal to the given radius.

4. The locus of the centre of a circle which touches a given circle at a given point is the join of the given point to the centre of the given circle.

5. The locus of the centre of a circle which touches a given line at a given point is the perpendicular to the line at the given point.

6. The locus of the centre of a circle which passes through two given points is the perpendicular bisector of the join of the given points.

7. The locus of the centre of a circle which touches two intersecting straight lines is the pair of bisectors of the angles between them.

8. The locus of the centre of a circle which touches two parallel straight lines is a line parallel to the given lines and midway between them.

9. All circles that pass through a fixed point and have their centres in a fixed straight line, pass through a second fixed point, *viz.*, its image.

MISCELLANEOUS EXERCISES ON THE CIRCLE.

1. If two chords of a circle bisect each other they must be diameters.

2. Circles are described on the two equal sides of an isosceles $\triangle$ as diameters; prove that they intersect at the middle point of the third side.

3. *A ladder gradually slips down a wall; find the locus of its middle point.*

4. Through the point of contact of the touching circles a st. line is drawn to cut the circles again in two points; prove that the radii drawn to these points are parallel.

5. If the diagonals of a trapezium are equal, it is cyclic.

6. *In a cyclic quadrilateral the bisectors of any angl and of the opposite exterior angle, intersect on the circle circumscribing the quadrilateral.*

7. A series of right-angled $\triangle$s is described on the same hypotenuse ; find the locus of the vertices of the right angles.

8. *A chord of constant length slides round the circumference of a given circle ; prove that (i) the locus of its middle point is a circle, (ii) the locus of a point fixed in the chord is a circle.*

Hint.— Let AB be the chord of given length which slides round the circumference of a $\odot$ with centre O. Let A′B′ be another position of the chord AB. From O draw OP, OP′ $\perp$ to AB and A′B′ $\because$ AB$=$A′B′ $\therefore$ OP$=$OP′. $\therefore$ the mid-point of AB in all positions remains at a constant distance from O. Hence the mid-point of AB lies on a $\odot$ of which the centre is O and radius OP.

(*ii*) Let Q be a fixed point in AB, join OQ $\because$ AB is a fixed length $\therefore$ its mid-point P is fixed and Q is also fixed. also PQ is fixed $\therefore$ its mid-point P is fixed and Q is also fixed, also PQ is fixed $\therefore$ $OQ^2=OP^2+PQ^2$ or OQ is constant. Hence the locus of Q is a $\odot$ whose centre is O and radius OQ.

9. Two circles intersect at A and B. Through A a line xy is drawn $\parallel$ the line of centres and meets the circles in x and y. Prove that xy is twice the line of centres.

10. Find the locus of the middle points of all chords of a circle passing through a point on its circumference.

11. AB, AC are two equal chords of a circle; prove that the bisector of the angle BAC passes through the *centre.*

12. If two circles touch each other externally, the st. lines which join the extremities of parallel diameters towards opposite parts, must pass through the *point of contact.*

13. AB and AC are two tangents to a circle whose centre is O ; show that AO bisects the *chord* of contact at right angles.

14. If two circles touch each other internally, and if the diameters of the inner circle is equal to the radius of the outer, then any *chord* of the outer circle drawn from the point of contact must be bisected by the circumference of the inner circle.

15. If two circles intersect at A and B and if through any point B on the circumference of one of them two st. lines PAC, PBD are drawn to cut the other circle at C and D, prove that the *chord* CD is parallel to the *tangent* at P.

16. If a tangent is parallel to a chord, the point of contact will bisect the arc cut off by the chord.

17. A parallelogram circumscribed about a circle must be a rhombus.

18. A rhombus cannot be inscribed in a circle.

19. Through one of the points of intersection of two circles diameters are drawn ; show that the other ends of the diameters and the other point of intersection are *collinear*.

20. The rectangle contained by the segments of any chord drawn through a given point within a circle is equal to the square on half the shortest chord drawn through that point.

21. ABC is a triangle right angled at C, and from any point D in AC, a perpendicular DE is drawn to the hypotenuse ; show that AB. AE=AC. AD.

22. If two circles cut one another, any two parallel st. lines drawn through the points of intersection to cut the circles are equal.

23. Of all st. lines drawn through a point of intersection of two circles and terminated by the circumferences, the greatest is that which is parallel to the join of centres.

24. The perpendicular bisectors of the sides of a △ are concurrent.

(For they pass through the centre of the circle which can be drawn through the angular points of the triangle).

25. The diameter of a circle bisects all chords which are parallel to the tangent at either of its extremities.

26. The angle between the tangents drawn from an external point to a given circle is supplementary to the angle between the radii to the points of contact.

27. Prove that the locus of a point, tangents from which to a given circle have a constant length, is a concentric circle.

28. A rectangle circumscribed about a circle must be a square.

29. If a quadrilateral is described about a circle, prove that the angles subtended at the centre by any two opposite sides are supplementary.

30. The bisectors of the *interior* or *exterior* $\angle$s of any quadrilateral form a *cyclic* quadrilateral.

31. In a circle the st. lines which join the extremities of parallel chords, (*i*) towards the same parts (*ii*) towards the opposite parts, are equal.

32. If a st. line touches a circle, and from the point of contact a chord be drawn, the perpendiculars drawn to the tangent and the chord from the middle point of the intercepted arc are equal.

Hint.—T'PT a tangent and PR a chord through P point of contact; from A the mid-point of arc PR, perpendiculars AB and AC are drawn to the chord and tangent. Join AP, AR. $\angle APC = \angle PRA = \angle RPA$ ($\because AR = AP$) and $\angle C = \angle R$ rt. $\angle$s. $\therefore \triangle$s ABP and APC are $\equiv$ $\therefore AB = AC$.

33. If H is the orthocentre of the $\triangle$ ABC, then any one of the four points, H, A, B, C is the orthocentre of the triangle whose vertices are the other three.

34. Parallel chords of a circle at the extremities of a diameter are equal.

35. AT is a tangent to a circle at A; find the locus of the mid-points of all chords parallel to AT.

36. The tangents to a circle at the ends of a chord include an angle double of that between the chord and the diameter through either point of contact.

Hint.—BP, AP are tangents at B, A ends of a chord AB and BOC, the diameter through B; (O the centre of $\odot$.)

RT. $\angle$OBP+rt. $\angle$OAP=2 rt. $\angle$s. $\therefore$ OBPA is cyclic $\therefore$ $\angle$P=$\angle$AOC=2 $\angle$CBA.

37. If two circles touch internally or externally and a st. line is drawn through the point of contact to cut both circles again, prove that (*i*) the tangents at its extremities are equal, (*ii*) the radii to its extremities are parallel.

38 Similar segments of circles on equal arcs are equal to one another. (Use methods of Superposition and Reductio and absurdum).

39. Chords of a circle whose centre is O pass through a fixed point P, either within or without the circle ; show that the locus of their mid-points is a circle described on OP as diameter.

40. If the bisectors of any two opposite angles of a cyclic quadrilateral cut the circle at A and B, prove that AB is a diameter of the circle.

41. The internal and external bisectors of the vertical angle of a triangle inscribed in a circle meet the circle at the extremities of the diameter which is perpendicular to the base.

42. *One side* of a triangle is equal to one side of another and the angles opposite to these sides are supplementary ; prove that the circle circumscribing the two $\triangle$s are equal.

43. In the triangle ABC the perpendiculars from B and C on the opposite sides intersect in P ; show that the circles circumscribing the $\triangle$s ABC, PBC are equal.

44. ABC is an equilateral triangle inscribed in a circle ; P is any point on the minor arc BC. Prove that PA=PB+PC.

Hint.—Produce CP to D, making PD=PB.

$\angle$BPD=$\angle$BAC=60° $\because$ BPCA is cyclic and $\triangle$ABC is equilateral. $\therefore$ $\triangle$BPD is equilateral. $\angle$ABP=$\angle$CBD (each=60°+$\angle$CBP) BA=BC, BP=BD $\therefore$ $\triangle$s ABP, CBD are congruent.
$\therefore$ PA=DC=PC+PB.

45. If the opposite sides of a cyclic quadrilateral be produced to meet in P and Q, and about the $\triangle$s so formed outside the quadrilateral $\odot$s be described intersecting again at R, show that P, Q, R are collinear.

Hint.—ABCD, a cyclic quadrilateral AD, BC meet in P and AB, DC meet in Q the $\odot$s about $\triangle$s CDP and BCQ intersect at R. Join CR. $\angle$CRP=$\angle$ADC and $\angle$CRQ=$\angle$ABC $\therefore$ $\angle$CRP+$\angle$CRQ=$\angle$ADC+$\angle$ABC =2 rt $\angle$. $\therefore$ P, Q, R are collinear.

46. If two circles intersect in P, Q and through any point R in their common chord PQ, two other chords AB, CD are drawn one in each circle, show that their four extremities A, B, C, D are concyclic.

47. *If two circles touch one another internally and a st. line is drawn to cut them, the segments of it intercepted between the circumference subtend equal angles at the point of contact.*

Hint.—A line ABCD cuts ths outer $\odot$ in A, D, and the inner $\odot$ in B, C. Through P, point of contact, draw TPT′ a common tangent to the $\odot$s $\angle$TPC=$\angle$PBC (alt. seg s.) $\angle$TPD=$\angle$PAB (alt. segs.) $\therefore$ $\angle$TPC$-$$\angle$TPD or $\angle$CPD=$\angle$PBC$-$$\angle$PAB=$\angle$APB.

48. ABCD is a cyclic quadrilateral whose diagonals AC, BD intersect at rt. $\angle$s in O ; AX is drawn $\perp$ to BC (produced if necessary) ; prove that DX is bisected by the perpendicular (produced if necessary) from O to BC.

Hint.—From O draw OY $\perp$ BC and produce YO to meet AD in Z, then Z is the mid-point of AD (Theorem of Brahmagupta) AX $\parallel$ YZ (both $\perp$BC) and in $\triangle$ADX, through Z, ZY is drawn $\parallel$ AX, $\therefore$ DX is bisected by ZY.

49. The diagonals of a cyclic quadrilateral are at right angles ; prove that the distance of the centre of the circumcircle from a side is half the opposite side.

Hint.—Diagonals AC, BD of a cyclic quadrilateral ABCD intersect at G at rt. $\angle$s. and from centre O, OE is drawn $\perp$ to AB, then OE$=\frac{1}{2}$ DC. Draw OF $\perp$ to CD. Join FG and produce it to meet AB in M$=$OE is $\perp$ AB, $\therefore$ E is the mid-point of AB. Join EG and produce it to meet CD at N $\therefore$ EGN is $\perp$ to CD. (Converse of Theorem of Brahmagupta). Similarly FGM is $\perp$AB and OE $\perp$ AB $\therefore$ OE $\parallel$ FG and OF $\perp$ to CD and EGN is also $\perp$ to CD $\therefore$ OF $\parallel$ EG $\therefore$ OFGE is a $\parallel^{\text{m}}$ $\because$ OE$=$GF but F is mid-point of hypotenuse CD $\therefore$ FG$=\frac{1}{2}$ CD. $\therefore$ OE$=\frac{1}{2}$ CD.

50. The diagonals of a cyclic quadrilateral are at rt. $\angle$s. Prove that the join of the intersection of the diagonals to the centre of the circumcircle is bisected by the join of the mid-points of the opposite sides.

Hint.—In Ex. 49 (last ex.) E is mid-point of AB and F the mid-point of opposite side CD, then OG is bisected by EF. $\because$ OFGE is a $\parallel^{\text{m}}$, (proved in last ex.) and EF and OG are its diagonals $\therefore$ OG and EF bisect each other.

51. The internal bisector of the vertical angle of a triangle and the perpendicular bisector of the base intersect on the circumcircle.

52. If the altitudes of a triangle are produced to cut the circumcircle, then the part of each intercepted between the orthocentre and the circumcircle is bisected by the corresponding side of the triangle.

Hint.—AP, BQ are two altitudes of $\triangle$ABC, which intersect at H, and AH produced cuts the circumcircle again in K. Then HP=PK $\angle$s at Q, P are rt. $\angle$s. $\therefore$ A, Q, P, B are concyclic $\therefore$ $\angle$PBH=$\angle$PAQ (in same seg.). But $\angle$PAQ=$\angle$PBK (in same seg. circle ABC) $\therefore$ $\angle$PBH=$\angle$PBK and $\angle$s at P are rt. $\angle$s. and BP is common $\therefore$ $\triangle$s PBH and PBK are congruent, $\therefore$ HP=PK.

53. AK is the diameter of the circumcircle through $\angle$A of the triangle ABC whose orthocentre is O ; prove that BOCK is a $||^m$ and BC and OK bisect each other.

Hint.—$\angle$s ABK, ACK being in a semicircle are rt. $\angle$s ; $\therefore$ CO is $||$ BK and BO to KC.

54. AK is the diameter of the circumcircle of a triangle ABC through A and O the orthocentre ; if S is the circumcentre and E, F, G the mid-point of BC, AB, AC, respectively, show that SE=$\frac{1}{2}$ OA. SF =$\frac{1}{2}$ OC and SG=$\frac{1}{2}$OB.

Hint.—BOCK is a $||^m$ (Ex. 53) $\therefore$ BK=CO. In $\triangle$ABK, S the mid-point of AK and F the mid-poin of AB $\therefore$ SF=$\frac{1}{2}$BK=$\frac{1}{2}$CO, etc.

55. ABC is a triangle ; D, E, F any points on the sides BC, CA, AB, respectively, prove that the circles AEF, BFD, CDE have one point common.

Hint .—Let circles AFE, CDE cut in O. Then $\angle$AFO$=\angle$OEC (ext. $\angle$ of cyclic quadrilateral AEOF= int. opp. $\angle$.

Similarly $\angle$OEC$=\angle$ODB $\therefore$ $\angle$AFO$=\angle$ODB

$\therefore$ BFOD is cyclic. $\therefore$ circle BFD goes through O.

56. AB, CD are perpendicular diameters of a circle, centre O ; the tangent at P (on the arc BD) to the circle cuts the line CD produced (in X and Y.) Prove that EX=EY.

Hint.—$\angle$s BOD and BPX are rt. $\angle$s. $\therefore$ BOXP is cyclic $\therefore$ $\angle$B=uPXD (ext. $\angle$ of cyclic quadrilateral =int. opp. $\angle$) Now $\because$ EP is a tangent and PA a secant $\therefore$ $\angle$EPX$=\angle$B$=\angle$PXD $\therefore$ PE=EX and $\therefore$ PE= EY hence EX=EY.

57. C is the centre, CA a radius of a circle ; B is a point on a perpendicular radius ; the line AB cuts the circle again in D ; the tangent at D cuts CB produced in E. Prove BE=DE.

Hint.—$\because$ DE is a tangent and CD a radius $\therefore$ $\angle$CDE=a rt. $\angle=\angle$ACB=uA$+\angle$ABC$=\angle$CDB$+\angle$DBE $\therefore$ $\angle$BDE>=DBE $\therefore$ BE=DE.

58. The perpendicular from the mid-points of the sides of a cyclic quadrilateral on the opposite sides meet in a point.

Hint .—L, M, N, P are mid-points of the sides AB, BC, CD, DA of the cyclic quadrilateral ABCD ;

Q, R are the ft. of the perpendiculars from L, N, on to CD, AB respectively ; LQ, NR cut in S. Then $\perp$s from P, M to BC, AD respectively go through S. The right-bisectors of AB and CD cut in the centre O of circle ABCD. Prove that LSNO a $\|^{m}$; $\therefore$ LN, OS bisect each other. Next, if the perpendiculars from P, M to BC, AD respectively cut S′, then similarly OS′, PM will bisect each other. But LN and PM bisect each other. Hence S and S′ coincide.

59. ABCD is a quadrilateral whose sides touch a circle in X, Y, Z, W ; XZ and YW cut in P. Prove that $\angle XPW = \frac{1}{2}(\angle ADC + \angle ABC)$.

Hint:—$\angle XPW = \angle PWZ + \angle PZW = \angle YZC + \angle AXW$ (*i*) ($\because \angle PWZ = \angle YZC$, $\angle PZW = \angle AXW$ (alt. sg.)

But tangent CZ = tangent CY $\therefore \angle YZC = \angle ZYC$, and $\angle YZC + \angle ZYC + \angle XCY = 2$ rt. $\angle$s.

$\therefore \angle YZC = \frac{1}{2}(2\text{rt. } \angle s - \angle ZCY)$.

Similarly $\angle AXW = \frac{1}{2}(2\text{rt. } \angle s - \angle XAW)$.

$\therefore \angle YZC + \angle AXW = \frac{1}{2}(4 \text{ rt. } \angle s - \angle ZCY - \angle XAW)$

$= \frac{1}{2}(\angle ADC + \angle ABC)$.

Sum of $\angle$s of quad. = 4 rt. $\angle$s.

$\therefore$By (*i*) $\angle XPW = \frac{1}{2}(\angle ADC + \angle ABC)$.

60. Prove that in a right-angled triangle the sum of the hypotenuse and the diameter of the inscribed circle is equal to the sum of the remaining sides. (*Bombay, 1918*).

Hint:—Let the inscribed $\odot$, centre O, touch the hypotenuse BC and sides CA, AB of the $\triangle$ ABC in D, E, F respectively. Join OA, OF.

$\therefore$ BC + Diameter = BC + OE + OF = BF + CE + OE + OF = BF + CE + AF + AE ($\because$ AEOF = a sq.) = AB + AC.

61. PR, SQ are two perpendicular chords of a circle. Prove that the tangents at P, Q, R, S form a cyclic quadrilateral.

(*Punjab, 1918*).

(See Hints on ex. 7 Prop. 80)

62. Four circles are placed in such a manner that each touches the two adjacent ones externally; prove that the four points of contact lie on a circle.

(*Punjab, 1917*)

Hint:—A, B, C, D are four points of contact and let the common tangents at A, B meet in E, and those at A, D in F. Now AE=BE and AF=DF.

$\therefore \angle EAB = \angle EBA$ and $\angle FAD = \angle FDA$.

$\therefore \angle BAD = \angle ABE + \angle ADF$. Similarly $\angle BCD = \angle CBE + CDF$.

$\therefore \angle BAD + \angle BCD = \angle ABC + \angle ADC = 2$ rt. $\angle$s.

$\therefore$ A, B, C, D are concyclic.

63. ABCD is a cyclic quad, whose diagonals AC and BD intersect at right angles in O and P is the mid-point of CD; prove that PO produced will be perpendicular to AB.

Hint:—Produce PO to meet AB at E.

$\angle BOE =$ vert. opp. $\angle POD = \angle PDO$ ($\because$ P is mid-point of hyp. CD). $=$ BAO (same seg.)

$\therefore \angle BOE + \angle AOE = \angle BAO + \angle AOE$.

$\therefore \angle BAO + \angle AOE = 1$ rt. $\angle$ $\because \angle AEO =$ a rt. $\angle$.

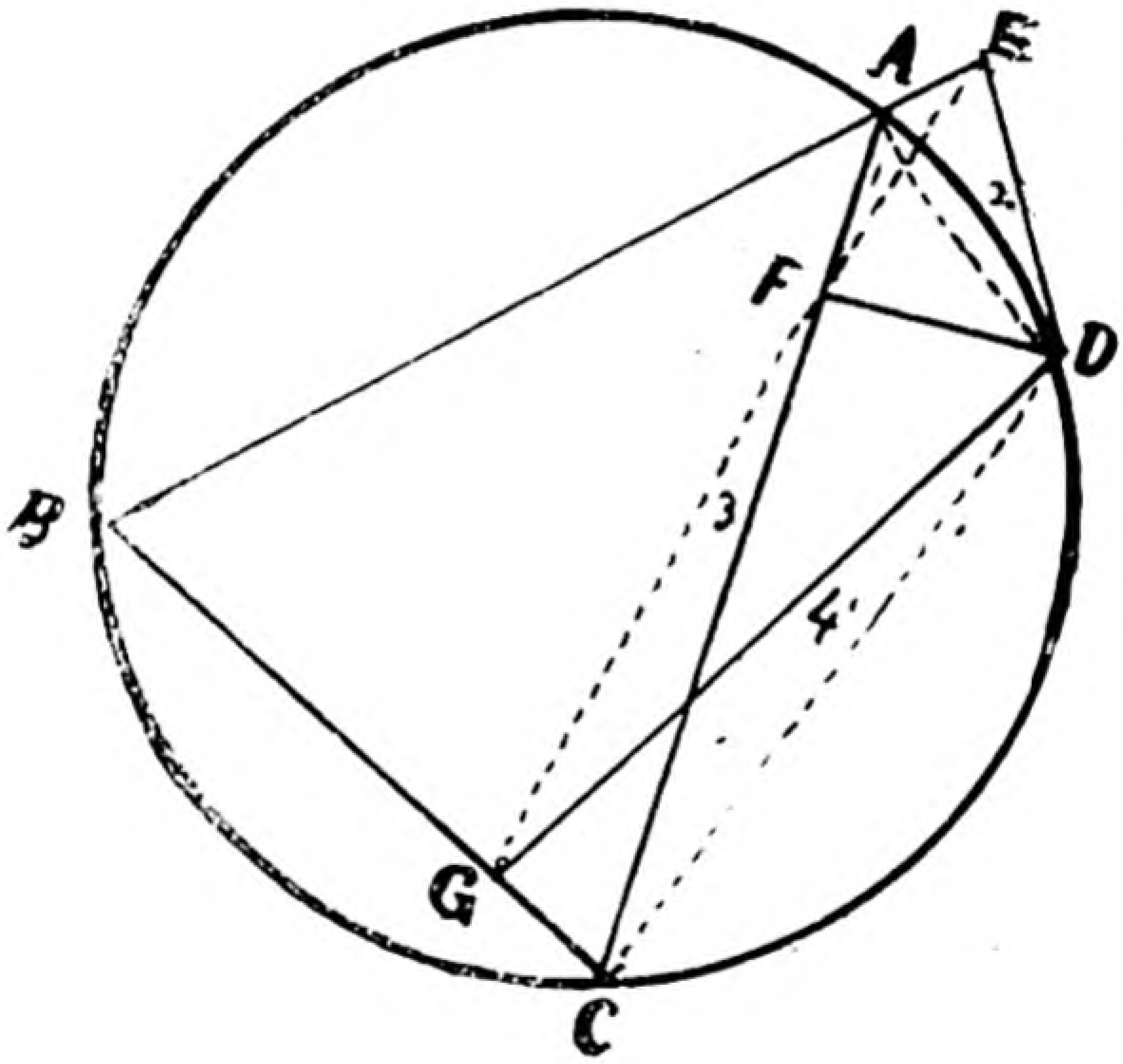
64. Three circles have external contact at P, Q, R; if PQ, PR are produced to meet the circle through Q and R in X and Y, then XY is a diameter of that circle and is parallel to the line of centres of the two circles.

Hint:—Let A, B, C be the centre of the ⊙s.

Let BX and BY be joined. Now, ∠CPQ=∠CQP(∵ CQ=CP radii). =Vert, opp. ∠XQB= ∠QXB (BX=BQ). But these are alternate ∠s ∴ CPA ∥ BX.

Similarly C?A is ∥ BY ∴ BX and BY are in the same st. line and as XBY passes through the centre B ∴ it is a diameter.

65. The feet of the perpendicular, drawn to the three sides of a △ from any point on its circumcentre are collinear.



Given :—A △ABC and a pt. D on its circumcircle. DE⊥AB, DF⊥AC, DG⊥BC. Join EF, FG.

To Prove :—EFG is a st. line.

Const. :—Join AD, CD.

Proof :—∵ AFDE is a cyclic qnad [∠DEA+ ∠AFD=2 rt. ∠s.

$\therefore \angle 1 = \angle 2$ (same segment).

$\because$ FGCD is a cyclic quad. $\left[\begin{array}{l}\angle DFC = 90^\circ. \\ \angle DGC = 90^\circ.\end{array}\right.$

$\therefore \angle 3 = \angle 4$ (same segment).

Now in $\triangle$s AED and DGC.

$\because \angle AED = \angle DGC$ (each rt. $\angle$.

and $\angle EAD = \angle DCG$ (Ex. $\angle$ of cyc. quad. $=$opp. Int. $\angle$.

$\therefore \angle 2 = \angle 4$ (3rd $\angle$s of respective $\triangle$s.

$\therefore \angle 1 = \angle 3$ ($\because \angle 1 = \angle 2$ and $\angle 3 = \angle 4$ proved.

$\therefore$ EFG is st. line.

Q. E. D.

Note :—The line EFG is called the pedal line or Simson line of $\triangle$ABC for the pt. D.

Method of Proof :—Converse of the theorem : If two st. lines intersect the vertically opp. $\angle$s are equal.

66. Enunciate and prove the converse of Ex. 65.

[Here we shall have another method of proving a set of four points concyclic.]

[Enunciation. If the three feet of the perpendiculars drawn from a point outside a $\triangle$ on its three sides be collineal, that point and the vertices, of the $\triangle$ are concyclic.

Proof :—Refer to the fig. under Ex. 65.

$\angle 1 = \angle 4$ (Ver. opp. $\angle$s.

But $\angle 1 = \angle 2$ (AEDF cyc. quad.

and $\angle 4 = \angle 3$ (DFGC cyc. quad.

$\therefore \angle 2 = \angle 3$

Now in $\triangle$s AED and DCG.

$\angle 2 = \angle 3$ [Proved

$\angle$DEA$=\angle$DGC. [Each rt. $\angle$

$\therefore$ $\angle$DAE$=\angle$DCG.

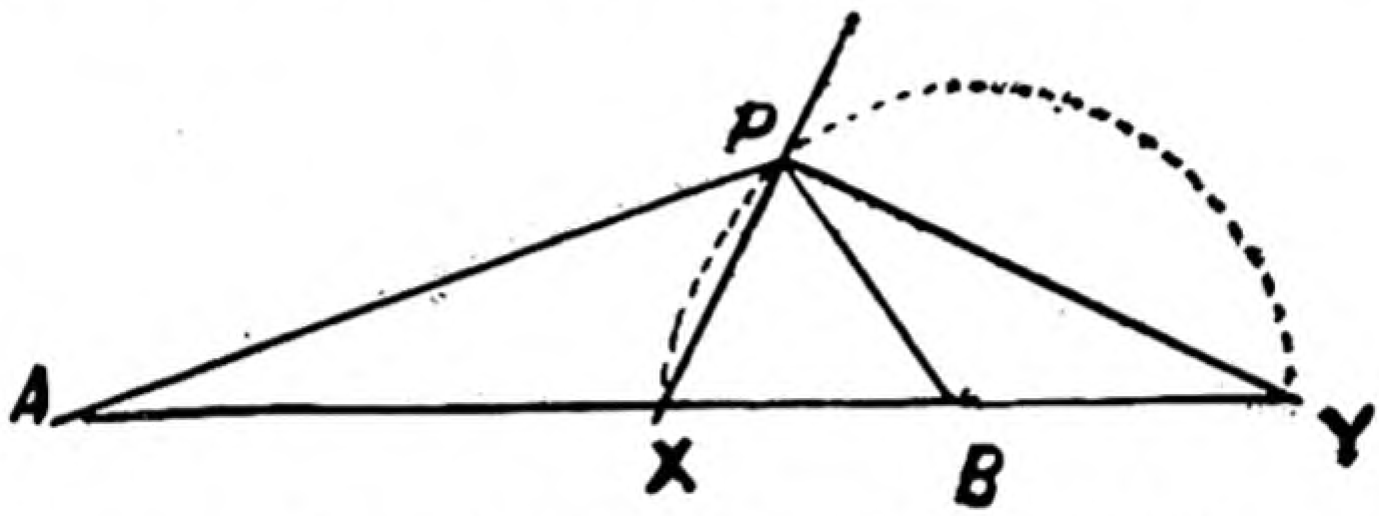
i.e. The ext. $\angle$DAE of the Quad.

ABCD=the int. opp. $\angle$DCG.

$\therefore$ the Quad. ABCD is cyclic.

Q. E. D.

67. The locus of a point the ratio of whose distances from two fixed points is constant, is a circle.



Given :—A and B are two fixed points and P is a point such that

PA : PB$=m:n$ (Say, where $m:n$ is constant.)

To prove that the locus of P is a circle.

Const :—Divide AB internally and externally at X and Y in the ratio $m:n$.

Join PX and PY. On XY as diameter draw a ⊙ which is the rqd. locus.

Proof :—In the $\triangle$APB.

PX divies AB in the ratio PA : PB.

PX is the internal bisector of $\angle$PX.

Similarly PY is the external bisector of $\angle$P.

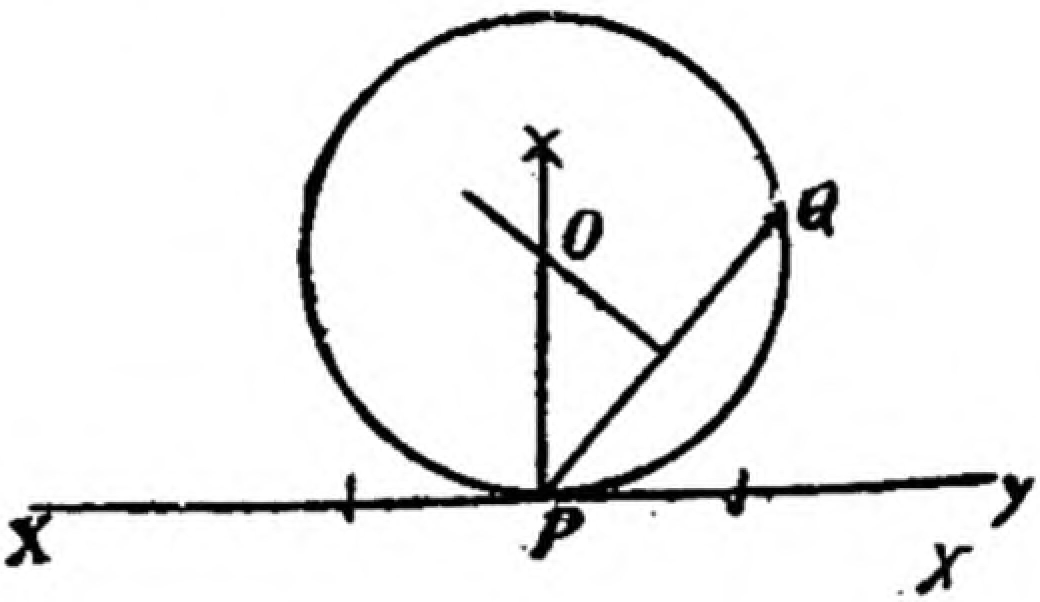
But the internal and external bisectors of an angle are at rt. $\angle$s.

$\therefore$ $\angle$PXY is a rt. $\angle$.

$\therefore$ the locus of P is a ⊙ drawn on XY as diameter.

Miscellaneous Construction.

1. *To draw a ⊙ passing through a given point and tou·hing a given straight line at a given point.*



XY a given a st. line with P a given point on it and Q another given point.

At P erect a ⊥ to YX. Join PQ.

Draw the perpendicular bisectors of PQ meeting the first perpendicular at O. Then O is the centre and PQ the radius of the required ⊙.

Ex. 1. Describe a circle which touches a given st. line at a given point and has its centre on another given circle.

Ex. 2. Draw a ⊙ of radius r passing through two given points.

Ex. 3. Draw a ⊙ to pass through two given pts. and having its centre on a given st. line.

2. *To draw a circle passing through a given point P* and touching a given circle at a given point Q.

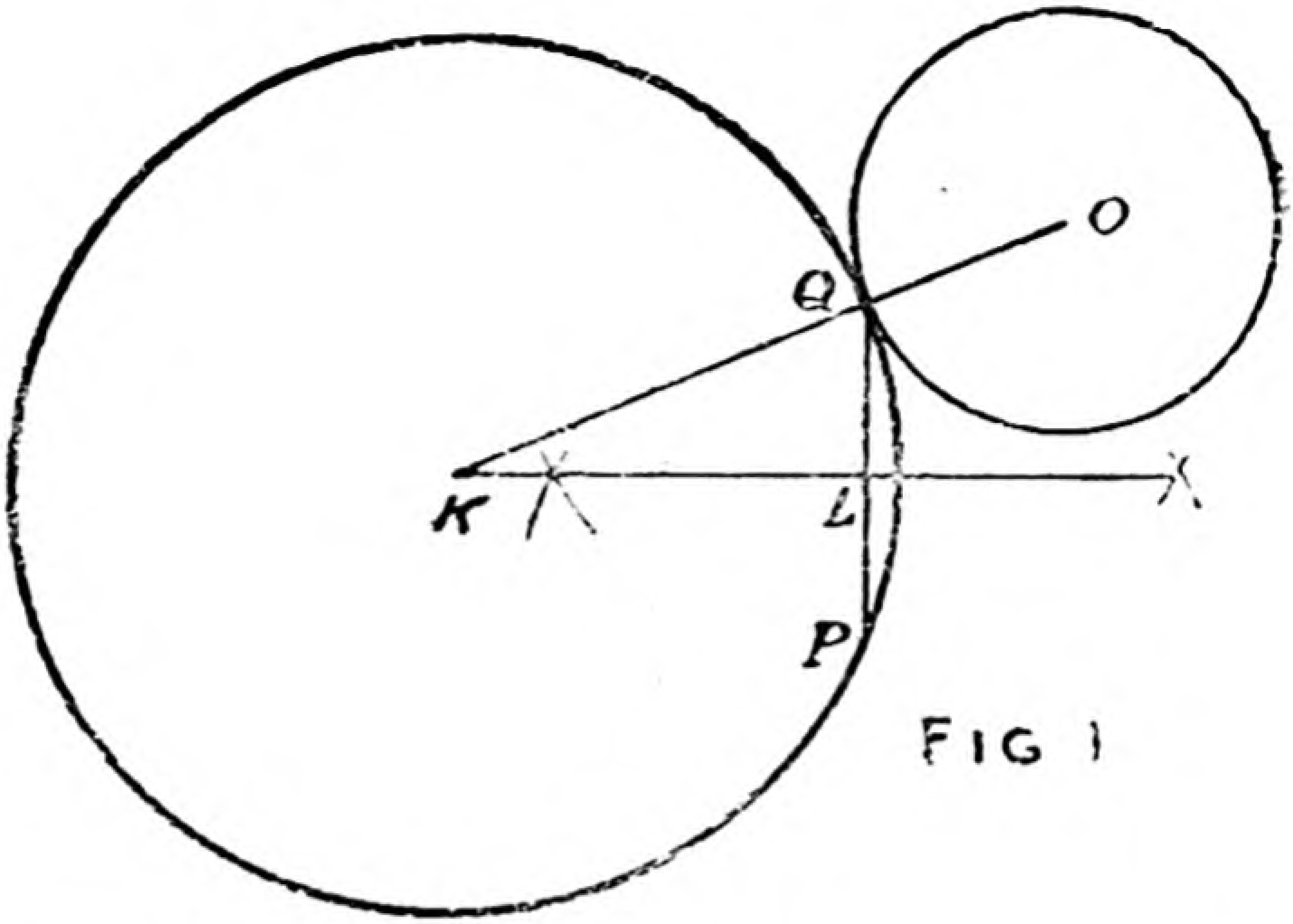


FIG 1

P the given point and Q another given point on the given ⊙ with centre O.

Join PQ and draw its perpendicular bisector, KL. Join OQ and produce it to meet KL at a point K. Then K is the centre and KQ the radius of the required ⊙.

If P, the given point, lies within the given circle, the second circle will touch the given circle at Q internally as in Fig. 2. The steps of construction are the same as above.

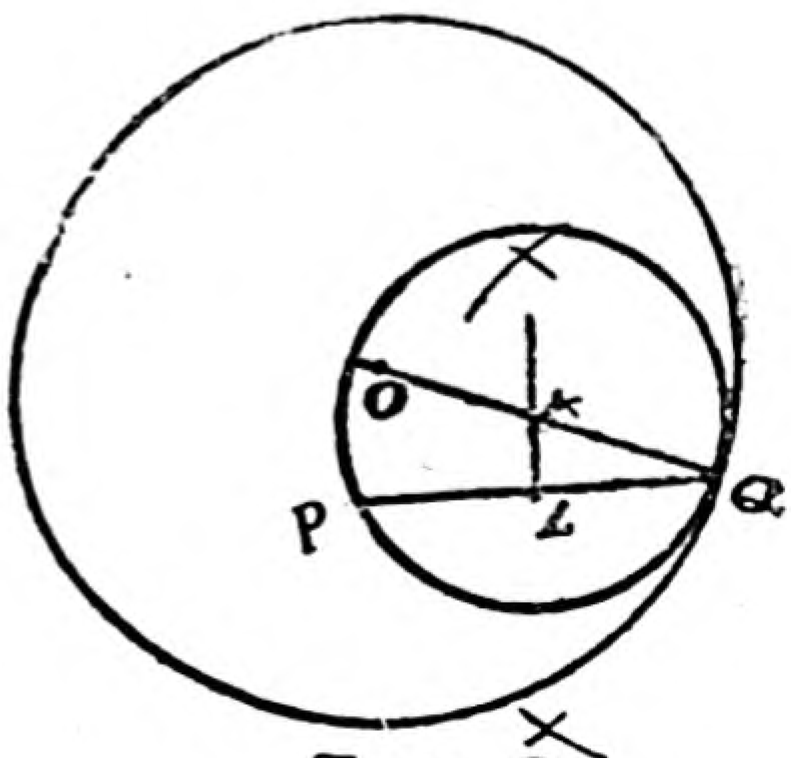
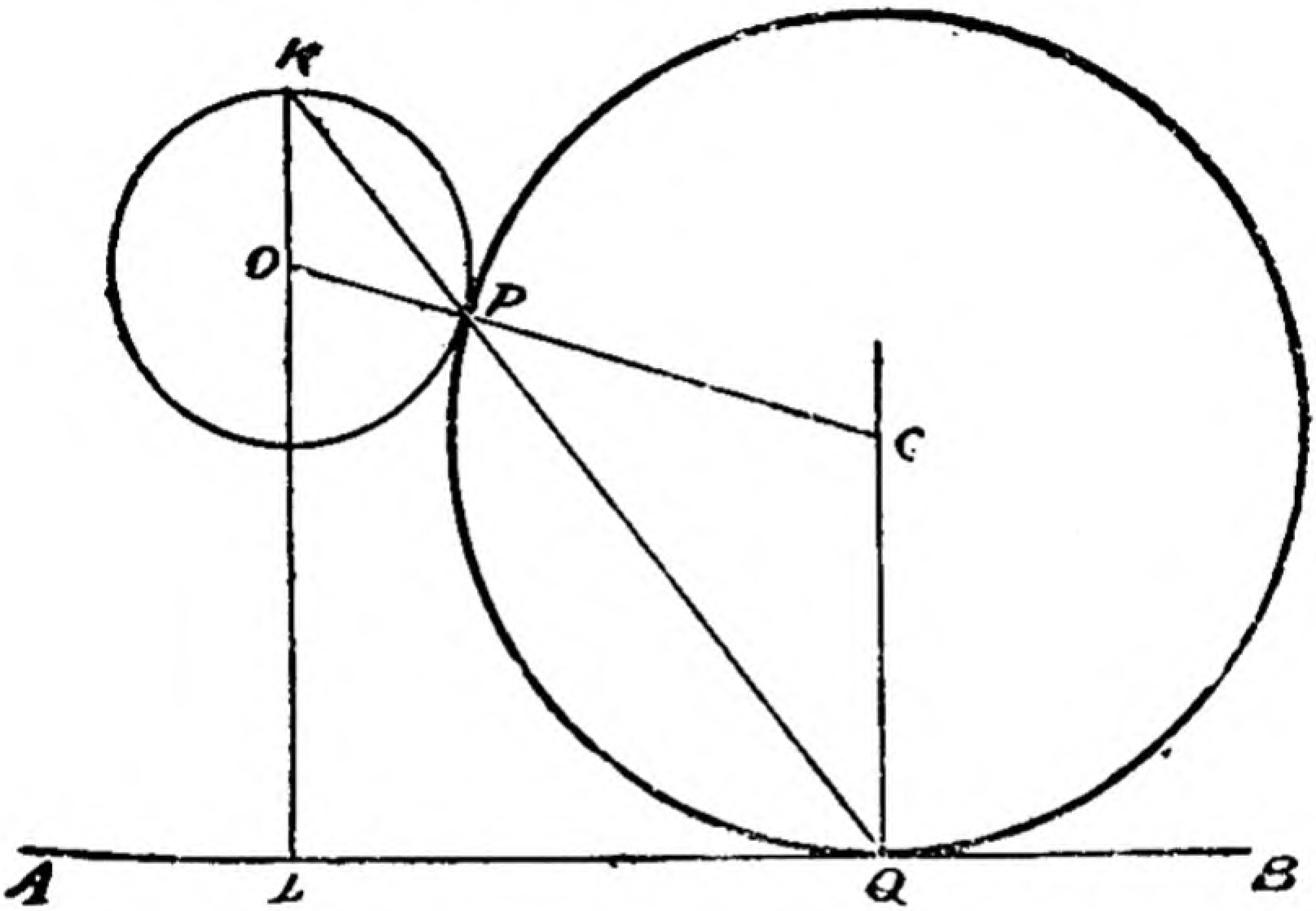


FIG. 2

3. *To draw a circle touching a given ⊙ and a given straight line at a given point.*

AB the given straight line and Q a given point on it. The circle with centre O the given ⊙.

From O drop perpendicular OL on AB.



Produce LO to meet the given ⊙ at K.

Join KQ cutting the given ⊙ at P.

At Q erect a perpendicular to AB and produce OP to meet this perpendicular at C.

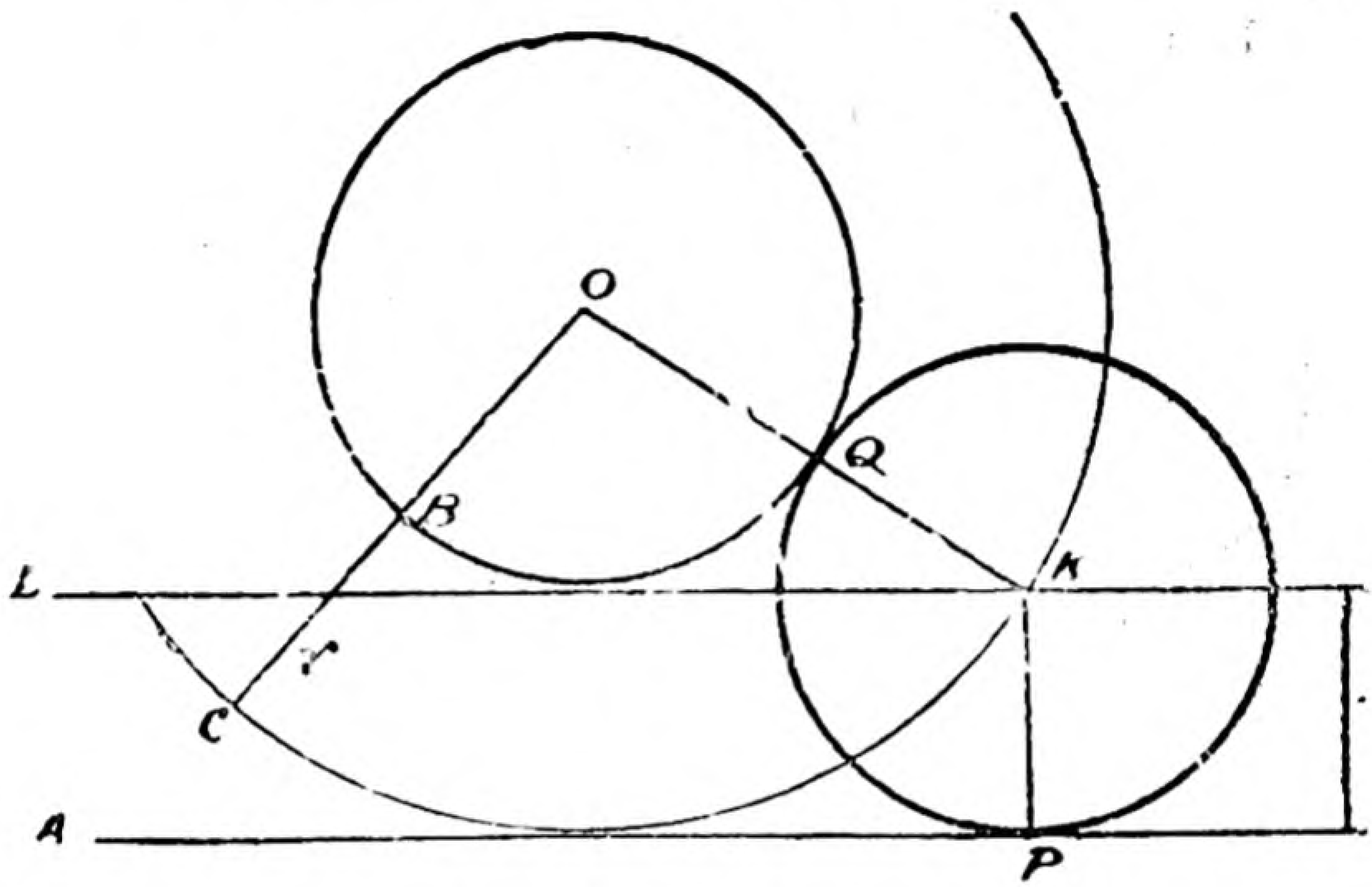
With centre C and radius = CP or CQ describe a circle. This is the required ⊙ which touches AB at Q and the given circle at P.

Ex. To draw a ⊙ touching a given st. line and a given ⊙ at a given point.

4. *To draw a ⊙ of a given radius tauching a given line and also touching a given circle.*

AB given line and ⊙ with centre O the given circle.

Draw a line LM ∥ AB at a distance r from it.



Draw any radius OB of the given circle and produce it to some point C such that $BC=r$.

With centre O and radius=OC describe a concentric circle to the given circle cutting line LM at K. Then K is the centre of the required circle. With K as centre and radius$=r$ describe a circle that will touch line AB at P and the given circle at Q.

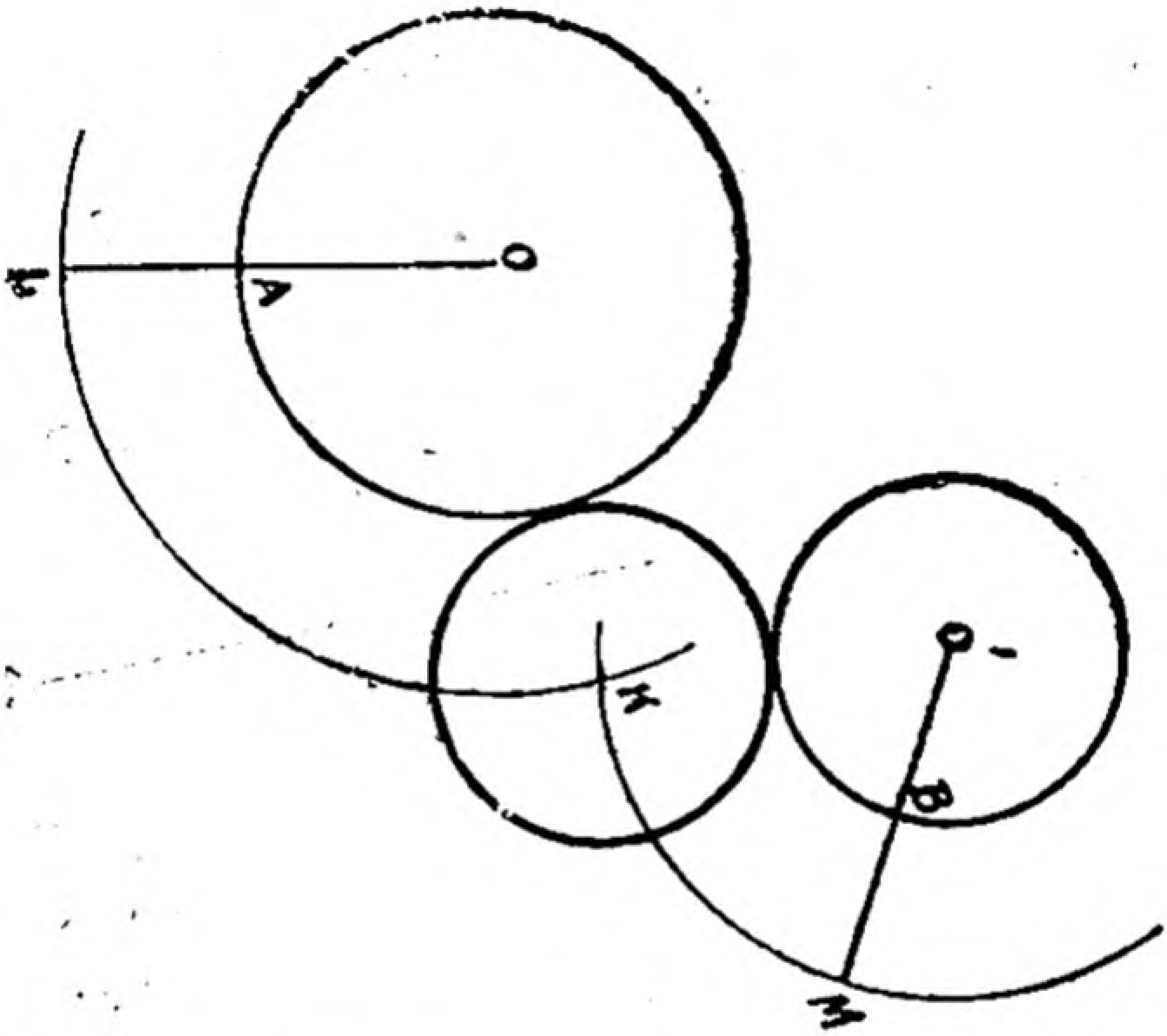
Ex. 1. Draw a ⊙ of given radius, having its centre on a given st. line and touching another given st. line.

Ex. 2 Draw a ⊙ of given radius, having its centre on a given st. line and touching a given ⊙.

Ex. 3. Draw a ⊙ of given radius to touch two given st. lines.

5. *To construct a $\odot$ of given radius touching two given circles.*

(*i*) Circles with centres O and O′ are the two given circles.



(*ii*) Draw two radii OA and O′B.

(*iii*) Produce them to L and M respectively cutting AL and BM each equal to the given radius of the required $\odot$.

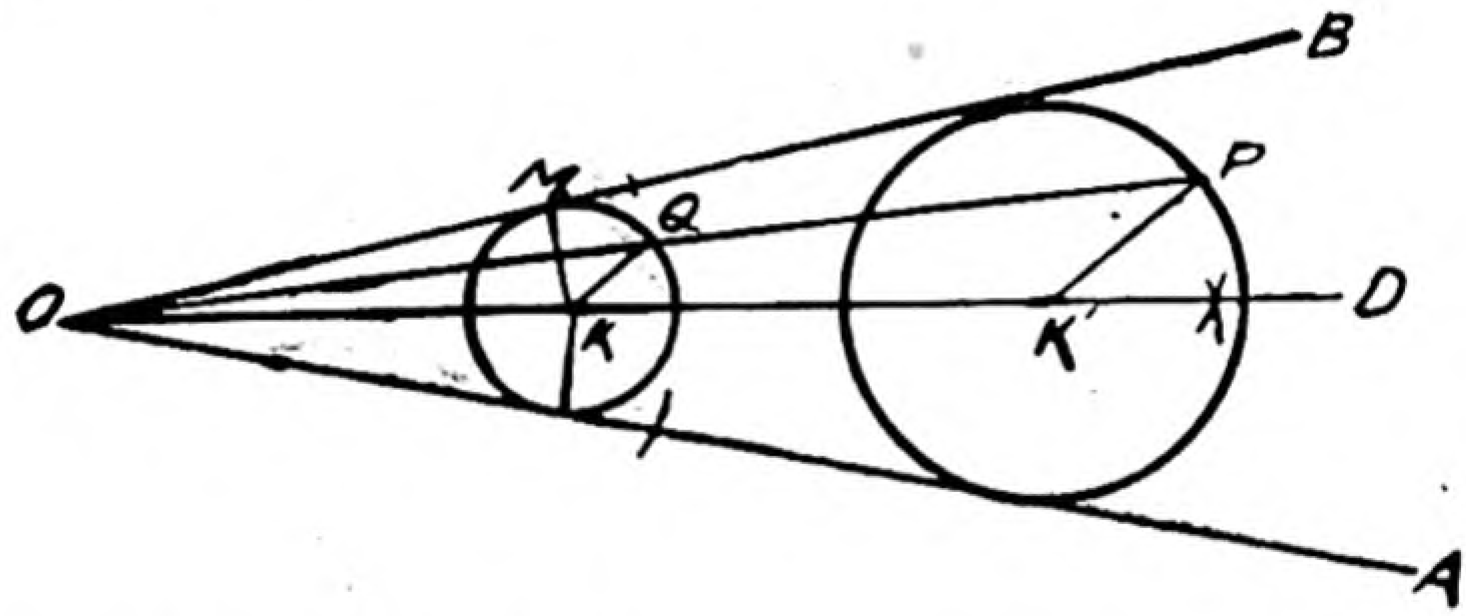
(*iv*) With O and O′ as centres and radii OL and O′M describe circles concentric with the given circles meeting at K. Then K is the centre of the required circle.

(*v*) With centre K and radius equal to the given radius of the required circle describe a circle. It will touch the given $\odot$s.

Ex. With radii $1\frac{1}{2}''$, $2''$ and $2\frac{1}{2}''$ describe three circles each touching the other two externally.

6. *To construct a ⊙ touching two intersecting straight lines and passing through a given point lying between the lines.*

(*i*) OA and OB two given intersecting straight lines.



P the given point.

(*ii*) Draw OD the bisector of the angle AOB.

(*iii*) Take any point K on OD and draw KL and KM perpendiculars on OA and OB respectively.

(*iv*) Describe a ⊙ with centre K and radius=KL or KM. It will touch the line OA and OB.

(*v*) Join OP cutting the circle at Q. Join KQ.

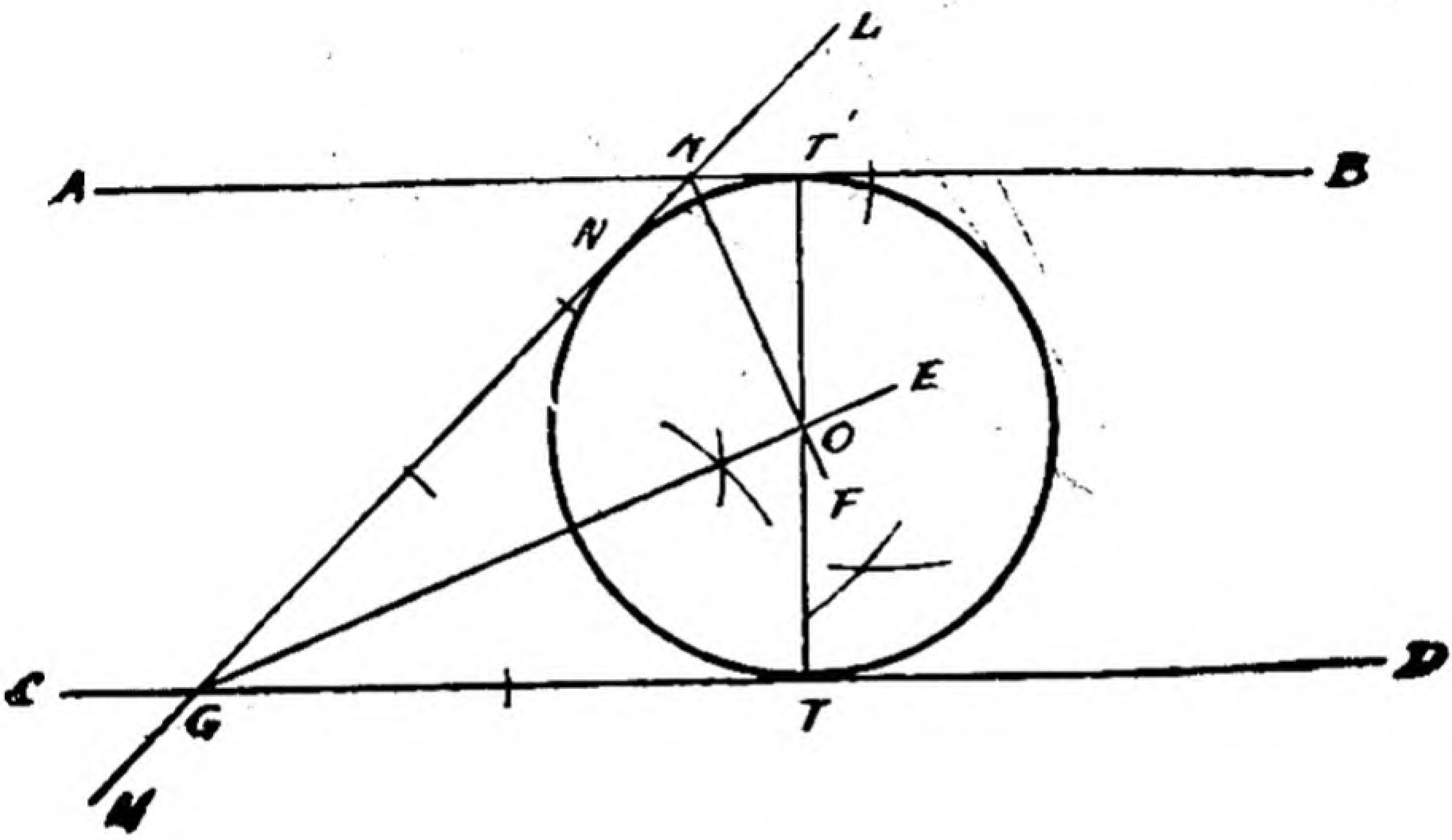
(*vi*) From P draw PK' ∥ KQ cutting OD at K'. Then K' is the centre of the required circle.

(*vii*) With K' as centre and K'P as radius describe the circle. It will touch OA and OB.

7. *To draw a ⊙ touching two parallel straight lines* $1{\cdot}4''$ *apart and a third line cutting them at an angle of* 45°. *Show that two such circles are possible.*

AB and CD two ∥ lines $1{\cdot}4''$ apart. LM third line meeting them at K and G at an angle of 45°.

(*i*) Draw GE and KF bisectors of the angles KGD and BKG meeting at a point O. Then O is the centre of the required ⊙.



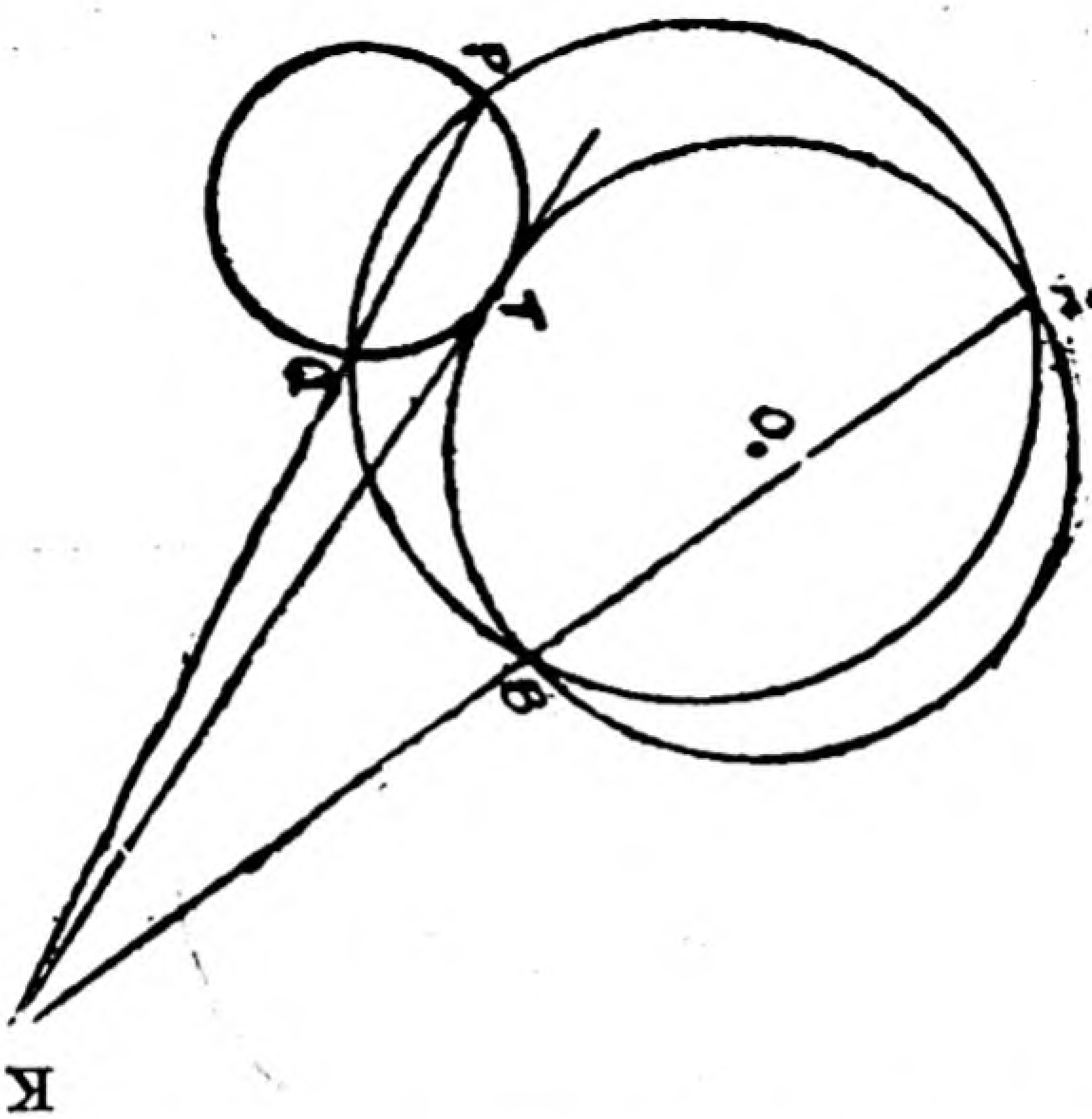
(*ii*) From O draw OT ⊥ GD.

(*iii*) With centre O and radius OT draw a circle. It will touch all the three lines at T, N and T′.

Another circle can be drawn on the other side of the line LM. Students should try to draw this themselves.

8. *To draw a circle passing through two given points and to touch a given circle.*

P and Q two given points, and a circle with the centre O given.



(*i*) Draw a circle passing through P and Q and cutting the given ⊙ at A and B.

(*ii*) Join PQ and AB and produce them to meet at K.

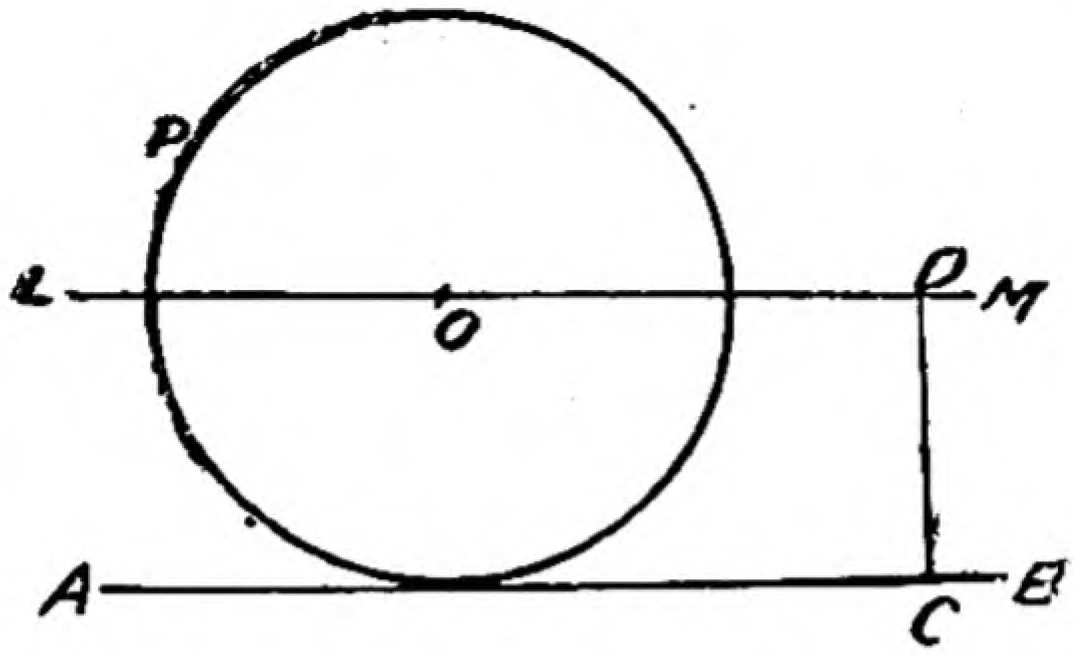
(*iii*) From K draw KT tangent to the given circle touching it at the point T.

(*iv*) Now draw a circle passing through three points P, Q and T. This is the required ⊙.

Prove thatth is circle touches the given circle.

9. *To construct a ⊙ of given radius touching a given straight line and passing through a given point.*

P a given point and AB a given straight line.



(i) At any point C on AB erect CD perpendicular to it and make CD equal to the given radius.

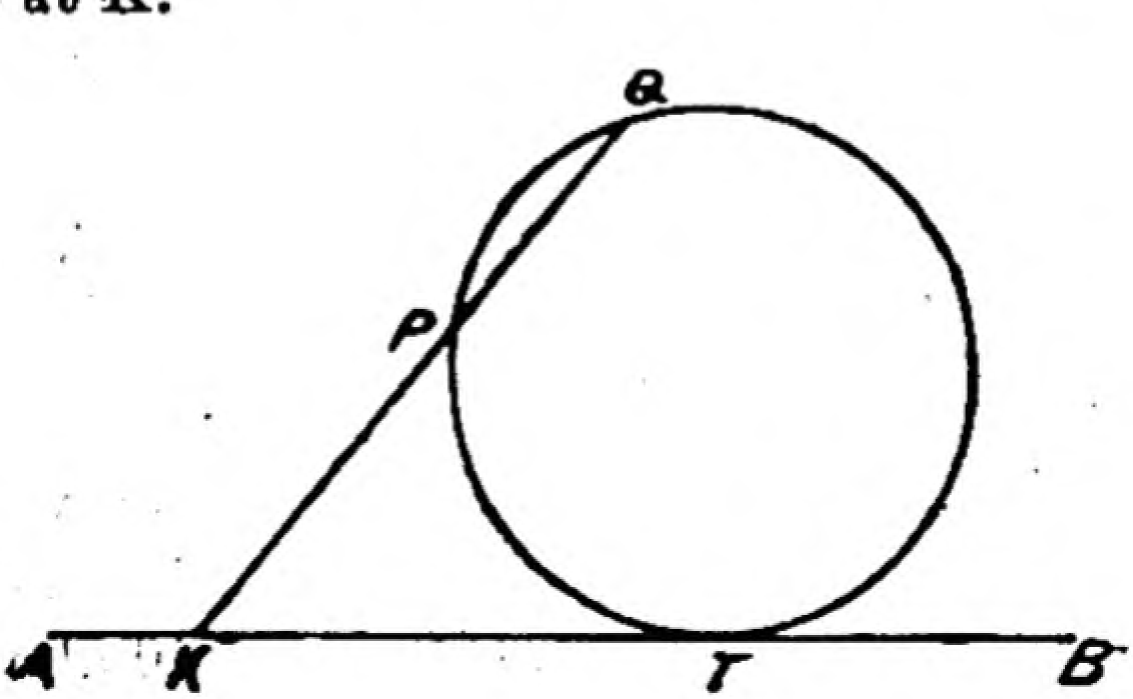
(ii) Through D draw LM ∥ AB.

(iii) With P as centre and radius equal to the given radius describe an arc cutting LM at O. Then O is the centre of the required circle. A ⊙ drawn with O as centre and radius OP which is equal to the given radius shall pass through P and shall touch the line AB.

10. *To describe a circle passing through two given points and touching a given straight line.*

P and Q two given points. AB a given st. line.

(i) Join QP and produce it to meet the given line AB at K.



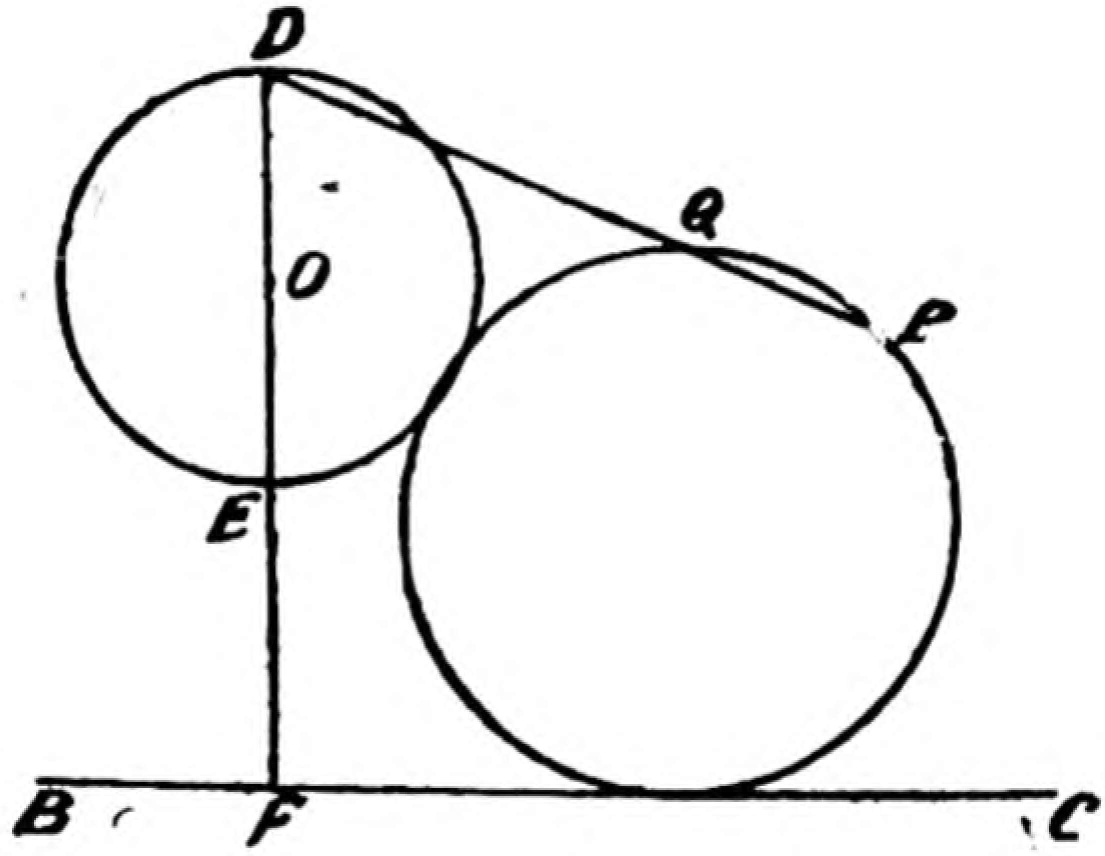
(*ii*) Cut off KT equal to the mean proportional between KQ and KP.

(*iii*) Describe a circle passing through the three points, P, Q and T. This is the required circle.

11. *To construct a circle passing through a given point and touching a given straight line and a given* ⊙.

Point P and line BC given and also circle with centre O is known.

Draw diameter DE of the given circle, ⊥ to BC and let it meet the line BC at F.



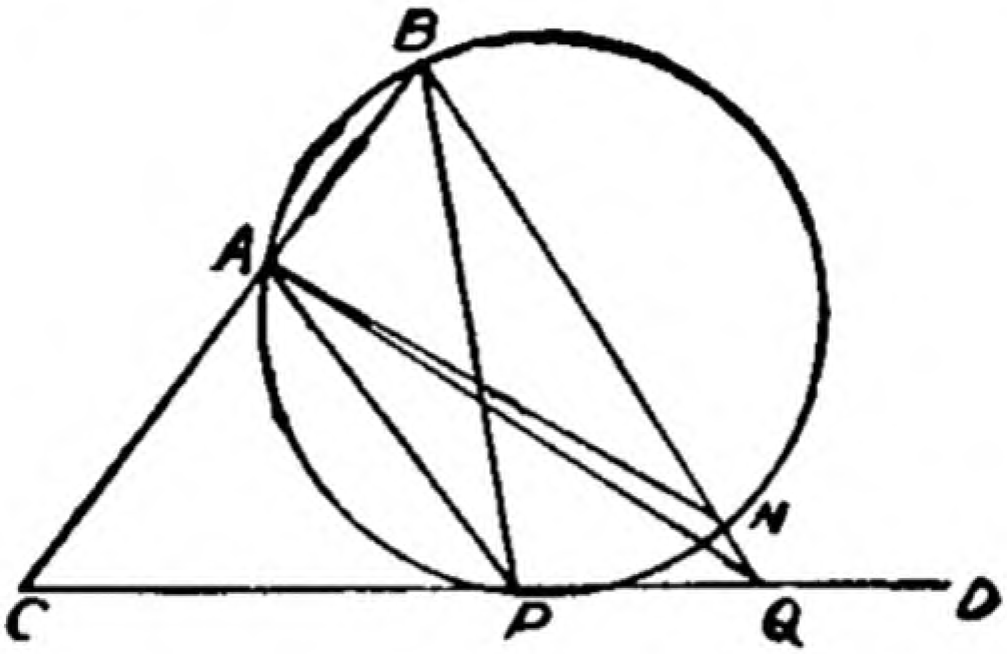
Point O and line BC being given, the lines DE DF are of known lengths. Join DP. P being known, DP also is a known line.

Determine a fourth proportional to DP, DE and DF. From DP cut off DQ equal to this fourth proportional. Then point Q is determined.

Now draw circle passing through two given points P, Q and touching the given straight line BC. This has already been explained in Example 10 above.

12. *A, B are two points on the same side of a st. line CD. Find a point P in CD such that the angle APB is the greatest possible.*

A and B two given points and CD given line.



Draw a ⊙ passing through points A and B and touching the st. line CD and P as in Example 10.

Then P is the required point.

Join PA and PB. Take any other point Q on the line CD and join AQ, BQ also.

Let BQ cut the circle at N. Join AN.

$\angle APB = \angle ANB$ (in the same segment)

But $\angle ANB > \angle AQB$

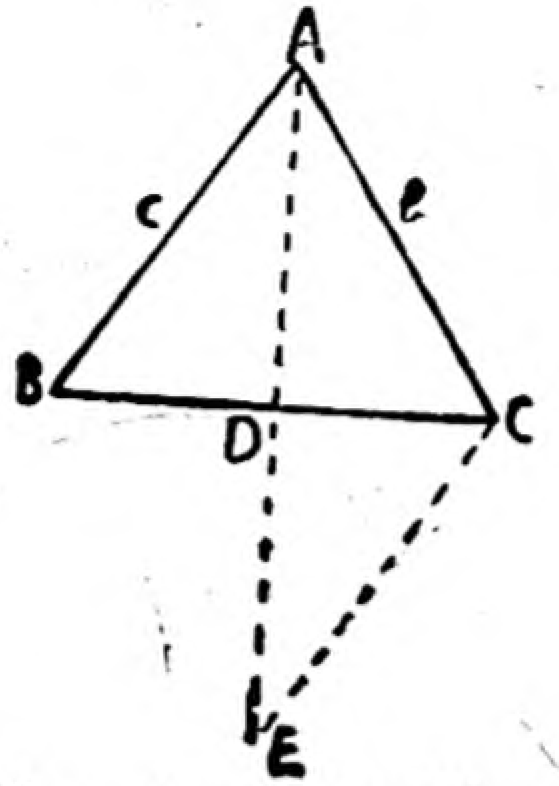
$\therefore$ $\angle APB > \angle AQB$.

Similarly it can be shown that $\angle APB$ is greater than an angle formed at any other point of the line CD.

$\therefore$ $\angle APB$ is the greatest.

13. *Construct a △ having given two sides and the included median.*

Given :—Two sides, c, b and the included median m_1.



(*i*) Construct the $\triangle$ACE with AE$=2m_1$, AC$=b$, CE$=c$.

(*ii*) Join C to the mid point of AE.

(*iii*) Produce CD to B making DB$=$CD.

(*iv*) Join AB.

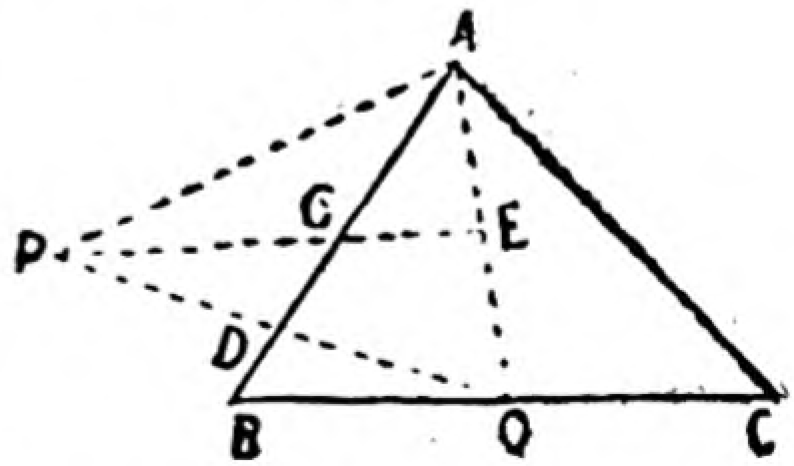
Then ABC is the required $\triangle$.

Ex. Construct a $\triangle$, having given a median, and the two $\angle$s into which the corresponding $\angle$ is divided by the median.

14. *Draw a $\triangle$ having given three medians.*

Let m_1, m_2, m_3 be the three medians.

(*i*) Draw a $\triangle$ APQ with AQ$=m_1$, AP$=m_2$, and PQ$=m_3$.



(*ii*) Draw AD, PE, the medians of $\triangle$APQ intersecting in G.

(*iii*) Produce AD to B so that GB$=$AG.

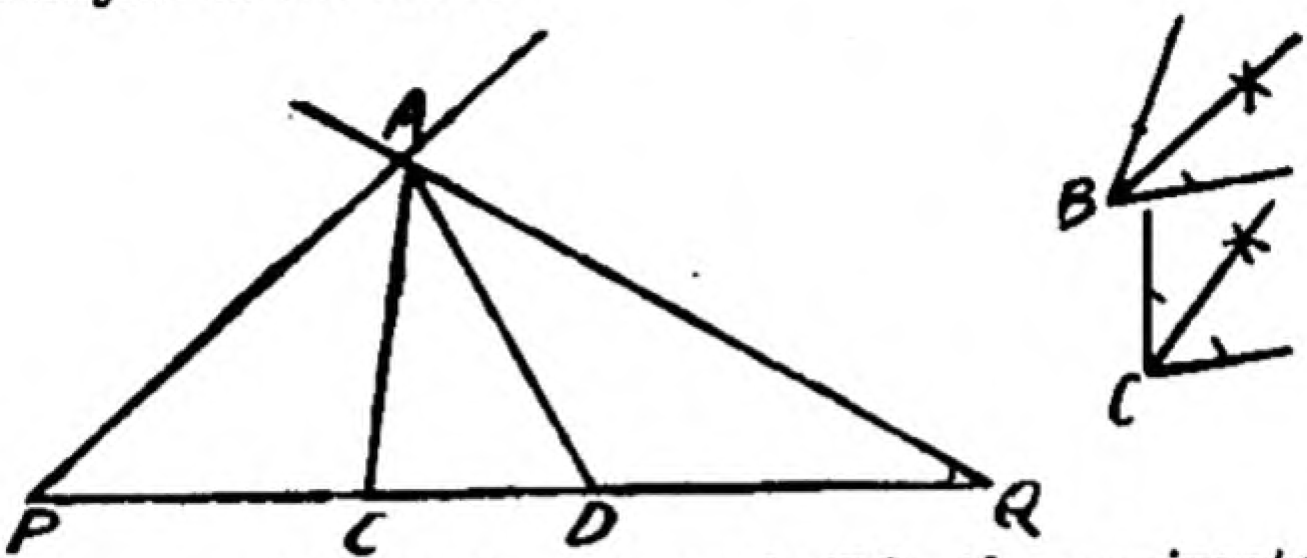
(*iv*) Join BQ and produce it to C so that QC$=$BQ.

(*v*) Join AC.

Then ABC is the required $\triangle$.

Ex. Construct a $\triangle$, having given the base, and the medians from the base $\angle$s.

15. *Construct a triangle having given the perimeter and the angles at the base.*



Given :—$\angle$B and $\angle$C and PQ the perimeter of the $\triangle$.

(*i*) At P and Q make $\angle$s equal to $\frac{1}{2}$ C and $\frac{1}{2}$ B respectively.

Let the other arms of these angles intersect at A.

(*ii*) With QA at A make $\angle QAB' = \angle Q$. Also with PA at A make $\angle PAC' = \angle P$. Then AB′C′ is the required $\triangle$.

Proof :—$\angle Q = \angle B'AQ \quad \therefore AB' = B'Q$.

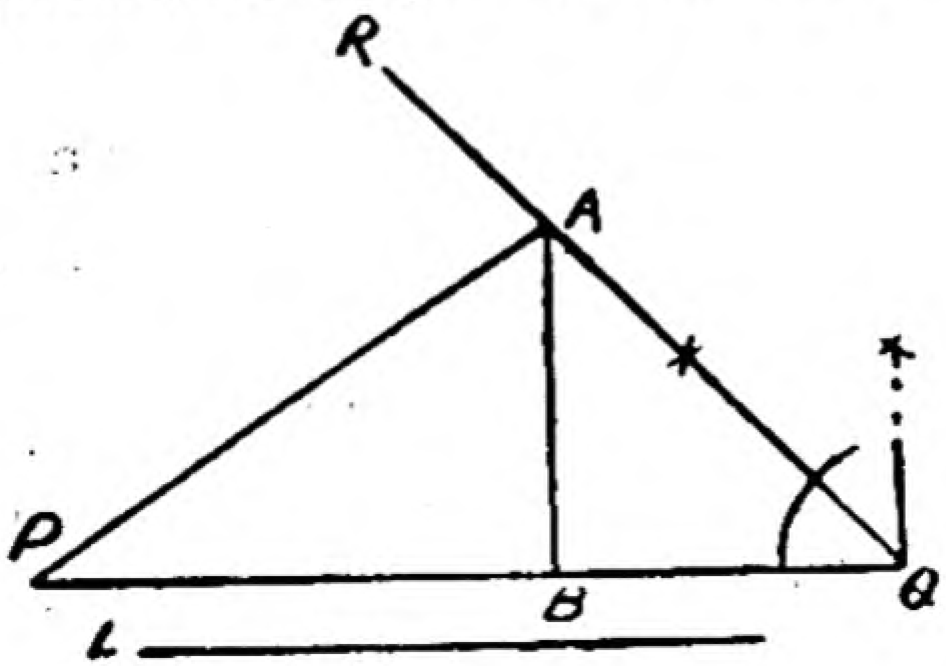
Also $\quad \angle P = \angle PAC' \quad \therefore AC' = C'P$.

$\therefore AB' + AC' + B'C' = PQ$. (The given perimeter).

$\angle AB'C' = 2\angle Q = \angle B$ (Construction).

$\angle AC'B' = 2\angle P = \angle C$.

16. *To construct a right-angled triangle having given the hypotenuse and the sum of the other two sides.*



Given :—PQ the sum of the two sides and **L** the hypotenuse.

(*i*) At Q erect QC $\perp$ PQ.

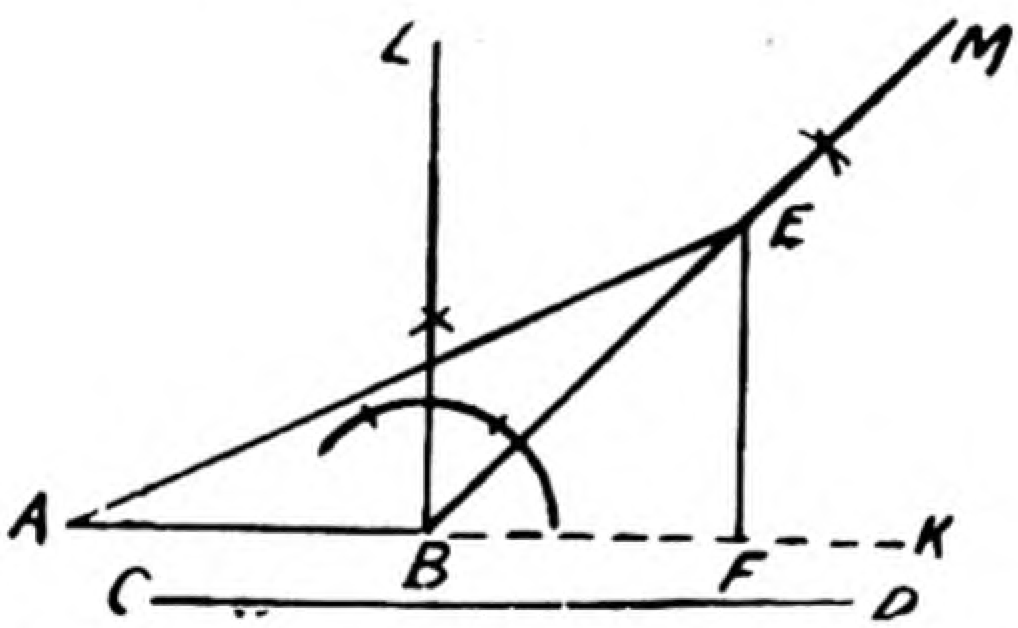
(*ii*) Draw QR the bisector of the $\angle$CQP.

(*iii*) With P as centre and radius=given length L describe a $\odot$ cutting QR at A.

(*iv*) From A draw AB $\perp$ PQ. Then ABP is the required triangle.

Supply proof.

17. *To construct a right angled triangle having given the hypotenuse and the difference of two sides.*



Given CD equal to the hypotenuse and AB the difference of two sides.

(*i*) At B erect a perpendicular BL, and produce AB to K.

(*ii*) Draw BM the bisector of the angle LBK.

(*iii*) With A as centre and radius=CD describe an arc cutting BM at E.

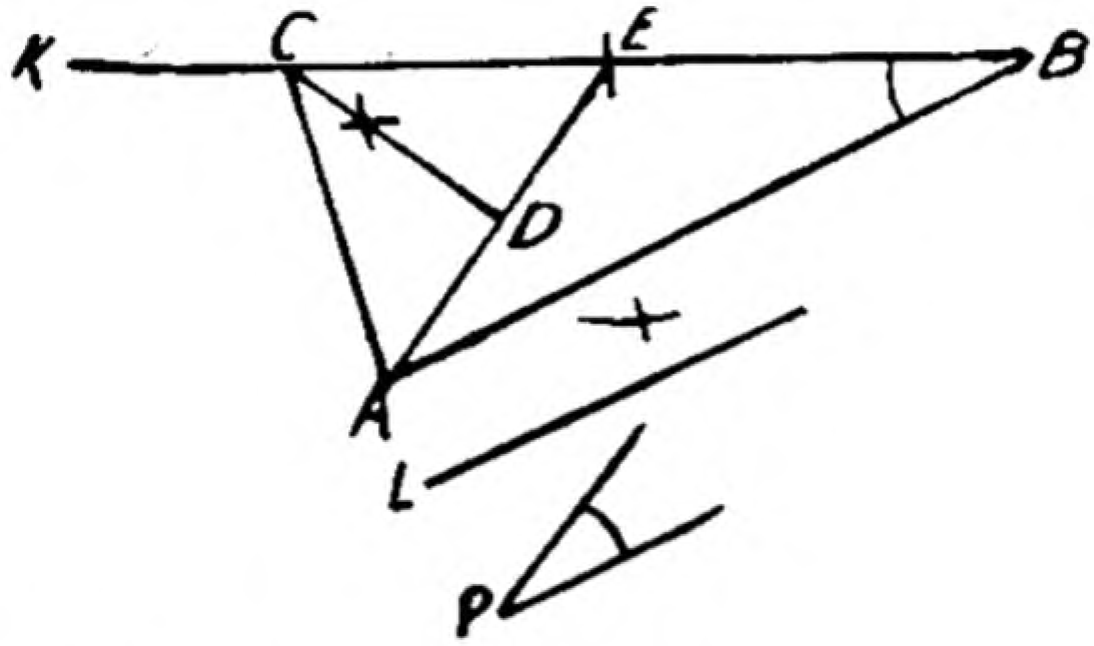
(*iv*) Join AE and draw EF perpendicular to BK.

Then AFE is the required rt. angled triangle.

Justify this construction by advancingpr oof.

18. *Given the base of a triangle, the difference of the other two sides and the lesser angle at the base; construct the triangle.*

Given base AB, the smaller of the base angles P and L, the difference of the two sides.



(*i*) At B make $\angle$KBA=$\angle$P.

(*ii*) From line KB cut off BE=given length L.

(*iii*) Join AE and draw the perpendicular bisector DC of the line AE meeting the line KB at C.

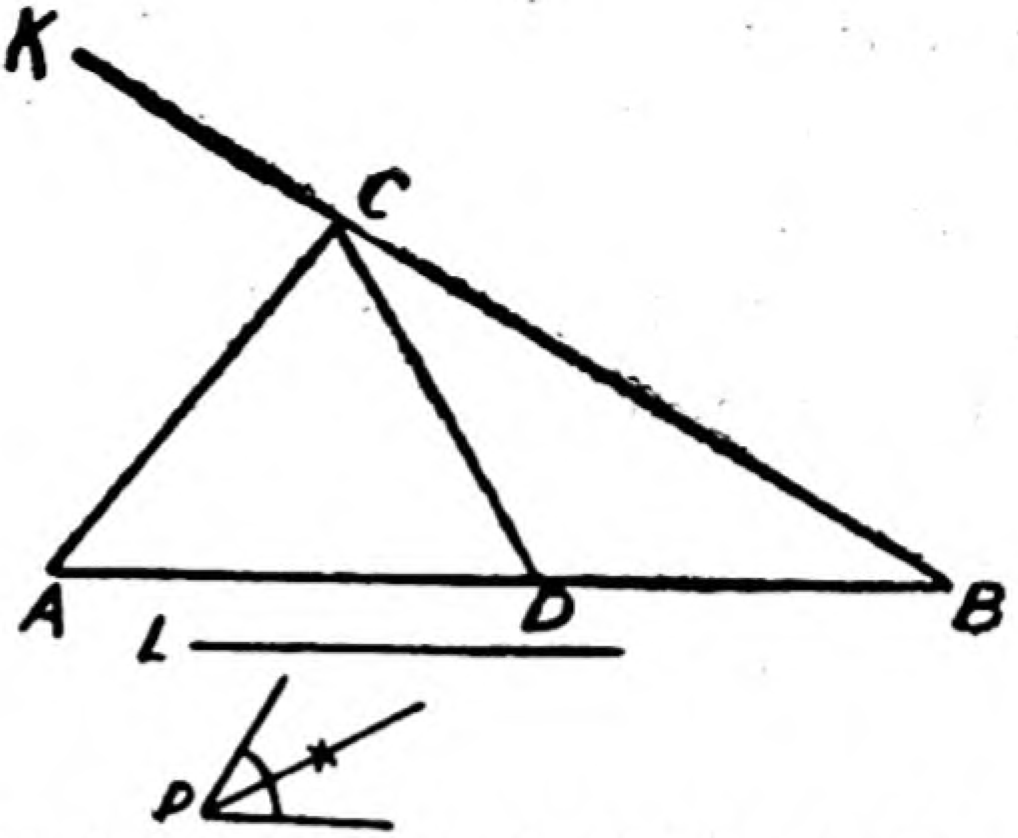
(*iv*) Join AC. Then ABC is the required $\triangle$.

Students should advance proof.

Ex. Construct a $\triangle$, having given the base, the sum of two sides, and one $\angle$ at the base.

19. *Construct a $\triangle$ having given a side, the angle opposite to the side, and the sum of the other two sides.*

Given one side equal to length L; angle opposite to this side equal to $\angle$P; the sum of the other two sides equal to AB.



(*i*) At B make an angle KBA$=\frac{1}{2}\angle$P.

(*ii*) With centre A and radius=length L describe an arc cutting KB at C

(*iii*) With BC at C make $\angle$BCD=$\angle$CBA.

Then ACD is the required $\triangle$.

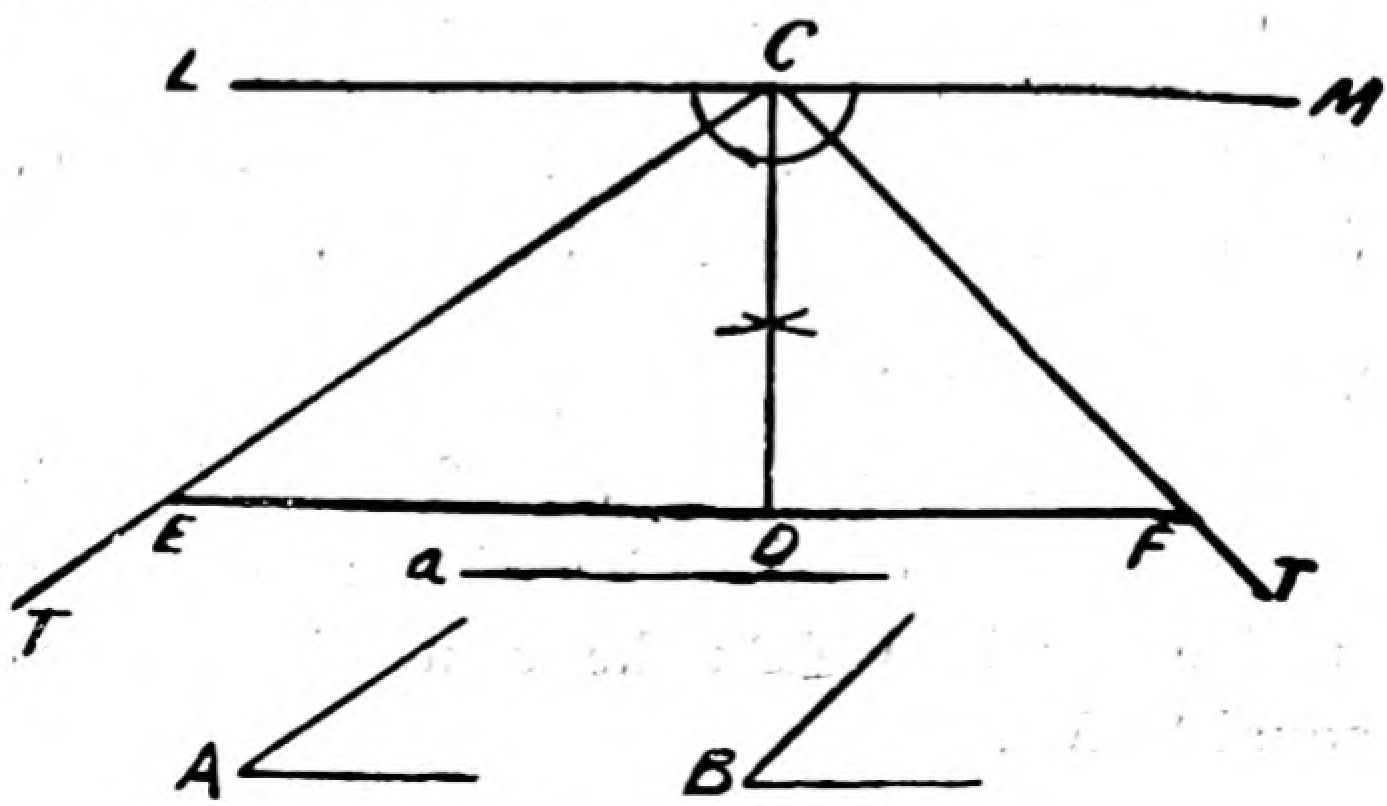
Proof :—$\triangle$BCD is an isosceles triangle.

$\therefore$ DB=DC also $\angle$ADC=2 $\angle$B=given $\angle$P.

Ex. Construct a $\triangle$, having given a side, the $\angle$ opposite to the side, and the difference of the other two sides.

20. *Construct a triangle having given the altitude and the angles at the base.*

Given the base angles equal to $\angle$s A and B and altitude=length *a*.



(*i*) Take any point C in an indefinite line **LM.**

(*ii*) Make angle MCT=$\angle$B and angle LCT$'$=$\angle$A.

(*iii*) Erect at C, CD perpendicular to LM and make it=given altitude *a*.

(*iv*) Through D draw a line $\parallel$ to LM meeting CT and CT$'$ at F and E.

Then ECF is the required $\triangle$.

Proof to be supplied by the student.

Ex. 1. Construct a $\triangle$ ABC having its altitude AL $=1{\cdot}2''$ and the sides AB, AC equal to $2{\cdot}1''$, $2{\cdot}3''$.

Ex. 2. Construct a $\triangle$, having given the base, the altitude, and one $\angle$ at the base.

Ex. 3. Construct a $\triangle$, having given the base the altitude and one of the remaining sides.

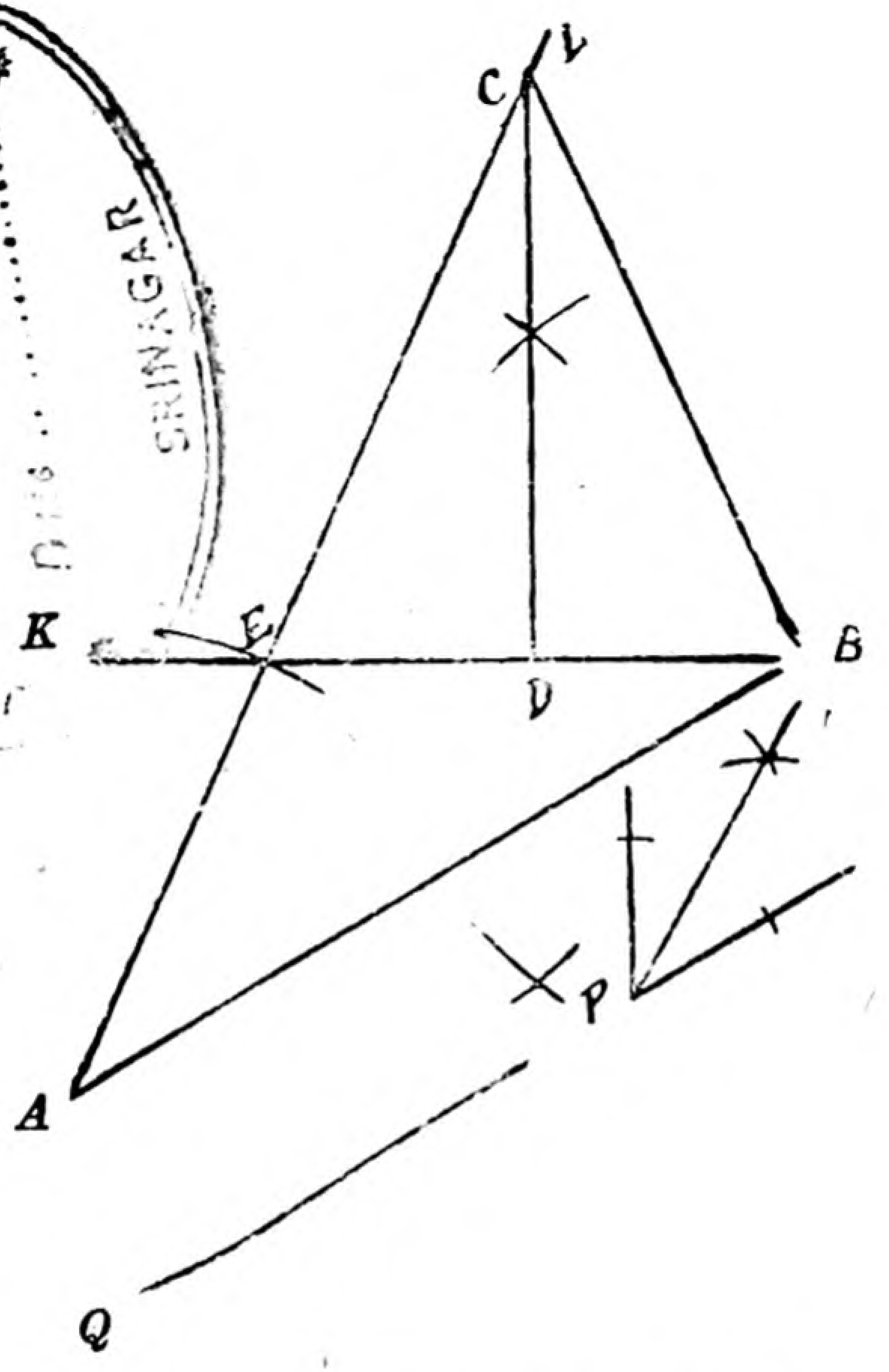
Ex. 4. Construct a $\triangle$, having given the base, the altitude, and the median bisecting the base.

Ex. 5. Construct an isosceles $\triangle$, given its base and height.

Ex. 6. Construct a $\parallel^m$ having given the base, the altitude and one of the following elements : other side, an $\angle$, a diagonal.

21. *Construct a triangle, having given the base the difference of the other two sides and the difference of the angles opposite to these sides.*

Given the difference of two sides equal to a given length *a* ; base equal to AB ; and the difference of two equal to P.



(*i*) At A make an angle KBA$=\frac{1}{2}\angle$P

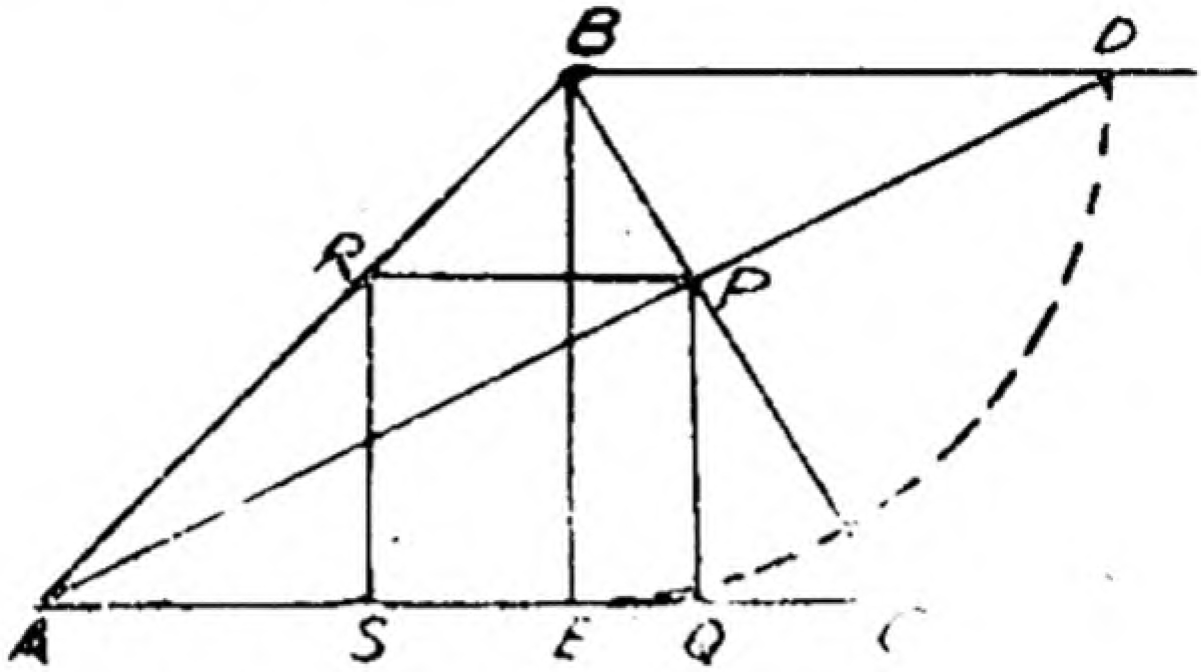
(*ii*) With centre A and radius$=a$ draw an arc cutting KB at E. Join AE and produce it to L.

(*iii*) Draw CD the perpendicular bisector of EB meeting AL at C. Join CB. Then ABC is the required triangle.

Proof :—As DC is the bisector of EB, $\triangle$ ECB is isosceles. $\therefore$ CB$=$CE ; also $\angle$CBE$=\angle$CEB $\therefore$ AC$-$CB $=$AE$=a$.

Also $\angle CBA - \angle CAB = \angle CBE + \angle EBA - \angle CAB$
$= \angle EBA + \angle CEB - \angle CAB.$
$= \angle EBA + \angle EBA = 2\angle EBA =$ given $\angle P$.

22. *In a given $\triangle$ inscribe a square.*



Let ABC be the $\triangle$.

(*i*) Drop BE $\perp$ AC.

(*ii*) Draw BL $\parallel$ AC and cut off BD $=$ BE.

(*iii*) Join AD cutting BC at P.

(*iv*) Draw PQ $\perp$ L and PR $\parallel$ AC.

Draw RS $\parallel$ PQ.

Then $\angle$QSR is the required square.

Proof :—Each angle of PQRS is a right-angle and opposite sides are $\parallel$, therefore the figure is a **rectangle.**

In $\triangle$ABD, RP $\parallel$ BD.

$$\therefore \frac{BD}{RP} = \frac{AB}{AR} \text{..........................} (i)$$

Also in $\triangle$ABE, RS $\parallel$ BE

$$\therefore \frac{AB}{AR} = \frac{BE}{RS} \text{..................... ....} (ii)$$

From (*i*) and (*ii*) $\dfrac{BD}{RP} = \dfrac{BE}{RS}$.

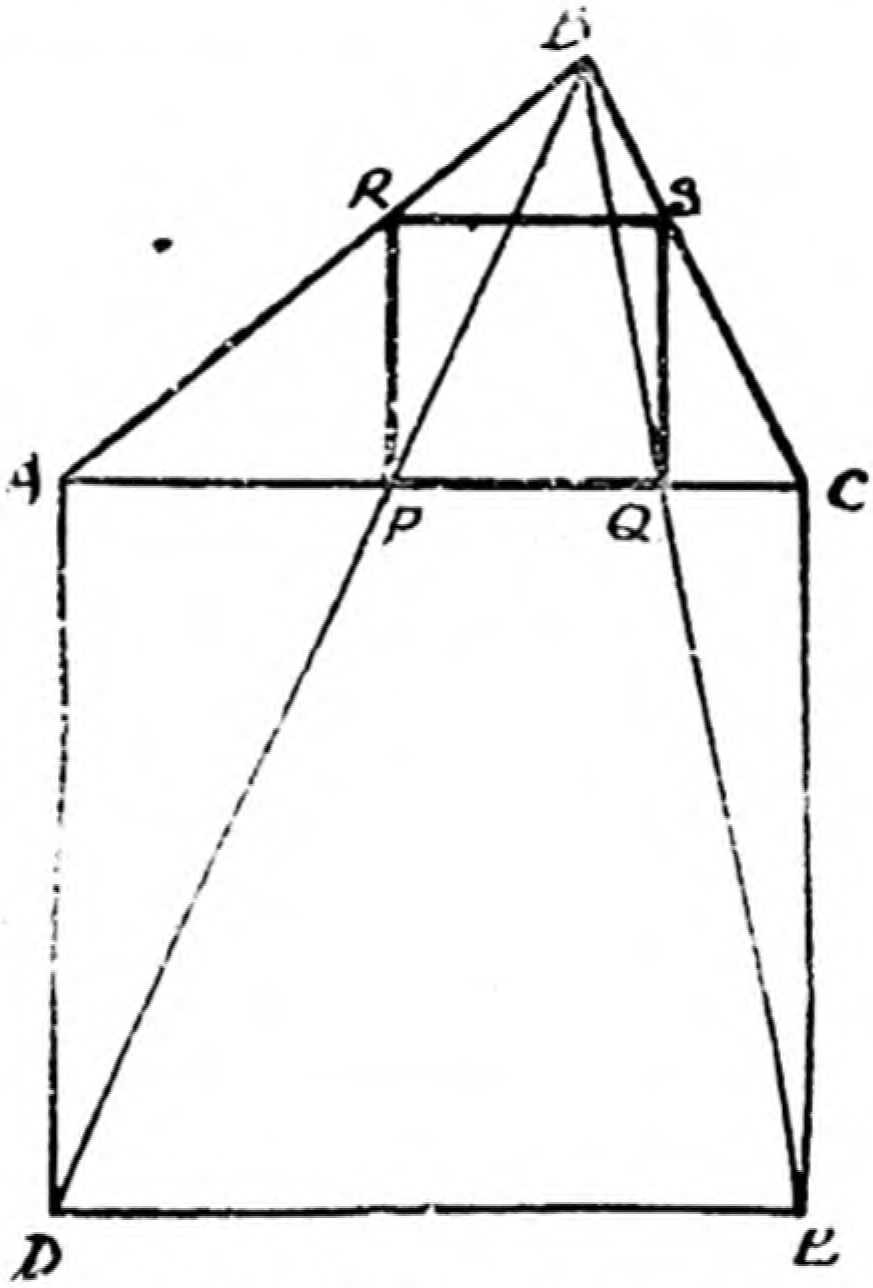
But BD $=$ BE (Const.)

$\therefore$ RP=RS.

$\therefore$ RSQP is a square.

Otherwise

(*i*) Construct ADEC a square on AC.



(*ii*) Join BD and BE cutting AC at P and Q respectively.

(*iii*) At P and Q erect perpendiculars PR and QS to line AC, meeting AB and CB at R and S.

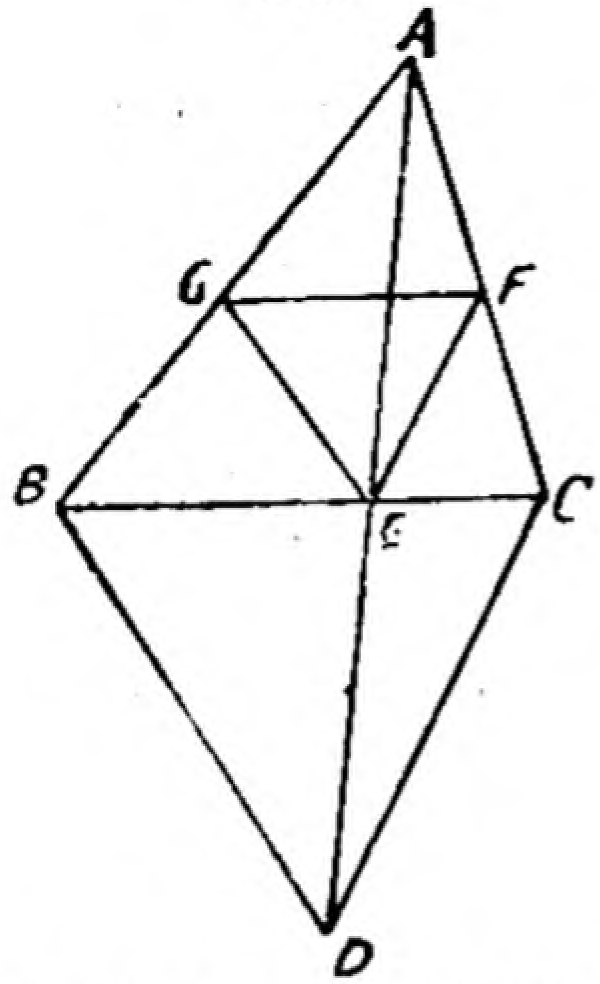
(*iv*) Join RS.

Then PQSR is the required square.

Supply proof.

23. *Construct an equilateral triangle in a given* $\triangle$.

ABC given triangle, apply second method explained in Example 22 above.



(*i*) Construct BCD, an equilateral △ on BC.

(*ii*) Join AD cutting BC at E.

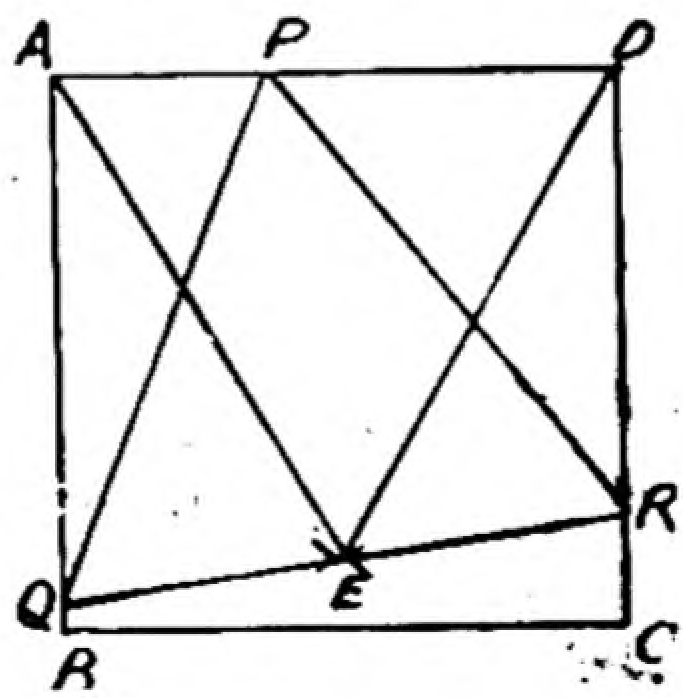
(*iii*) From E draw EF ∥ CD and EG=BD.

(*iv*) Join GF. Then EFG is the required equilateral triangle.

24. *Inscribe an equilateral triangle in a given square having the vertex at a given point in one of the sides of the square.*

ABCD is the given square and P the given point in AD.

(*i*) Construct equilateral △ADE on the side AD.



(*ii*) Join PE and through E draw QER at right angles to PE.

(*iii*) Join PQ, PR. Then PQR is the required equilateral $\triangle$.

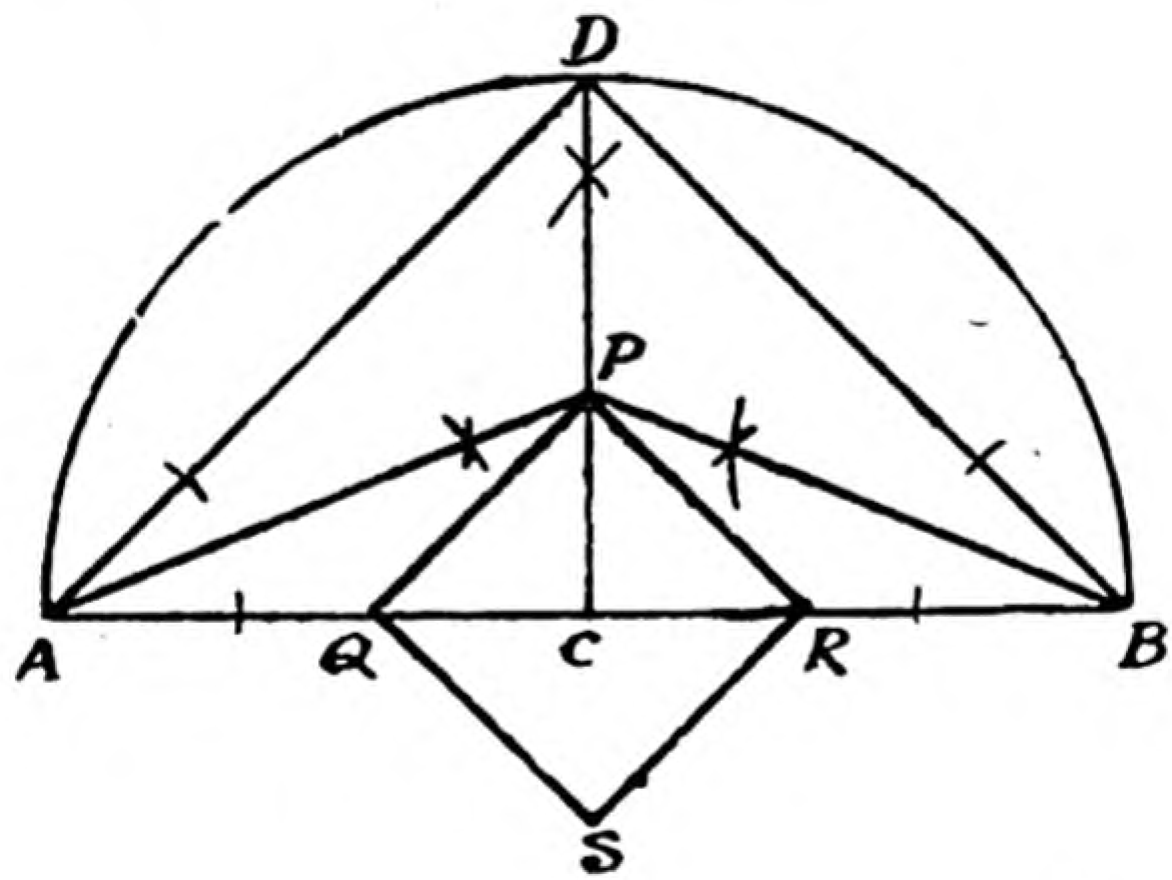
Proof :—Figure PERD is cyclic because $\angle$PDR and $\angle$PER are both right angles.

$\therefore$ $\angle PRE = \angle PDE = 60^\circ$.

also $\angle PQE = \angle PAE = 60^\circ$ as APEQ is cyclic.

25. *Construct a square having given the sum of the diagonals and two sides.*

Given :—AB the sum of two sides and one diagonal of the square.



Construction :—(*i*) On AB as diameter describe an arc of a circle.

(*ii*) Draw CD the perpendicular bisector of AB meeting the arc at D.

Join AD, DB.

(*iii*) Bisect the angles DAB and DBA. The bisectors BP and AP shall meet each other at P on the line CD.

(*iv*) From P draw PQ and PR $\parallel$ to AD and BD respectively. PQ and PR are the two sides of the required square.

Complete the square PQSR.

Proof :—AD $\parallel$ to QP and DB $\parallel$ PR.

$\therefore\ \angle QPR=\angle ADB=90^\circ$.

$\angle PAQ=\angle DAP=\angle APQ \quad \therefore\ AQ=PQ$.

Similarly RB=PR.

$\therefore\ AQ+RB+QR=PQ+QR+PR$

Or $AB=PQ+PR+QR$.

Now in $\triangle ACD$, PQ $\parallel$ AD

$\therefore\ \dfrac{PQ}{AD}=\dfrac{CP}{CD}$ ....(*i*)

Also in $\triangle BCD$, PR $\parallel$ BD.

$\therefore\ \dfrac{PR}{BD}=\dfrac{CP}{CD}$............................(*ii*)

$\therefore\ \dfrac{PQ}{AD}=\dfrac{PR}{BD}$ from (*i*) and (*ii*)

But AD=BD

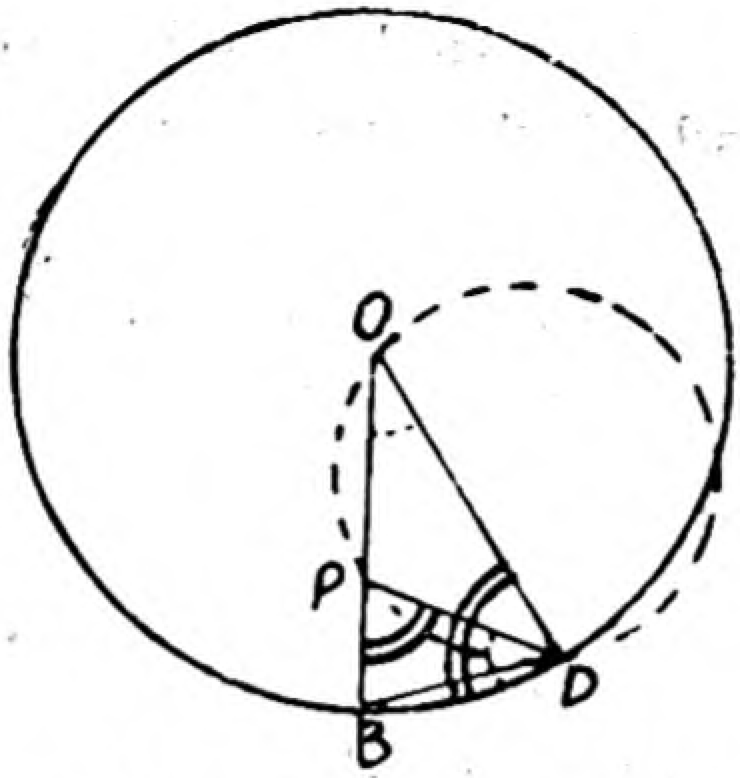
$\therefore\ PQ=PR$

$\therefore$ PQRS is a square.

26. *To construct an isosceles $\triangle$ having each of the base angles double the vertical angle.*

(*i*) Take a circle with centre O and radius OB.

(*ii*) Divide OB at point P in medial section so that $OP^2=OB.BP$.



(*iii*) In the circle place a chord BD equal in length to OP. Join OD and PD,

Then OBD is the required triangle.

Proof :—Describe circumcircle of $\triangle$OPD.

Now $OP^2 = OB.BP$, but $OP = BD$.

$\therefore \quad BD^2 = OB.BP$

$\therefore$ BD is a tangent to the $\odot$ POD.

$\angle BDP = \angle DOP$ (in the alternate segment of the smaller circle).

$\angle BPD = \angle POD + \angle PDO$

$\quad = \angle PDB + \angle PDO = \angle BDO$

$\quad = \angle OBD \;\therefore\; \triangle OBD$ is isosceles

$\therefore \; PD = BD = OP$

$\therefore \; \angle POD = \angle ODP$.

Again $\angle BPD = \angle POD + \angle PDO$.

$\quad = 2\,\angle POD$.

But $\angle BPD = \angle OBD$ (Proved above).

$\therefore \; \angle OBD = 2\,\angle POD$.

27. *Construct an isosceles $\triangle$ on a base 1·5″ long, each of whose base angles is double of the verticle angle.*

Hint.—In the fig. of the last Ex., draw a $\triangle$ equi-angular to $\triangle$OBD on a base$=1{\cdot}5$.

Or alternatively

Draw $AB = 1{\cdot}5''$ and bisect it in P.

Erect $BQ \perp AB$ and equal to it.

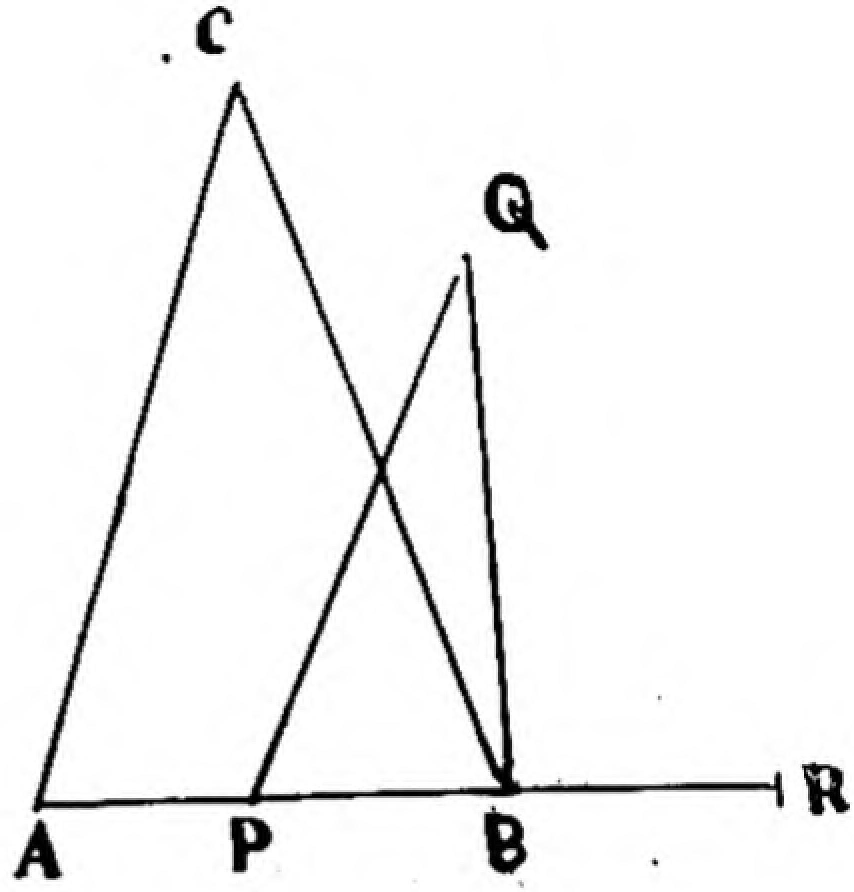
Join PQ.

Produce AB and cut off $PR = PQ$.

With A and B as centres and radius $= AR$, draw arcs intersecting at C.

Join AC, BC.

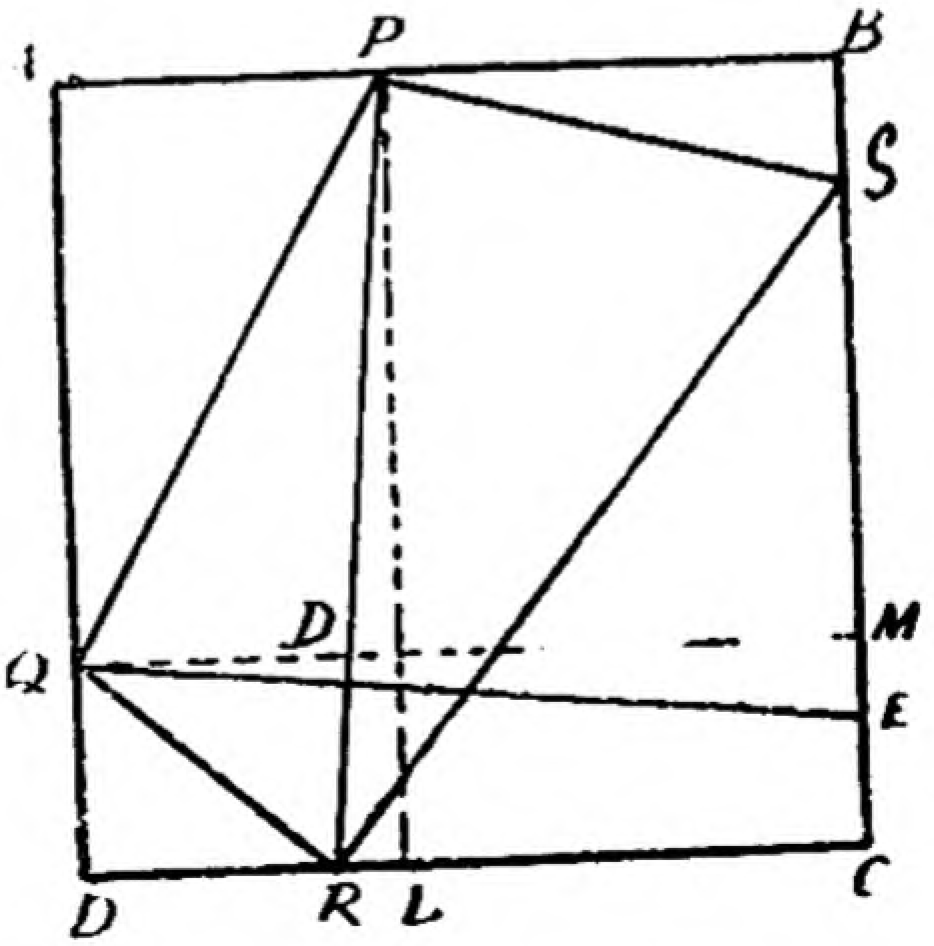
Then ABC is the rqd. $\triangle$.



28. *To describe a square about a given quadrilateral.*

PQRS is the given quadrilateral.

(*i*) Join PR.



(*ii*) From Q drop QD perpendicular on PR and produce it to E making QP=PR.

(*iii*) Join SE.

(*iv*) Through Q draw a line ∥ to SE.

(*v*) From P and R draw lines perpendicular to SE meeting SE produced at B and C and its parallel through Q at A and D.

Then ABCD so formed is the required square.

Proof :—From the construction ABCD is a rectangle. Drop PD and QM perpendiculars from L and Q on DC and BC.

In the △s PRL and QEM.

∠PLR=∠QME (Right angles).

Also ∠RPL=∠EQM.

Hyp. PR=hyp. QE.

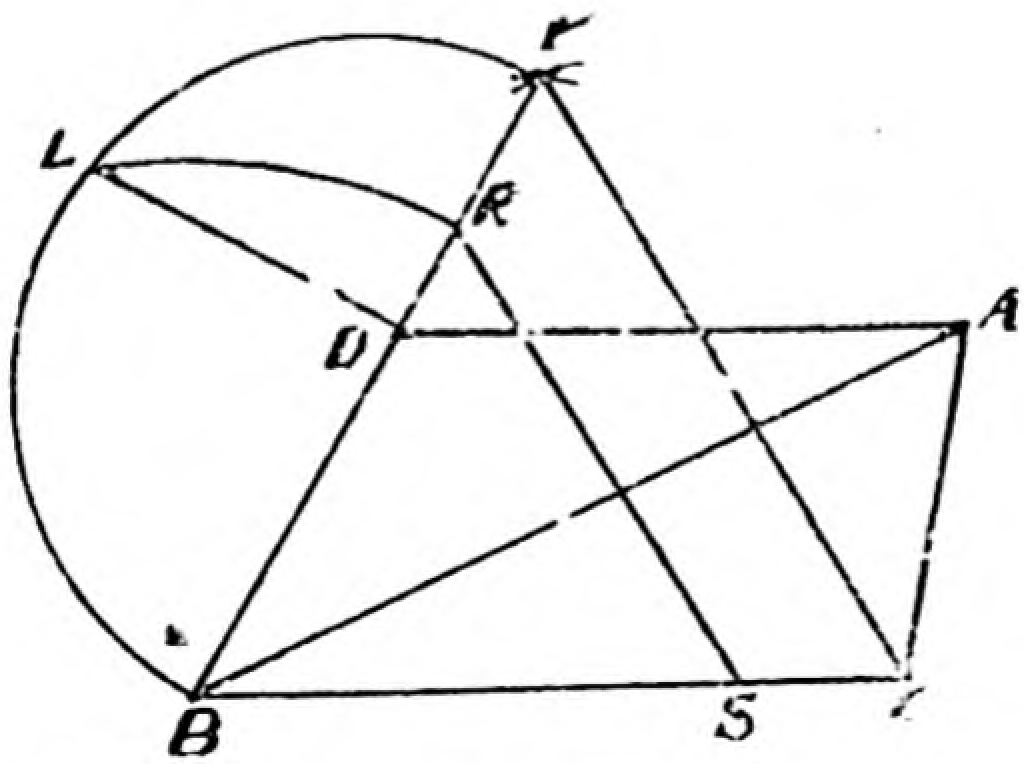
∴ △s are congruent.

∴ PL=QM.

Or AD=AB.

∴ ABCD is a square.

29. *Describe an equilateral triangle equal to a given triangle.*



ABC, the given triangle.

(*i*) On the base BC of the given $\triangle$ construct an equilateral triangle PBC.

(*ii*) On PB describe a semi-circle.

(*iii*) From A draw AD $\parallel$ BC, cutting PB at D; also draw DL $\perp$ PB meeting the semi-circle at L.

(*iv*) With B as centre and radius=BL draw an arc cutting BP in R.

(*v*) From R draw RS $\parallel$ PC. Then RSB is the required equilateral $\triangle$.

Proof :— $\dfrac{\triangle RBS}{\triangle PBC}=\dfrac{BR^2}{BP^2}=\dfrac{BL^2}{BP^2}$

But $BL^2=BD.BP$.

$\therefore \dfrac{\triangle RBS}{\triangle PBC}=\dfrac{BD.BP}{BP^2}=\dfrac{BD}{BP}$

But $\dfrac{BD}{BP}=\dfrac{\triangle BCD}{\triangle BCP}=\dfrac{\triangle ABC}{\triangle BCP}$ (AD $\parallel$ BC).

$\therefore \quad \dfrac{\triangle RBS}{\triangle BCP}=\dfrac{\triangle ABC}{\triangle BCP}$

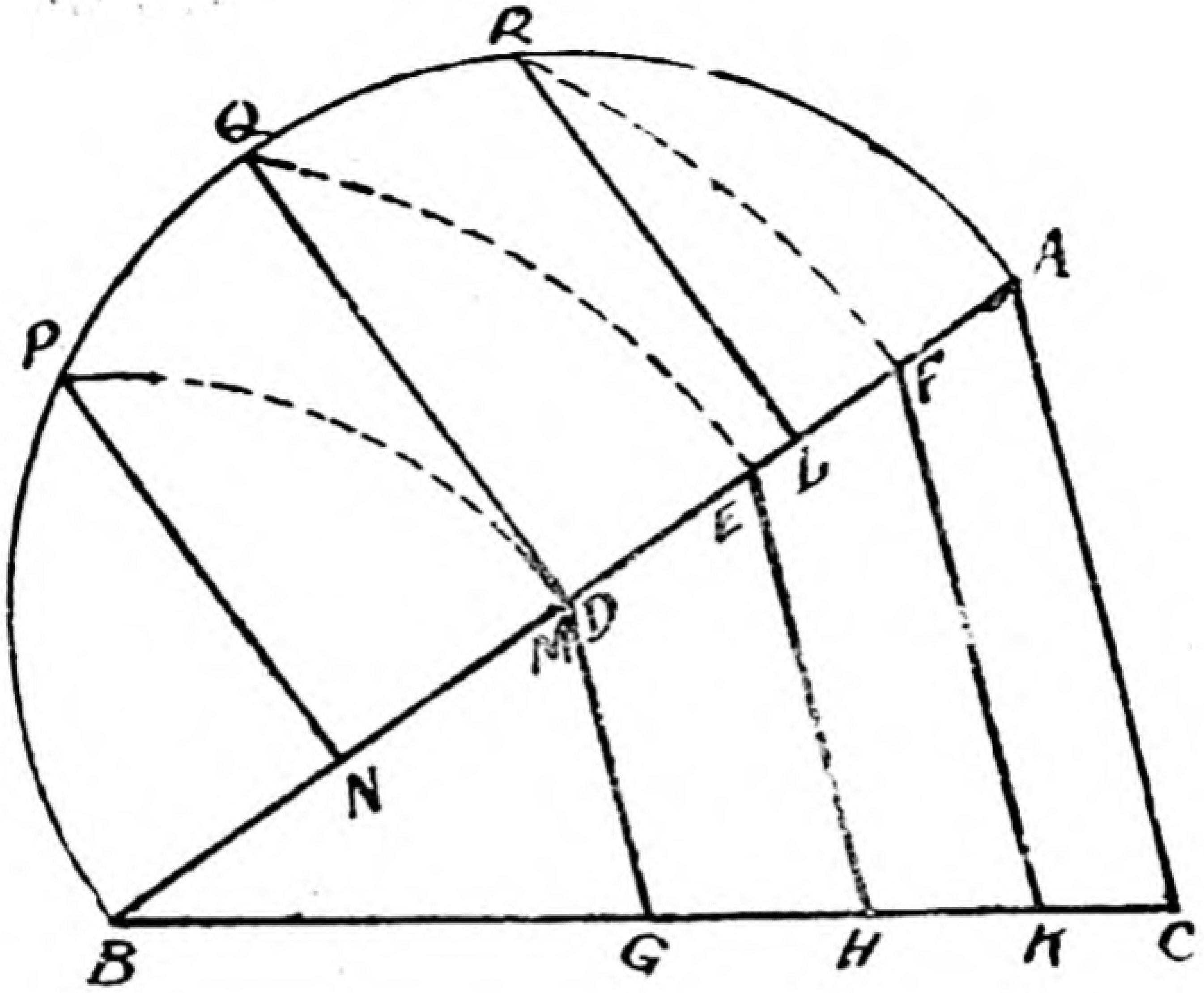
$\therefore \quad \triangle RBS=\triangle ABC.$

Now angle $S=\angle C=60^\circ$ and $\angle R=\angle P=60^\circ$

$\therefore$ SR $\parallel$ PC.

$\therefore \quad \triangle RBC$ is an equilateral triangle and equal to $\triangle ABC$.

30. *Divide a triangle into any number of equal parts by drawing lines parallel to one of its sides.*



(*i*) On the side AB of the $\triangle$ABC describe a semicircle.

(*ii*) Divide AB into as many equal parts as the number of parts into which triangle is to be divided.

Suppose we want to divide the $\triangle$ into four equal parts. The points of division of AB are L, M and N.

(*iii*) Erect perpendiculars at L, M and N meeting the semi $\odot$ at R, Q, and P.

(*iv*) With B as centre and radii equal to BP, BQ and BR draw arcs cutting AB at D, E, and F, respectively.

(*v*) From D, E, and F draw DG, EH and FK $\parallel$ to AC.

The $\triangle$ is split up into four equal parts AFKC FEHK, EDGH and DBG.

Proof :— $\frac{\triangle BGD}{\triangle ABC}=\frac{BD^2}{BA^2}=\frac{BP^2}{BA^2}$

$$=\frac{BN.BA}{BA^2}=\frac{BN}{BA}$$

$$=\tfrac{1}{4}$$

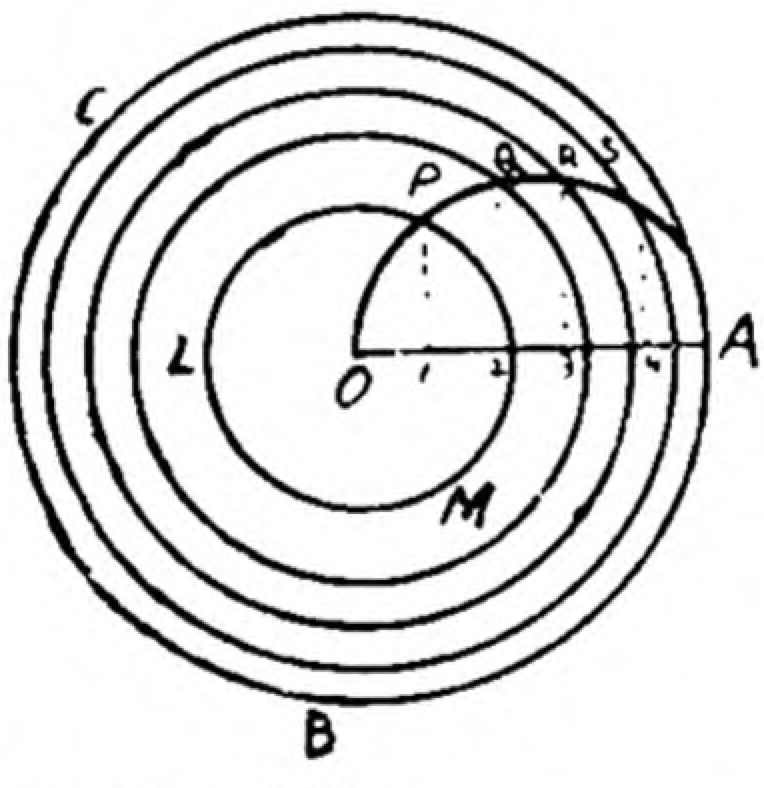
$\therefore$ $\triangle BGD=\frac{1}{4}\triangle ABC$.

Similarly it can be shown that

$\triangle EBH=\frac{1}{2}\triangle ABC$.

and $\angle FBK=\frac{3}{4}\triangle ABC$.

31. *Divide a circle into any number of equal concentric rings.*



Let ABC be the circle.

(*i*) On OA radius describe a semi-circle, and divide OA into as many equal parts as the number of portions into which the circle is to be divided.

Suppose it is required to divide the $\odot$ into 5 equal parts ; then divide OA into 5 equal parts, the points of division being 1, 2, 3 and 4.

(*ii*) Erect perpendiculars to OA at 1, 2, 3 and 4, meeting the semicircle at P, Q, R and S respectively.

(*iii*) With centre O and radii equal to OP, OQ, OR and OS draw circles. These circles will divide the given ⊙ into 5 equal parts, *i.e.* four rings and the fifth innermost circle.

Proof :— $\frac{\odot PLM}{\odot ABC}=\frac{\pi OP^2}{\pi PA^2}=\frac{OP^2}{OA^2}$

But $OP^2=OI.OA^2$.

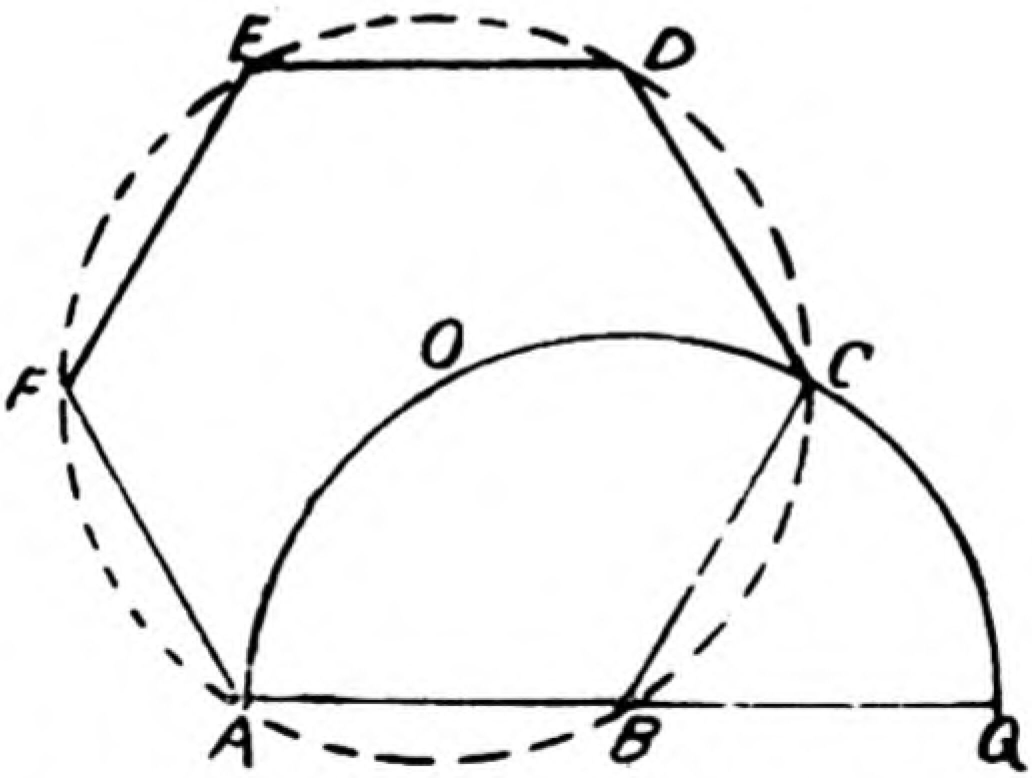
$\therefore \frac{\odot PLM}{\odot ABC}=\frac{OI.OA}{OA}=\frac{OI}{OA}=\frac{1}{5}$

Or $\odot PLM=\frac{1}{5}\odot ABC$.

Similarly it can be shown that the second innermost, the third innermost, etc......⊙s are equal to $\frac{2}{5}$th, $\frac{3}{5}$th, etc. of the given circle.

Hence the result.

32. *On a finite straight line construct a regular hexagon.*



AB given finite st. line.

(*i*) Produce AB to Q making BK=AB.

(*ii*) On AQ as diameter construct a semi-circle.

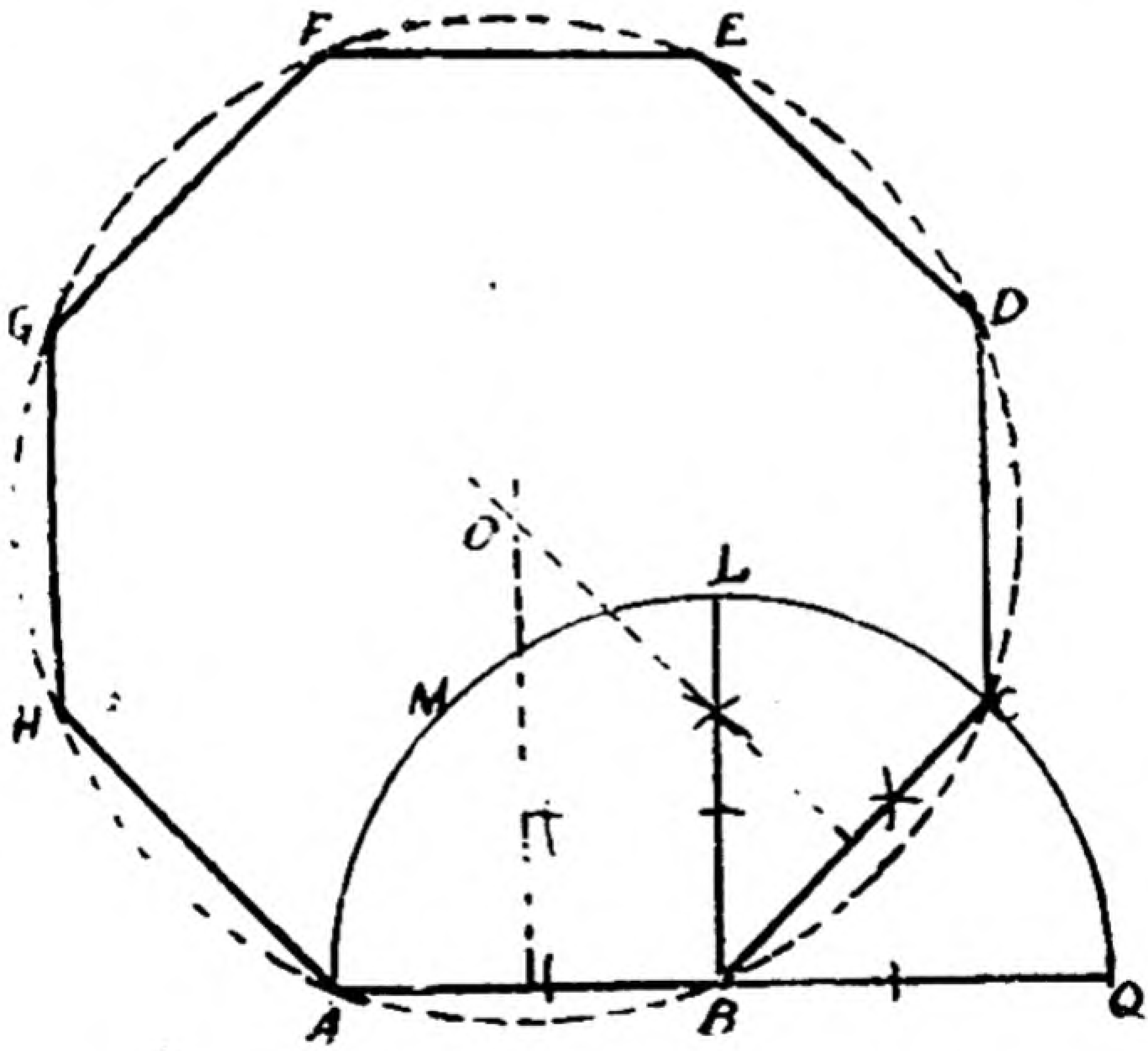
(*iii*) Divide the circumference of this semi-circle into three parts at C and O.

(iv) Join BC and describe the circumcircle of the triangle ABC. Second point of division O will be the centre of this circle and BC=AB.

(v) In this circle place chords CD, DE, EF and FA each equal to AB or BC.

Then ABCDEF is the required haxagon described on the finite line AB.

33. *On a finite straight line construct a regular octagon.*



AB given straight line of finite length.

(i) Produce AB to Q and make BQ=AB.

(ii) On AB describe a semi-circle.

(iii) Divide the ⊙ ce of the semi-circle into four equal parts at C, L, M.

(*iv*) Join BC. Now AB = BC.

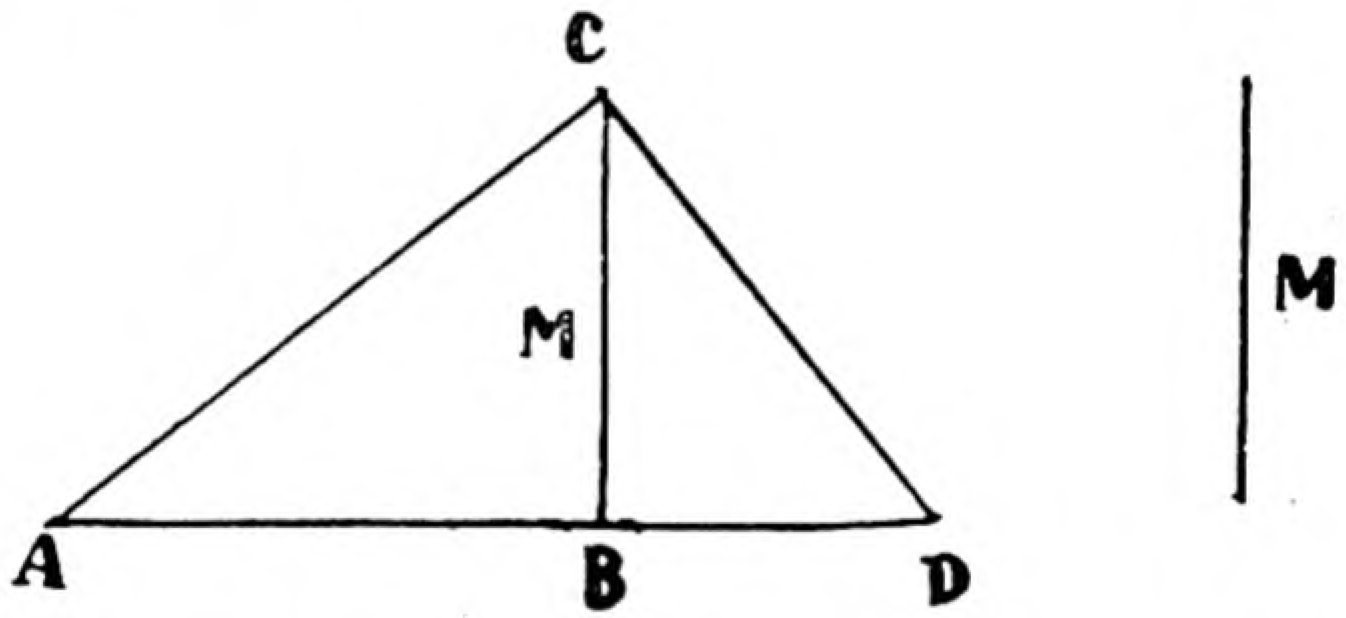
Draw the circumcircle of the $\triangle$ABC.

(*v*) In this circle place CD, DE, EF, FG, GH, HA chords equal to AB or BC.

Then ABCDEFGH is the rqd. octagon described on the line AB.

Q. E. F.

34. On a given st. line AB as base draw a rect. equal to a given square.



Hint :—Let AB be the given side of the rect., and M the side of the given square.

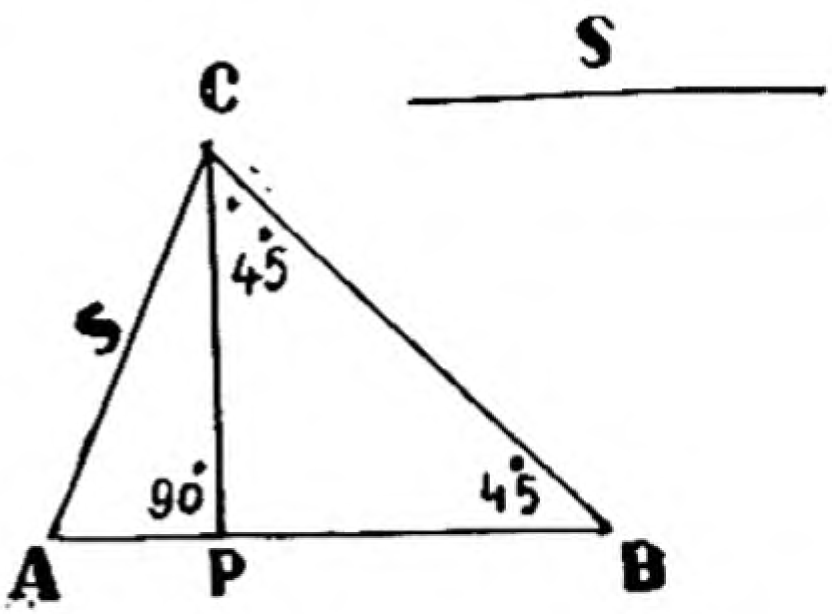
Draw BC $\perp$ AB and $=$ M

Join AC and draw CD $\perp$ AC meeting AB produced in D. Then BD is the other side of the reqd. rect.

Proof :—$BC^2 = AB.BD$ in the rt. $\angle$d. $\triangle$ACD. in which CB is $\perp$ AD.

35. Divide a st. line into two parts such that the sum of their squares is equal to a given square.

Given :—A st. line AB and S the side of a square.



Required:—To divide AB in P so that $AP^2 + PB^2 = S^2$.

Construction :—

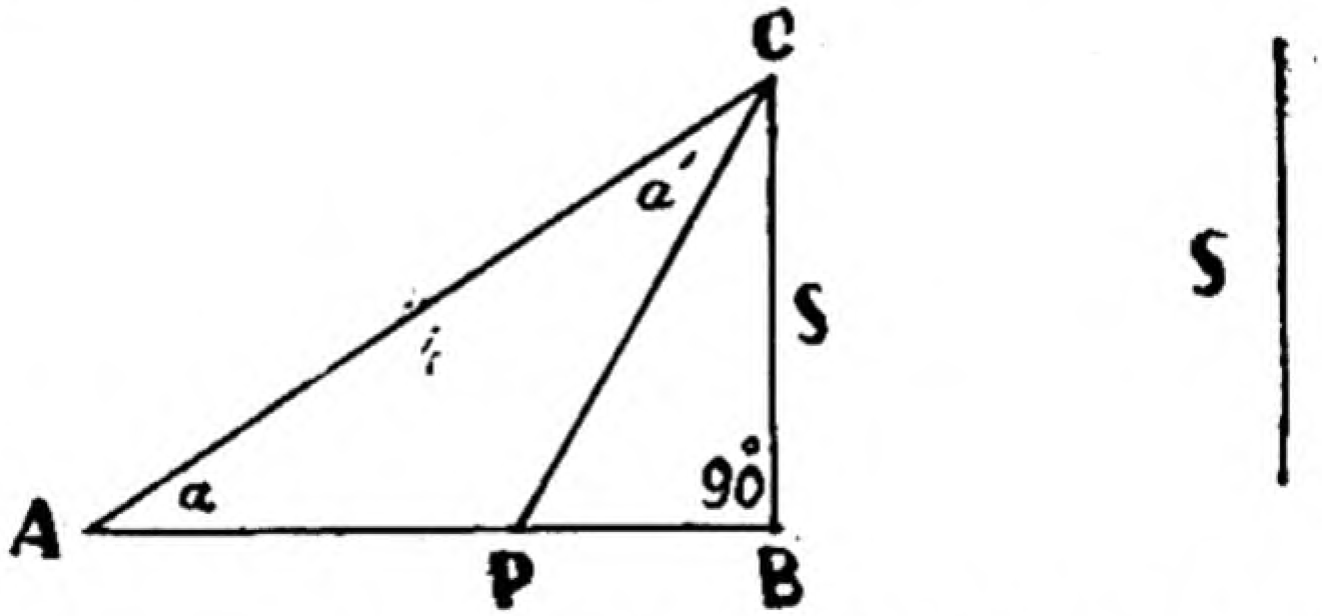
Draw $\angle ABC = 45°$.

Cut off $AC = S$

Draw $CP \perp AB$. Then P is the reqd. pt.

Proof :—$S^2 = AP^2 + PC^2 = AP^2 + PB^2$.

36. Divide a st. line into two parts so that the difference between their squares is equal to a given square.



Given :—A st. line AB and S the side of the given square.

Required :—To find a pt. P in AB such that $AP^2 - PB^2 = S^2$

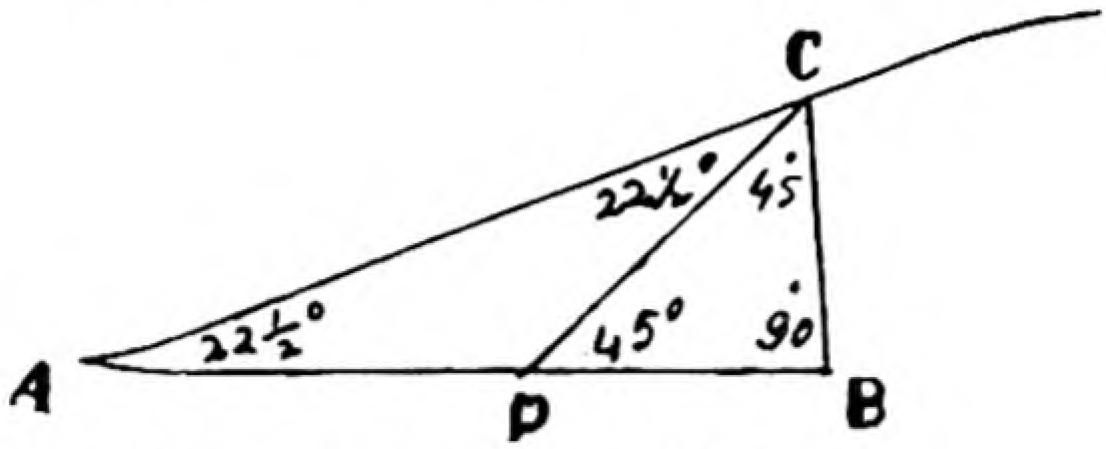
Construction :—Draw $BC \perp AB = S$

Join AC. Make $\angle a' = \angle a$.

Then P is the rqd. pt.

Proof :—In rt. $\angle$d $\triangle PBC$
$S^2=Pu^2-PB^2=AP^2-PB^2$.

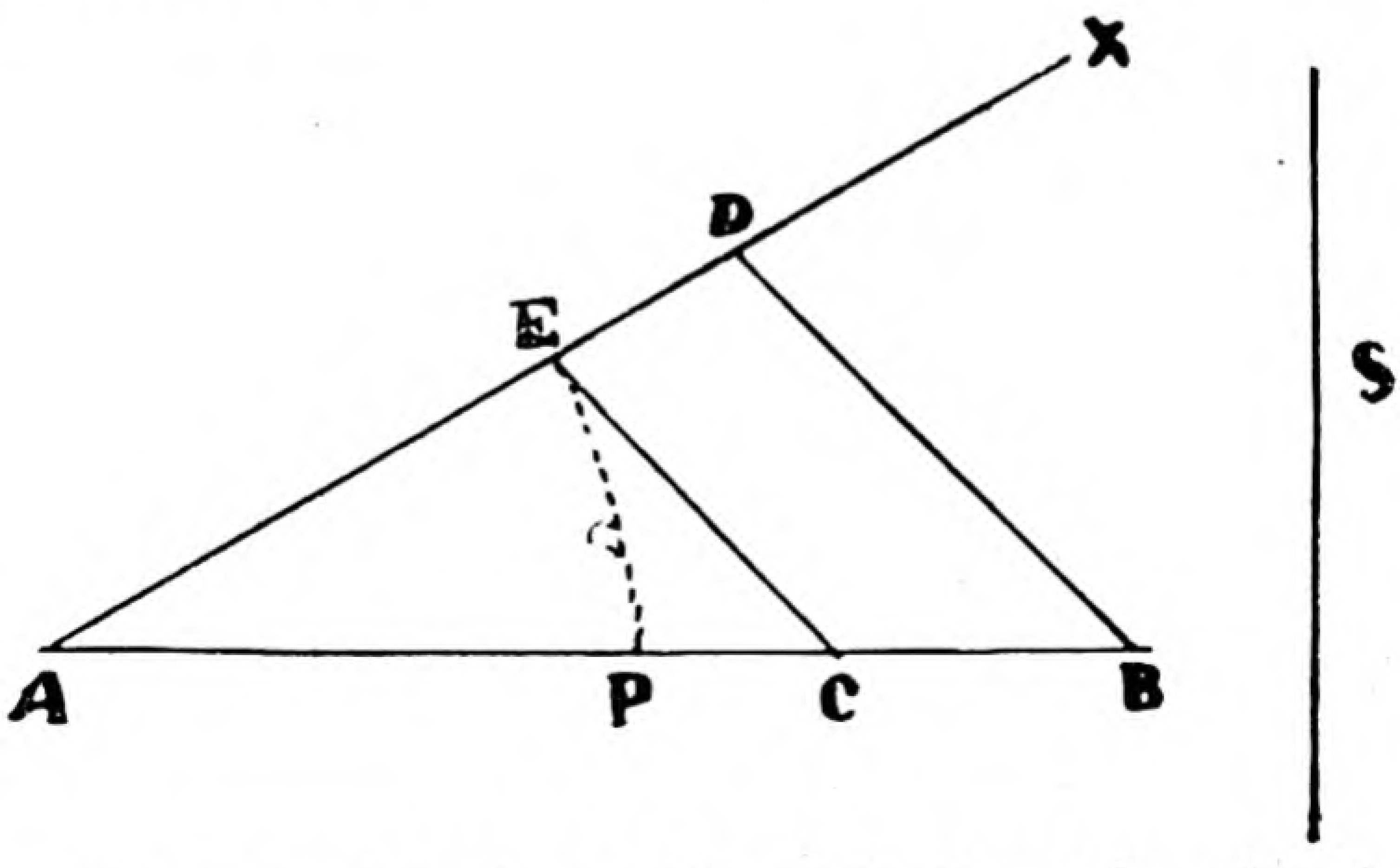
37. Divide a st. line AB at P so that $AP^2=2PB^2$.



Construction :—Make $\angle BAC=22\frac{1}{2}°$
Draw $BC \perp AB$ meeting AC in C.
Make $\angle ACP=22\frac{1}{2}°=\angle A$.
Then P is the rqd. pt.

Proof :—$AP^2=PC^2=PB^2+BC^2=2PB^2$

38. Divide a st. line into two parts such that the rect. contained by the line and one part may be equal to a given square.



Given :—A st. line AB and S the side of a given square.

Required :—To find a pt. P in AB such that AB.AP $= S^2$.

Construction :—Draw AX making any $\angle$ with AB.

Cut off AC=AD=S.

Join BD and draw EC $\parallel$ BD.

Cut off AP=AE,

Then P is the reqd. pt.

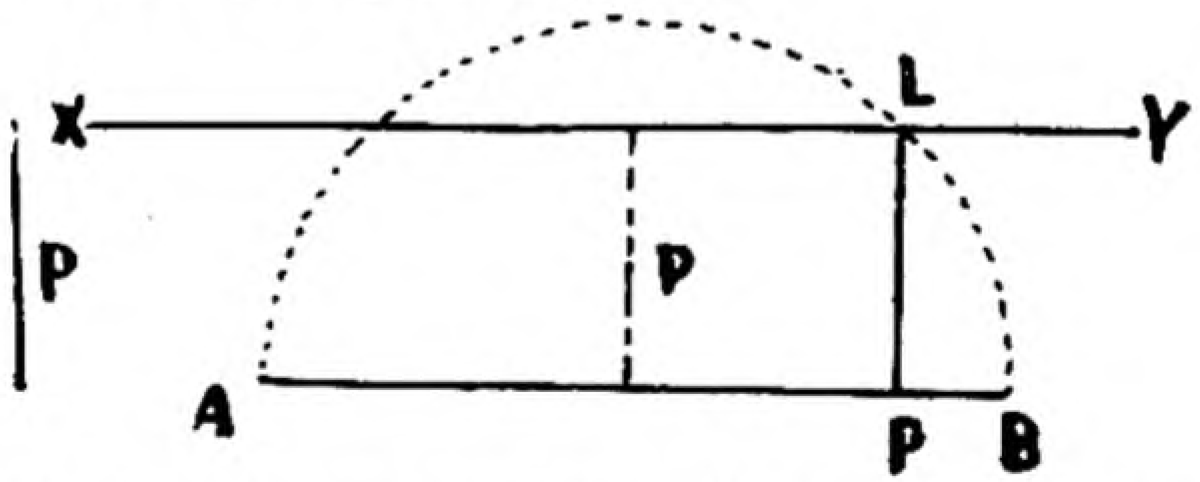
Proof : – $\triangle$s AEC and ADB are similar

$\therefore \quad \frac{AB}{AC} = \frac{AD}{AE}$

$\therefore \quad AB.AE = AC.AD$

$\therefore \quad AB.AP = S^2 \qquad [AC = AD = S]$

39. Divide a given st. line internally into two parts such that the rect. contained by the segments may be equal to a given square.



Given :– A st. line AB and p, the side of a given square.

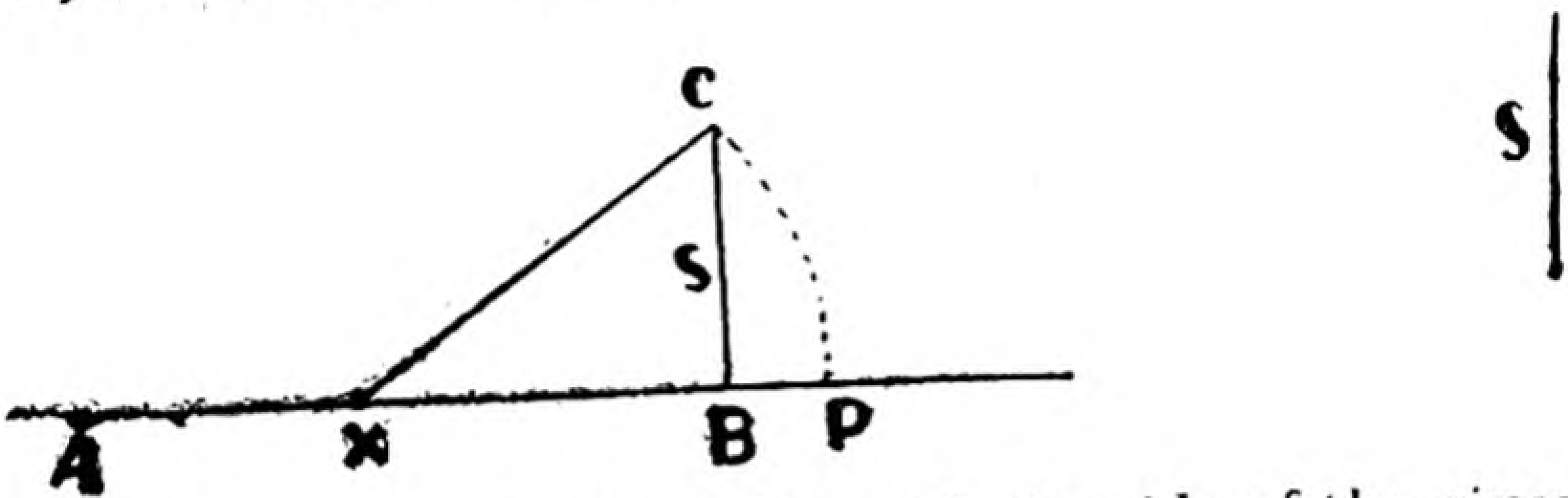
Required :—To divide AB at P such that AP.BP $= P^2$.

Construction :—Draw a semicircle on AB. Draw XY $\parallel$ AB at a distance of p from AB, cutting the semicircle at L. Draw LP $\perp$ AB.

Then P is the rqd. pt.

$AP.PB = PL^2 = p^2$

40. Divide a given st. line externally into two parts such that the rect. contained by the segments is equal to a given square.



Given :—A st. line AB and S the side of the given square.

Required :—To divide AB externally at P so that $AP.BP=S^2$.

Construction :—Bisect AB in X.
Draw $BC \perp AB$ and equal to S.
From AB produced cut off $XP=XC$.
Then P is the rqd. pt.

Proof :—$AP.BP=(XP+AX)(XP-XB)$
$=(XP+XB)(XP-XB)$
$=XP^2-XB^2$
$=XC^2-XB^2$
$=BC^2=S^2$

41. Divide a st. line $AB=2''$ internally at X so that the rect. contained by its parts is equal to a square $\frac{1}{4}$sq. in. in area. Measure the parts. Verify by calculation.

Hint.—We have $AX+BX=2''$ and side of the given sq. $\frac{1}{2}''$. Find a point X in AB so that $AX.XB=(\frac{1}{2})^2$. Vide Ex. 39 above.

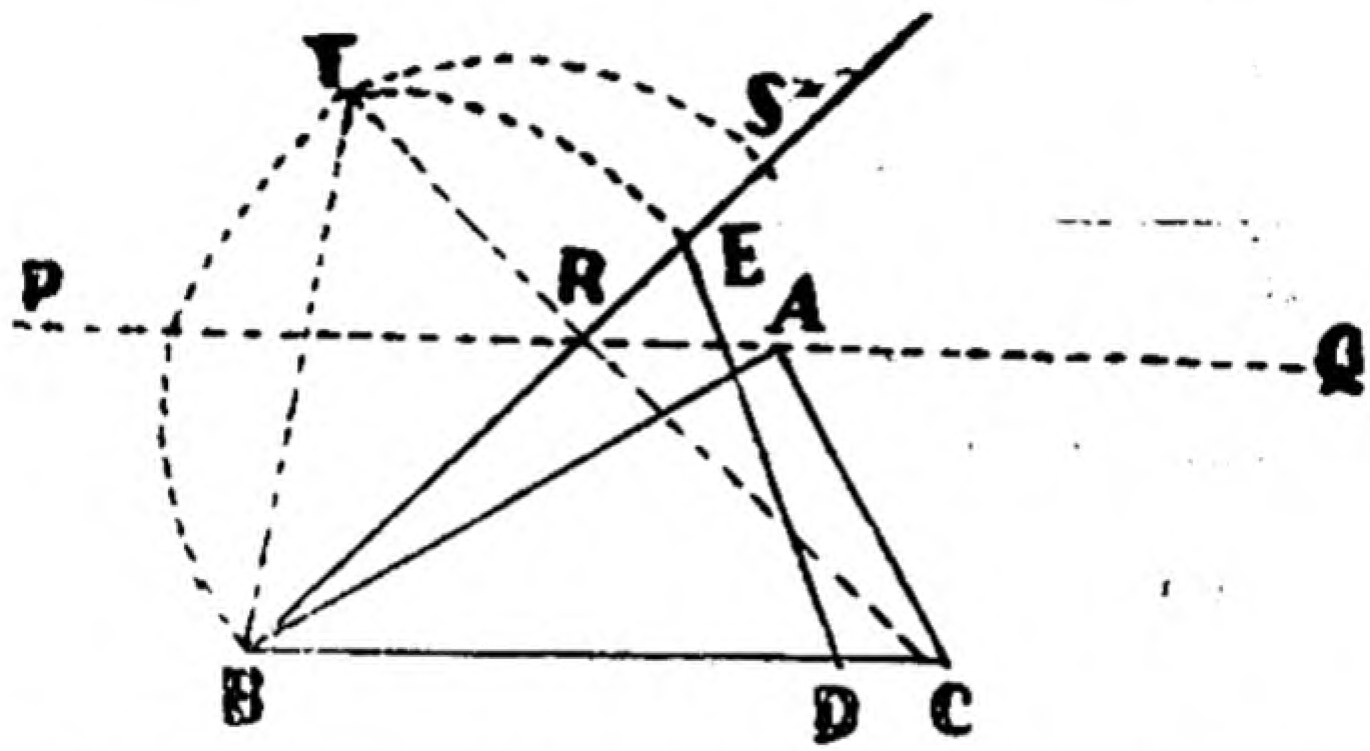
42. Solve graphically the equation $X+Y=7{\cdot}5''$ and $XY=(2{\cdot}5)^2$

43. Draw an isosceles $\triangle$ having a given vertical angle and equal in area to a given $\triangle$.

Given :—A $\triangle ABC$ and an $\angle x$.

Required :—To draw an isosceles $\triangle = \triangle ABC$ and having its vert. $\angle = \angle x$.

Note :—The following proof is beyond the matric syllabus.



Construction :—Through A draw PQ $\parallel$ BC.

Make $\angle CBR$ = the given $\angle x$ cutting Q in PR.

Join RC.

Produce BR to S making BS = BC.

On BS draw a semi-circle.

Draw RT $\perp$ BS. Cut off BD = BE = BT.

Join DE.

Then BDE is the reqd. $\triangle$.

Proof —$\triangle BDE = BD.BE \sin. x.$

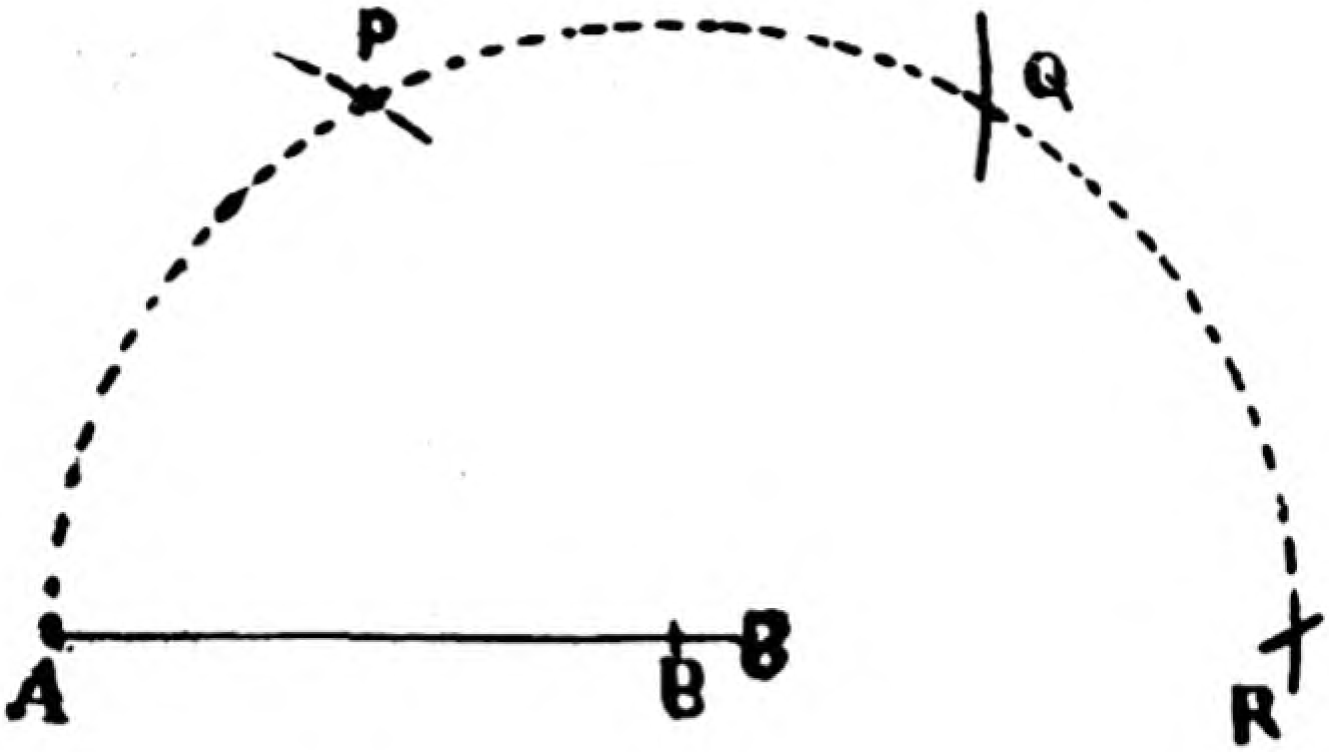
$= BT^2 \sin x$ [$\because$ BD = BE = BT]

$= BR.BS \sin x$ [$\because$ BT is mean prop. between BR, BS].

$= BR.BC \sin x$ [BS = BC]

$= \triangle BCR = \triangle ABC.$ [$\triangle$s BCR = ABC]

44. To bisect a given finite st. line by means of a pair of compasses only.



Given :—A finite st. line AB.

Required :—To bisect AB by means of compasses only.

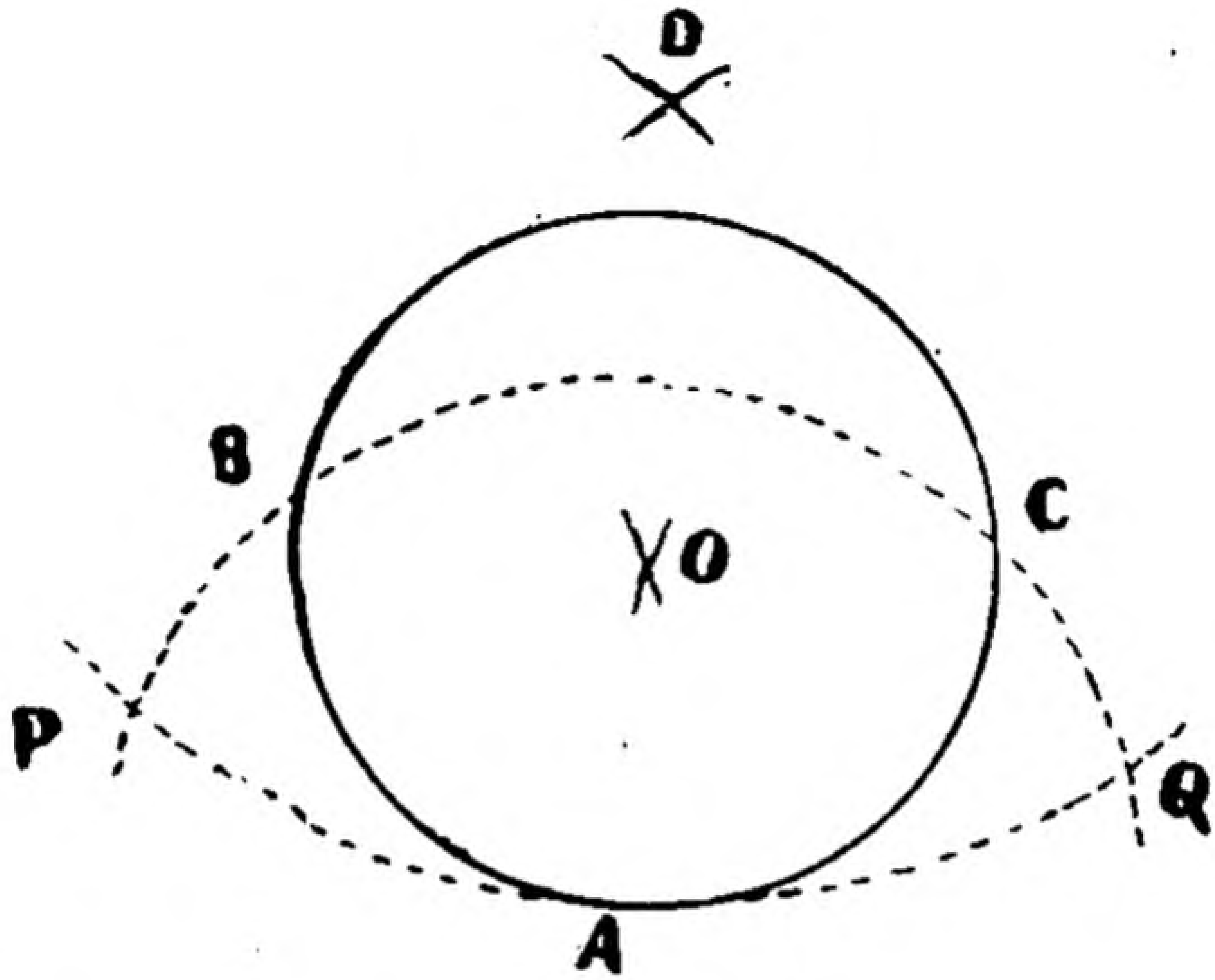
Construction :—With B as centre and radius=AB draw an arc.

Step off AP=PQ=QR=AB.

With A as centre and radius=AR draw an arc. With R as centre and radius=AQ describe another arc cutting the previous arc at S. With S as centre and radius=AQ describe an arc cutting the line AB in T. Then T is the middle point of AB.

Q.E.F.

45. Find the centre of a given circle without using st. lines.



Given :—Circle whose centre is unknown.

Required :—To find the centre without using a st. line.

Construction :—Take any pt. A on the $\odot^{ce}$. With A as centre and any convenient radius draw an arc cutting the circle in B and C.

With B and C as centres and radius=AB draw arcs cutting in D. With D as centre and radius=DA draw an arc cutting the arc BC produced in P and Q. With P and Q as centre, and radius=AB, draw arcs cutting each other at O and the circle. Then O is the centre of the given circle.

A. REVISION PAPERS.

Revision Paper No. 1.

MATHEMATICS PAPER B.

Time allowed : 3 hours. Maximum Marks : 100.

(Angles at a pt., || st. lines, $\triangle$s and rectilineal figures and elementary constructions).

PART I.

I. (*a*) Draw two st. lines OA and OB making an $\angle$ of 60° with each other. Make OA = 1·6″ and OB = 2″. Join AB and draw OP $\perp$ AB.

(*b*) Draw a line $\frac{3}{5}$ of 4.5″.

II. (*a*) If the sides of a convex polygon be produced in order, prove that the sum of all the exterior angles so formed is equal to 360°.

(*b*) The alternate sides of a convex pentagon are produced to meet, so as to form a star-shaped figure ; prove that the sum of the internal angles at the vertices is equal to two right $\angle$s.

Find also the value of the corresponding sum when the pentagon is replaced by a polygon of n sides.

III. (*a*) If the sum of two adjacent $\angle$s be equal to two rt. $\angle$s, prove that their arms are in the same straight line.

(*b*) Four st. lines OA, OB, OC and OD are drawn in order from a pt. O so that $\angle$AOB = $\angle$COD and $\angle$BOC = $\angle$DOA. Prove that AOC and BOD are st. lines.

Or

If the bisectors of two adjacent $\angle$s are at rt. $\angle$s., Prove that their external arms are in the same st. line.

IV. (*a*) Prove that if two lines intersect, the vertically opposite ∠s are equal.

(*b*) Two st. lines AB, CD intersect at E. If the bisector of ∠AEC be produced towards E, prove that it will bisect the ∠BED.

Or

(*a*) If a st. line cuts two other st. lines so as to make the alternate or the corres. ∠s equal, prove that the two st. lines are parallel.

(*b*) AB and CD are two diameters of a ⊙, prove that AC ∥ BD and AD ∥ BC.

PART II.

V. (*a*) Draw a △ having its sides equal to 2·5, 3, 5 cm. Bisect the smallest ∠, and draw the greatest altitude and the right bisector of the greatest side. Measure the length of the altitude.

(*b*) Draw a △ ABC with sides 3, 4, 5 cm.; construct a △, so that A, B, C are the middle pts. of its sides.

Or

Draw a rt. ∠d △ when hypt. $c=5{\cdot}6$ cm., $a=3{\cdot}2$ cm. Draw a ⊥ from the rt. ∠ upon the hyp. and measure it.

VI. (*a*) Prove that two st. lines in the same plane ∥ to a third st. line are ∥ to each other.

(*b*) A regular hexagon ABCDEF is given, show that its opposite sides are ∥, *e.g.*, AB ∥ ED, etc.

VII. (*a*) Prove that the sum of the ∠s of a △ is equal to two rt. ∠s.

(*b*) Prove that in any $\triangle$ XYZ, the $\angle$ between the bisector of the vertical $\angle$X and the $\perp$ drawn from the vertex X to the base YZ is equal to half the difference between the base $\angle$s Y, Z.

Or,

In the $\triangle$ABC, BD is the internal bisector of $\angle$B and CD the external bisector of $\angle$C. Show that 2 $\angle$BDC=$\angle$A.

VIII. (*a*) Prove that if a st. line stands on another st. line, the sum of the two adjacent $\angle$s so formed=2 rt. $\angle$s.

(*b*) A regular pentagon ABCDE is inscribed in a $\odot$ with centre O. From O the perpendicular OL is drawn on CD and OA is joined. Prove that AOL is a st. line.

Or,

(*a*) If a st. line cuts two parallel st. lines, prove that it makes the alternate $\angle$s equal.

(*b*) If from a pt. two st. lines are drawn perpendicular to the arms of an $\angle$, prove that the $\angle$ between them is either equal or supplementary to the given $\angle$.

IX. Do two of the following :—

(*i*) Construct a $\triangle$ having given the perimeter and the base $\angle$s.

(*ii*) Construct a $\triangle$, given two sides and the median which passes through their point of intersection.

(*iii*) Construct a $\triangle$, having given the lengths of its three medians.

(*iv*) Construct a $\triangle$, having given the altitude and the base angles.

(*v*) Construct a $\triangle$, having given the base, the altitude and the median to the base.

Revision Paper No. 2.

MATHEMATICS PAPER B.

Time allowed : 3 hours. Maximum Marks : 100.

(*Congruency of $\triangle$s, inequalities, $\|^{ms}$; transversals and elementary constructions.*)

PART I.

I. (*a*) The sides AB, BC, of a $\triangle$ ABC are $2{\cdot}8''$ and $2{\cdot}1''$, and the $\angle$BAC is $30°$. Construct two different $\triangle$s with the given data. Measure the length of the $\perp$ from B to the opposite side, and verify.

(*b*) Construct a quad. ABCD having $AB=2''$, $BC=3''$, $CD=3{\cdot}5''$, $DA=3''$, $BD=2{\cdot}5''$. Measure the perpendicular distances of A and C from BD.

II. (*a*) Prove that if two $\triangle$s have two $\angle$s of the one equal to two $\angle$s of the other, each to each, and also one side of the one equal to the corresponding side of the other, the two $\triangle$s are congruent.

(*b*) The vertices of two $\triangle$s drawn on the opposite sides of a common base are equidistant from the base. Show that the base bisects the line joining their vertices.

III. (*a*) Prove that if three or more $\|$ st. lines intercept equal parts on one transversal, they intercept equal parts on every other transversal.

(*b*) In $\triangle$ABC, D and E are the middle pts. of AB, AC respectively; show that DE ∥ BC and $=\frac{1}{2}$ BC.

IV. (*a*) Prove that the opposite sides and ∠s of a ∥^{m} are equal.

(*b*) On the sides AB, BC, CD, DA of ∥^{m} pts. A′, B′, C′, D′ are taken in such a way that AA′ is one third of AB, BB′ is one third of BC, CC′ one third of CD and DD′ one third of DA. Prove that the figure A′B′C′D′ is a ∥^{m}.

Or.

(*a*) If two $\triangle$s have the three sides of the one equal to the three sides of the other, each to each, prove that they are equal in all respects.

(*b*) If two $\triangle$s have two sides of the one equal to the two sides of the other and the ∠s opposite to a pair of equal sides equal, then the ∠s opposite to the other pair of equal sides are either equal or supplementary.

PART II.

V. (*a*) Draw a $\triangle$ABC having BC=5 cm., CA=6 cm. and AB=7 cm. Find a point K in BC equidistant from AB and AC. Measure KB, KC.

(*b*) Draw a ∥^{m} whose two diagonals are 4 cm. and 8 cm. and one of the sides 5 cm. Measure the altitudes of the ∥^{ms} and hence find its area.

Or,

Draw ∥^{m} whose diagonals are 4 cm. and 8 cm. long, the ∠ between them is of 60°. Measure the altitudes of the ∥^{m} and hence find its area.

VI. (*a*) Two $\triangle$s LMN, PQR are such that LM=PQ, LN=PR, $\angle$MLN=$\angle$QPR. Prove that the $\triangle$s are congruent.

(*b*) A′BC, B′CA, C′AB are equilateral $\triangle$s drawn on the sides BC, CA, AB of the $\triangle$ABC and externally to it. Prove that AA′=BB′=CC′.

VII. (*a*) Prove that, if two sides of a $\triangle$ are unequal, the angle opposite to the longer of these sides is greater than the angle opposite to the shorter side.

(*b*) The sum of the median of a $\triangle$ is greater than its semi-perimter but less than its perimeter.

Or,

Any two sides of a $\triangle$ are together greater than the third.

Or,

The difference between any two sides of a $\triangle$ is less than the third side.

VIII. (*a*) If two right angled $\triangle$s have their hypotenuses equal, and one side of the one equal to one side of the other, prove that the $\triangle$s are congruent.

(*b*) If the $\perp$s from the vertices of a $\triangle$ on the opposite sides are equal, prove that the $\triangle$s are equilateral.

IX. Do two of the following :—

(*i*) Construct a $\triangle$, given the base, sum of other two sides and one of the $\angle$s at the base.

(*ii*) Construct a $\triangle$, given the base, difference of the other two sides and the lesser $\angle$ at the base.

(*iii*) Construct a $\triangle$, given the base, vertical $\angle$, and the ratio of other two sides.

(*iv*) Construct a $\triangle$ having given the base and the $\perp$s from the extremities of the base on the opposite sides.

Revision Paper No. 3.

MATHEMATICS PAPER B.

Time allowed : 3 hours. **Maximum Marks . 100.**

(*Area and algebraic identities. Constructions*).

1. (*a*) Construct a quad. ABCD such that AB=1·2". BC=1·5" CD=1·7", DA=1·3" and AC =1·9". Construct a rectangle equal in area to the $\triangle$ABC and measure its sides.

(*b*) Construct a $\triangle$ABC such that AB=1·4", BC=1·5" and CA=1·8" and construct a $\triangle$ of equivalent area having its base of length 2·3" and lying on AB. Draw the altitudes of these two $\triangle$s, measure them, and verify that the areas of the two $\triangle$s are equal.

II. (*a*) Prove that $\parallel^{ms}$ on the same base and between the same $\parallel^{s}$ are equal in area.

(*b*) Prove that the area of a $\parallel^{m}$ is equal to that of a rectangle on the same base and having the same altitude.

III. (*a*) Prove that the equal triangles on equal bases have equal altitudes.

(*b*) In $\triangle ABC$, D and E are the middle points of AB, AC. F is any point in BC. Show that $\triangle DBF = \frac{1}{4}\triangle ABC$.

IV. (*a*) Prove that the area of a $\triangle$ is half the area of a rectangle of the same base and having the same altitude.

(*b*) In $\triangle$s ABC, XYZ, AB=XY, BC=YZ, $\angle ABC + \angle XYZ = 180°$, prove that $\triangle ABC = \triangle XYZ$.

Or,

(*a*) State, illustrate and explain the geometrical theorem corresponding to the algebraical identity : $(a+b)^2 = a^2 + 2ab + b^2$.

(*b*) Show geometrically that $(2a)^2 = 4a^2$.

PART II.

V. (*a*) Draw a quad. ABCD when AB=60 mm. BC=18 mm, CD=23 mm., DA=65 mm. and, $\angle B = 90°$. On side AD const. a $\triangle$=the quad. : Measure its greatest side.

(*b*) Construct a quad. ABCD, AB=3·5 cm. DC=4·5 cm., AD=5 cm., and the angles ABC and ACD are right $\angle$s. Draw a square equal in area to this quad. ; and measure its sides.

VI. Construct a $\triangle ABC$ having AB=4″, BC=3″, CA=2″. Find a point within ABC equidistant from A and C and also equidistant from AB and AC.

VII. (*a*) Prove the geometrical theorem corresponding to the algebraical identity :

$$x(a+b+c) = xa + xb + xc$$

(*b*) If A, B, C, D are four points taken in order on a st. line prove that $AB.\ CD+DC.AD=AC.BD$.

VIII. (*a*) Prove geometrically the formula :—

$$(a-b)^2=a^2+b^2-2ab.$$

(*b*) Prove geometrically : $(a+b)^2+(a-b)^2=2(a^2+b^2)$.

Or,

(*a*) State, illustrate and explain the geometrical theorem corresponding to the algebraic identity :—

$$a^2-b^2=(a+b)(a-b).$$

(*b*) If a st. line AB is bisected at P, and also divided internally or externally into two unequal segments at Q, then $AQ^2+QB^2=2(AP^2+PQ^2)$.

IX. Do two of the following :—

(*i*) Reduce a $\triangle ABC$ to a $\triangle$ of equal area having its base BD of a given length.

(*ii*) Construct a $\triangle$ equal in area to a given $\triangle$ and having a given altitude.

(*iii*) On a given st. line construct a rectangle equal in area to a given rectangle.

Revision Paper No. 4.

MATHEMATICS PAPER B.

Time allowed : 3 hours. Maximum Marks : 100

(*Squares on the sides of $\triangle$s. Constructions*).

I. Draw a regular hexagon on a side 2·0" long. Draw a square equal to it in area.

II. (*a*) The difference of the squares on the two sides of a $\triangle$ is equal to twice the rectangle contained

by the third side and projection on it of the median which bisects the third side.

(*b*) A and B are two fixed points and P is a point such that $PA^2 - PB^2$ is constant. Show that the locus of P is a st. line $\perp$ AB.

III. (*a*) Prove that in any rt. $\angle$d $\triangle$ the square on the hypotenuse is equal to the sum of the squares on the sides containing the rt $\angle$.

(*b*) ABC is a $\triangle$, in which C is a rt. $\angle$ and $AC = \sqrt{3}.BC$. Prove that the $\angle ABC = 60°$.

IV. (*a*) Prove that in any $\triangle$ the square on the side opposite to an acute angle is equal to the sum of the squares on the sides containing the acute angle diminished by twice the rectangle contained by one of these sides and the projection on it of the other.

(*b*) In a $\triangle ABC$ the $\angle$s at B and C are acute; if BE and CF be drawn $\perp$s to AC and AB respectively prove that $BC^2 = AB.BF + AC.CE$.

PART II

V. (*a*) Take a st. line 1·6″ long and divide it internally in extreme and mean ratio.

(*b*) Draw a st. line 7·5 cm. long. Divide it so that the rectangle contained by the segment may be equal to a square whose side is 2·5 cm. Measure the greater segment. Verify it by calculation.

VI. (*a*) In $\triangle ABC$, AD is the median. Prove that $AB^2 + AC^2 = 2BD^2 + 2AD^2$.

(*b*) ABCD is a $\|^m$. Prove that $AC^2 + BD^2 = 2AB^2 + 2BC^2$.

VII. (*a*) The $\angle$ABC of the $\triangle$ ABC is obtuse, and the perpendicular from A to BC meets BC produced in D. Show that the square on AB exceeds the sum of the squares on BC and CA by twice the rectangle BC.CD.

(*b*) The sides AB and AC of a $\triangle$ABC are equal, and D is any point on BC. Prove that the square on AB is equal to the square on AD together with the rectangle BD.DC.

VIII. (*a*) If the square on one side of a $\triangle$ is equal to the sum of the squares on the other sides, prove that the $\triangle$ is rt. $\angle$d.

(*b*) AD is an altitude of an equilateral $\triangle$ABC ; prove that $4AD^2=3BC^2$.

IX. (*a*) Bisect a $\triangle$ by drawing a st. line through a given point in one of its sides.

(*b*) Bisect a quad. by drawing a st. line through one of its angular points.

Revision Paper No. 5.

MATHEMATICS PAPER B.

Time allowed : 3 hours Maximum Marks : 100.

(*Locus and concurrency, Constructions.*)

I. Draw a transversal cutting two || st. lines 1·3″ apart at an $\angle$ of 45°. Find a point equidistant from the three st. lines. How many such pts. are there ?

II. (*a*) Prove that the medians of a $\triangle$ are concurrent.

(*b*) If two medians of a $\triangle$ are equal, show that the $\triangle$ is isosceles.

III. (a) Prove that the locus of a pt. which is equidistant from two fixed pts. is the right bisector of the st. line joining the two fixed pts.

(b) Find the locus of the centres of ⊙s passing through two fixed points.

IV. (a) Enunciate and prove that a^2-b^2
$=(a+b)(a-b)$.

(b) If a st. line be divided equally and unequally, prove that the rect. contained by the unequal segments is equal to the square on half the line minus the square on the line between the pts. of section.

PART II.

V. (a) Draw the plan of a quad. piece of land PQRS, when PQ=80 yds., QR=150 yds., RS=120 yds., SP=114 yds. and the $\angle PQR=90°$. Mark the position of a stone pillar which is 120 yds. from the corner S and 50 yds. distant from the side PQ. Measure its distance from P.

(b) A and B are two points 2·5″ apart and 1·7″, 2·3″ distant from a given st. line and on the same side of it. Through A and B draw two st. lines meeting on XY and also equally inclined to it.

VI. (a) Prove that the locus of a point which is equidistant from two intersecting st. lines is a pair of st. lines.

(b) $AC \perp AB$. A variable line DE of fixed length moves with its ends on AC and AB. Find the locus of O, its mid-pt.

VII. (*a*) Prove that the perpendicular bisectors of the three sides of a $\triangle$ are concurrent.

(*b*) If L, M, N be the feet of the $\perp$s from the vertices A, B, C respectively of an acute $\angle$d $\triangle ABC$ to the opposite sides, prove that AL bisects the $\angle MLN$.

VIII. (*a*) "The bisectors of the $\angle$s of a $\triangle$ are concurrent." Prove it.

(*b*) Prove that the sides of a $\triangle$ subtend obtuse $\angle$s at the incentre.

Or,

(*a*) "The $\perp$ bisectors of the sides of a $\triangle$ are concurrent." Prove it.

(*b*) O is the circum-centre of the $\triangle ABC$ and AD is an altitude; show that $\angle OAD = \angle B \sim \angle C$.

Revision Paper No. 6.

MATHEMATICS PAPER B.

Time allowed : 3 hours. Maximum Marks : 100.

(*Elementary Theorems on* $\odot$*s. Constructions.*)

I. (*a*) Draw a $\odot$ of radius $0{\cdot}7''$; draw a line through the centre; on it take two pts. each $2{\cdot}3''$ from the centre and draw tangents to the $\odot$ from these points.

(*b*) Draw a $\odot$ of radius $1''$. Draw two tangents to it inclined at an $\angle$ of $60°$ to each other.

Or,

Draw direct common tangents to two $\odot$s each of $0{\cdot}8''$ radius, having their centres $2{\cdot}1''$ apart. Measure one of them.

II. (*a*) Prove that one and only one ⊙ can be drawn through three given points not in a st. line.

(*b*) If the non-parallel sides of a trapezium are equal, show that a ⊙ can be described to pass through its angular points.

III. (*a*) Prove that the tangent at any point of a ⊙ and the radius through that point are perpendicular to each other.

(*b*) Find the locus of the centres of all the ⊙s which touch a given st. line at a given point.

Or,

Find the locus of point from which the tangents drawn to a given ⊙ are of given length.

IV. (*a*) If two tangents are drawn to a ⊙ from an external pt., prove that (*i*) they are equal (*ii*) they subtend equal ∠s at the centres and (*iii*) they are equally inclined to the diameter through the point.

(*b*) In any quad. circumscribed about a ⊙, the sum of one pair of opposite sides is equal to the sum of the other pair.

Or,

(*a*) If in two equal ⊙s two arcs are equal, prove that they subtend equal ∠s at the centres.

(*b*) Prove that the arcs intercepted between two parallel chords of a ⊙ are equal.

PART II.

V. (*a*) Draw two ⊙s of radii 3 cm. and 4·5 cm., whose centres are 5·7 cm. apart. Draw their common chord and a common tangent. Measure the length of the common chord and the common tangent. Verify your results.

VI. (*a*) In equal ⊙s if two chords are equal, prove that they cut off equal arcs.

(*b*) Prove that the st. lines joining the alternate angular points of a regular polygon inscribed in a ⊙ are equal.

VII. (*a*) Prove that equal chords of a ⊙ are equidistant from the centre.

(*b*) AB and AC are two equal chords of a ⊙ show that the bisector of the ∠ABC passes through the centre.

Or,

If two equal chords of a ⊙ intersect, show that the segments of the one are respectively equal to the segments of the other.

VIII. (*a*) A st. line drawn from the centre of a ⊙ bisects a chord which is not a diameter is at rt. ∠s to the chord.

(*b*) Prove that the line joining the middle points of two parallel chords of a ⊙ passes through the centre.

Or,

(*a*) Prove that if two ⊙s touch each other either externally or internally, the st. line which joins their centres passes through the point of contact.

(*b*) Two equal ⊙s touch externally; find the locus of the centre of a third touching them both.

IX. Do two of the following :--

(*i*) Draw a ⊙ to pass through a given point and to touch a given st. line at a given point.

(*ii*) Construct a ⊙ of given radius passing through a given point, and touching a given st. line.

(*iii*) The transverse common tangents of two ⊙s of radii 1″, $1\frac{1}{2}$″ respectively are at rt. ∠s to each other. Construct the ⊙s explaining clearly the steps of construction.

(*iv*) Draw a ⊙ of radius 2·7 cm., having its centre 4 cm. from a given st. line; draw another ⊙ of radius 2·4 cm. touching both the line and the ⊙. Measure the distance between the points of contact.

Revision Paper No. 7.

MATHEMATICS PAPER B.

Time allowed : 3 hours. Maximum Marks : 100

(*Circles. Constructions.*)

I. (*a*) Draw a △ whose sides are respectively 3·1″, 4·2″ 5·3″. Describe the inscribed and circumscribed ⊙s and measure the distance between their centres.

(*b*) Draw a △ whose sides are 3″, 4″, 5″. Draw a ⊙ touching the smallest side and the other two sides produced. Measure its radius. Verify your result.

II. (*a*) Prove that if the line joining the two pts. subtends equal ∠s at two other pts. on the same side of it, the four points lie on a ⊙.

(*b*) ABCD is a cyclic trapezium, in which side AB ∥ CD. Prove that AD=BC.

III. (*a*) "If two chords of a ⊙ intersect inside it, the rectangle contained by the parts of the

one is equal to the rectangle contained by the parts of t e other." Prove it.

(*b*) Prove that the common chords of three intersecting ⊙s taken two by two are concurrent.

IV. (*a*) Prove that the ∠ at the centre of a ⊙ is double the ∠ at the circumference standing on the same arc.

(*b*) Two equal ⊙s intersect in A and B. Any st. line through A meets the ⊙ces in P and Q. Show that PB=BQ.

Or,

(*a*) Prove that the ∠ in a segment less than a semi ⊙ is greater than a rt. ∠.

(*b*) Prove that the ⊙ described on the hypotenuse of a rt. ∠d △ as diameter, passes through the rt. ∠.

PART II.

V. (*a*) Construct a rhombus whose diagonals are 6 cm. and 8 cm. respectively. Inscribe a ⊙ in it, measure its radius and verify the result.

(*b*) In a ⊙ of radius 2·8 cm. inscribe a regular octagon and measure one of its sides.

Or,

About a ⊙ of 1·5″ radius describe a regular hexagon. Measure one of its sides.

VI. (*a*) Prove that the opposite ∠s of a cyclic quad. are supplementary.

(*b*) Two ⊙s cut at A, B and a st. line PAQ cuts the ⊙s at P, Q. If the tangents at P, Q meet in T, prove that the pts. P, B. Q, T are concyclic.

Or,

PQ is a common tangent to two ⊙s PAB, QAB. Prove that ∠PAQ+∠PBQ=2rt. ∠s.

VII. (*a*) If the opposite ∠s. of a quad. are supplementary, prove that the quad. is cyclic.

(*b*) If one side of quad. be produced and the exterior ∠ so formed is equal to the interior opposite ∠, prove that the quad. is cyclic.

VIII. (*a*) ABCD is a cyclic quad., whose opposite sides AB, DC, produced meet in P, and BC, AD produced meet in Q. Prove that if the bisectors of the ∠s AQB, APD meet in O, then ∠POQ is a rt.∠.

(*b*) Show that every parallelogram inscribed in a ⊙ is a rectangle.

Or,

(*a*) Prove that ∠s in the same segment of a ⊙ are equal.

(*b*) ABC is a △ inscribed in a ⊙. AD and BE, the perpendiculars from A and B to the opposite sides intersect in H, and AD is produced to meet the ⊙ in X. Show that HD=DX.

IX. Do two of the following :—

(*i*) Construct a ⊙ passing through a given pt. and touching a given ⊙ at a given point.

(*ii*) Draw a ⊙ of radius 1·2″ with centre O; draw any diameter AOB and produce it to C, making BC=BA; draw CD ⊥ CA, and cut off CD=AC. Construct another ⊙ touching CD at D and also touching the first ⊙. Measure its radius.

(*iii*) Draw a ⊙ through two given pts. A, B to touch a given st. line CD.

(*iv*) Draw a ⊙ to pass through two given pts. and to touch a given ⊙.

(*v*) Construct a ⊙ touching two intersecting st. lines and passing through a given point between them.

Revision Paper No. 8.

MATHEMATICS PAPER B.

Time allowed : 3 hrs. Max. Marks : 100.

(*Ratio and Proportion and Similar △s. Constructions*)

I. (*a*) Divide a st. line 2·8″ long internally, and externally in the ratio 4 : 3. Measure the parts and check your result by calculation.

(*b*) Find graphically a line which is a fourth proportional to the three lines whose lengths are 4″, 3″, 5″. Measure it and verify your result by calculation.

Or,

Find a line equal in length to $(1\cdot4)^2$ inches.

(*ii*) Prove that the st. line bisecting the vertical ∠ of a △ internally divides the base in the ratio of the adjacent sides.

(*b*) The side BC of a △ is bisecting at D, and ∠s ABD and ADC are bisected by DE and DF respectively, meeting AB in E and AC in F. Show that EF is parallel to the base BC.

Or, O is any pt. in the △ LMN. OD bisects the ∠MON, OE bisects the ∠NOL and OF bisects the ∠MOL. Prove that MD × NE × LF = DN × EL × FM.

III. (*a*) If a st. line be drawn parallel to one side of a △, prove that it must divide the other two sides in the same ratio.

(*b*) LP, the bisector of the base of the △ LMN, is bisected in Q, and MQ cuts LN in R; prove that LR : RN = 1 : 2.

IV. (*a*) If three sides of one △ are proportional to the three sides of another △, prove that the two △s are equiangular.

(*b*) The base AB of an isosceles △ ABC is produced both ways to D and E such that AD.BE $=AC^2$. Show that △s ACD and BCE are similar.

Or,

(*a*) Prove that the st. line which divides the two sides of a △ proportionally is parallel to the third.

(*b*) In △ABC, O is any pt. LM ∥ AB, meeting OA in L and OB in M; MN ∥ BC meeting OB in M and OC in N. Prove that NL ∥ AC.

PART II.

V. Find by geometrical construction a mean proportional between ·9″ and 1·6″. Verify it by caloulation.

(*b*) Draw geometrically a st. line equal to 8″ in length.

VI. (*a*) If two △s are equiangular, prove that their corresponding sides are proportional.

(*b*) If one of the diagonals of a trapezium divides the other in the ratio of 1 : 2, prove that one of the parallel sides is twice the other.

Or,

ABC is rt. ∠d. △ having the rt. ∠ at A ; AD is drawn ⊥ BC. Prove that $AD^2 = BD.DC$.

VII. (*a*) If two △s have an ∠ of the one equal to an ∠ of the other, and the sides about these equal ∠s proportional, prove that the △s are equiangular.

(*b*) From a point O outside a ⊙ two st. lines OAB and OT are drawn meeting the ⊙s. in A, B and T, so that $OT^2 = OA.OB$; show that OT is a tangent to the ⊙.

VIII. (*a*) Prove that the areas of similar △s are in the ratio of the squares on the corresponding sides of the △s.

(*b*) The tangent at the vertex A to the circumcircle of the △ABC meets the side BC in the pt. T. Prove that $TB : TC :: AB^2 : AC^2$.

Or,

Show that similar △s are to one another in the duplicate ratio of :—

(*i*) their altitudes ;

(*ii*) their corresponding medians ;

(*iii*) the bisectors of the corresponding ∠s.

Or,

(*a*) Prove that the external bisector of an ∠ of a △ divides the opposite sides externally in the ratio of the sides containing the ∠.

(*b*) Show that any st. line parallel to the parallel sides of a trapezium will cut the non-parallel sides proportionally.

IX. Do two of the following :—

(*i*) Draw a quad. ABCD with AB=3″, BC =1·9″, DA=1·8″, $\angle$A=45°, $\angle$B=60°. On a st. line 3·8″ long const. a quad. similar to it.

(*ii*) Bisect a given $\triangle$ by drawing a st. line parallel to the base.

(*iii*) Draw tangents to a circle from an external pt. without finding the centre.

B. MODEL TEST PAPERS.

Model Test Paper No. 1.

MATHEMATICS PAPER B.

Time allowed : 3 hours. Max. Marks : 100.

PART I.

I. (*a*) Construct a quadrilateral ABCD having given the sides AB, BC, and CD equal to 1″, ·5″ and 1·6″ respectively and $\angle$B=135°, $\angle$C=90° ; measure the length of AD and verify it by calculation.

(*b*) Draw a $\triangle$ABC with a=34 mm. $\angle$B=60°, $\angle$C=45°. Inscribe in a $\odot$ of radius 1·8″ a $\triangle$ similar to it.

II. (*a*) Prove that in a $\triangle$ the greater side has a greater angle opposite to it.

(*b*) Prove that the sum of the distances of any point within a triangle from its vertices is less than the perimeter of the triangle.

Or,

Prove in any triangle the sum of the medians is less than the perimeter.

Or,

Prove that any two sides of a △ are together greater than the third.

Or,

Prove that the difference of any two sides of a △ is less than the third side.

III. (*a*) Prove that the locus of a point which is equidistant from two given points is the right bisector of the straight line joining the given points.

(*b*) Prove that the rt. bisectors of the sides of a triangle are concurrent.

IV. (*a*) Give illustration and explanation of the geometrical theorem corresponding to the algebrical identity $(a+b)^2=a^2+b^2+2ab$.

(*b*) Draw a diagram to show that square on a line equals four times the square on half the line.

PART II.

V. (*a*) Draw two ⊙s of radii 3 cm. and 4·5 cm., whose centres are 5·7 cm. apart. Draw all the common tangents. Measure and verify by calculation their lengths.

(*b*) Construct a parallelogram whose two diagonals are 4 cm. and 8 cm. and one of the sides 5 cm. Measure the altitudes of the ||^m.

VI. (*a*) From a point O outside a circle two st. lines OAB and OT are drawn meeting the circle in A, B, and T respectively, such that $OT^2=OA.OB$. Show that OT is a tangent to the circle.

(*b*) Hence show how you would draw a circle passing through two given points and touching a given straight line.

VII. (*a*) "If two $\triangle$s have one angle of the one equal to one angle of the other; and the sides about these equal angles proportional the triangles are similar." Prove it.

(*b*) In a $\triangle ABC$, CN is drawn perpendicular to AB. If $CN^2 = AN.NB$, show that $\angle C$ is a right angle.

VIII. (*a*) How will you inscribe a sq. in a triangle ?

Or,

Construct a rectangle equal to a given rectangle, having one of its sides equal to a given line.

(*b*) How will you inscribe an equilateral $\triangle$ in a $\triangle$, having one of its sides $\parallel$ to the base of the original triangle ?

Or,

How will you draw tangents to a circle, from a point outside it, without using its centre.

Model Test Paper No. 2.

MATHEMATICS PAPER B.

Time allowed : 3 hours Max. Marks : 100.

PART I.

I. (*a*) About a circle of 1·2". radius describe a right angled isosceles $\triangle$.

(*b*) Construct a quadrilateral ABCD, when $AB = 60$ mm., $BC = 18$ mm., $CD = 23$ mm., $DA = 65$ mm. and $\angle B = 90°$. On side AB construct a $\triangle$ equal to the quadrilateral. Measure its greatest side.

II. (*a*) The sides of a convex polygon are produced in order ; prove that the sum of the angles so formed is equal to 360°.

(*b*) One of the interior $\angle$s of a regular polygon is of 120°. How many sides has the polygon ?

Or,

The sides of a pentagon, with no pair of parallel sides, are produced both ways, so as to form a star-shaped figure. Show that the sum of the angles at the vertices of the star is two right angles.

III. (*a*) Give geometrical illustration and explanation of the algebric identity $(a+b)(a-b) = a^2-b^2$.

(*b*) If a line be divided equally and unequally, the rectangle contained by the unequal parts together with the square on the line between the points of division, equals the square on half the line.

IV. (*a*) PQR is triangle with an obtuse angle at R. PS $\perp$ QR produced. Prove that $PQ^2=PR^2+RQ^2+2QR.RS$.

(*b*) If $\angle R=120°$, prove that
$PR.RQ=PQ^2-BR^2-AQ^2$.

Or,

Prove that the sum of the squares on the sides of a quad. is equal to the sum of the squares on its diagonals plus four times the square on the line joining the middle points of the diagonals.

PART II.

V. (*a*) Draw one transverse common tangent to two circles whose radii are 15 mm. and 24 mm. and whose centres are 44 mm. apart. Measure its length and verify by calculation.

(*b*) Find a line which is a fourth proportional to the three given lines whose lengths are 1·7", 2·3" and 2·9". Also measure its length.

Or,

Find out the value of $\frac{1{\cdot}5\times3{\cdot}8}{5{\cdot}8}$ graphically.

VI. (*a*) If a st. line touches a ⊙, and from the point of contact a chord be drawn, prove that the ∠s between the chord and the tangent are equal to the ∠s in the alternate segment.

(*b*) Two ⊙s touch each other and a chord is drawn through the point of contact, prove that it cuts off similar segments.

Or,

The diagonals of the parallelogram ABCD inter sect in E ; prove that the circumcircles of the triangles EBC, EAD touch each other at E.

VII. (*a*) What are similar figures ? State the various conditions in which two △s can be similar. Prove one of them.

(*b*) In the right-angled triangle ABC, in which the angle at A is a right angle, and AL is drawn ⊥ BC prove that either

$AB^2=BL.BC$; or, $AC^2=CL.CB$;

Or,

$AL^2=BL.LC$.

Or,

Prove that if from any point without a circle a secant and a tangent be drawn to the circle, the rectangle contained by the secant and its external segment is equal to the square on the tangent.

VIII. Only two of the following are to be attempted :—

(1) Construct a circle to touch a given st. line at a given point and pass through a given point outside it.

(2) Construct a circle to touch a given circle at a given point and pass through a given point outside it.

(3) Draw a ⊙ to pass through two given pts. and to touch a given ⊙.

Model Test Paper No. 3.

MATHEMATICS PAPER B.

Time allowed : 3 hours. Maximum Marks : 100.

PART I.

I. (*a*) Draw a △ having its sides equal to 4·5 cm , 5 cm and 3·5 cm. Bisect the smallest angle. Draw the greatest altitude and the right bisector of the greatest side. Measure the length of the altitude and that of the shortest median.

(*b*) Divide a straight line 5·6 cm. long internally in medial section, or externally in the ratio of 3 : 4. Measure in each case the length of its segments.

II. If a straight line cuts two parallel straight lines, prove that (*i*) alternate angles are equal (*ii*) corresponding angles are equal and (*iii*) interior angles on the same side of the transversal are supplementary.

(*b*) State and prove the converse of the above theorem.

Or,

Prove that angles which have their arms parallel can either be equal or supplementary. State clearly when they will be equal and when supplementary.

III. (*a*) Prove that parallelograms on equal bases and between the same parallels are equal in area.

(*b*) Prove that the perimeter of a square is less than that of an equal parallelogram on the same base.

IV. (*a*) In a $\triangle$ PQR, if $PS \perp QR$ and Q is an acute angle, prove that $PR^2 = PQ^2 + QR^2 - 2QS\ QR$.

(*b*) If angle $Q = 60°$, prove that $PQ^2 + QR^2 = PR^2 + PQ.QR$.

Or,

If in $\triangle PQR$, PD is a median, prove that $PQ + PR^2 = 2QD^2 + 2PD^2$.

PART II.

V. Draw a $\triangle$ with sides 2·1″, 2·8″, 3·5″. Inscribe a circle and also escribe a circle to touch the shortest side and the other two sides produced. Measure and calculate the radius in each case.

Or,

Draw a $\triangle$ with sides 3″, 2·8″, 2·6″. Construct a square equal to it in area. Measure and calculate the length of its diagonal.

VI. (*a*) Prove that the locus of the vertices of triangles which stand on the same side of a common base and have their vertical angles equal, is an arc

of a circle passing through the extremities of the base.

(c) If the diagonals of a cyclic quadrilateral are at right angles, the perpendicular from their point of intersection on any side being produced bisects the opposite side.

Or,

Given three points on the circumference of a circle, find a fourth point without finding the centre.

Or,

Two circles intersect at P, Q ; and through P any two straight lines APA′, BPB′ are drawn terminated by the circumferences. Show that the arcs AB, A′B′ subtend equal angles at Q.

VII. (*a*) The areas of similar $\triangle$s are proportional to the squares on the corresponding sides.

(*b*) In a right-angled triangle the equilateral $\triangle$ on the hypotenuse is equal to the sum of the equilateral $\triangle$s on the sides.

VIII. (*a*) Draw a circle to pass through a given point and to touch two given straight lines AB, AC.

Or,

Draw a circle to touch a circle at a given point and pass through a given point outside it.

Or,

Construct two circles of given radii to touch each other and a given straight line on the same side of it.

(*b*) Describe a circle which touches a given straight line at a given point and has its centre on another given straight line.

Or,

Describe a circle which touches a given circle at a given point, and has its centre on another given circle.

Model Test Paper No. 4.

MATHEMATICS PAPER B.

Time allowed : 3 hrs. Maximum Marks : 100.

PART I.

I. (*a*) Draw a straight line 5·8″ long and divide it in the ratio of 3 : 4 : 5. Construct a triangle with these segments and measure its greatest angle.

(*b*) Describe about a circle of 2·4 cm. radius a regular hexagon. Measure one of its sides.

II. (*a*) What are congruent $\triangle$s ? State the conditions in which two $\triangle$s are congruent. Prove that if two $\triangle$s have three sides of the one equal to three sides of the other, each to each, the two $\triangle$s are congruent.

(*b*) If two isosceles $\triangle$s have a common base, the line joining their vertices bisects the base at rt. $\angle$s.

III. (*a*) State and prove the geometrical theorem corresponding to the algebraic identity :—

$$(a-b)^2=a^2-2ab+b^2.$$

(*b*) Read geometrically and illustrate by a diagram either $(a+b)^2-(a-b)^2=4ab$, or,

$$(a+b)^2+(a-b)^2=2(a^2+b^2)$$

IV. (*a*) Prove that in a right-angled $\triangle$ the square on the hypotenuse is equal to the sum of the squares on the other two sides.

(*b*) P is a point in any rectangle ABCD; prove that $PA^2+PC^2=PB^2+PD^2$.

Or,

From any point L in a st. line AB or AB produced a perpendicular is drawn to AB, if P be any point in the perpendicular, then $AP^2-BP^2=AL^2-BL^2$.

PART II.

V. (*a*) Circumscribe a $\odot$ about a $\triangle$ ABC with $B=60^\circ$, $c=2{\cdot}3''$, $b=3{\cdot}2''$. Measure its radius and hence find out the area of the $\triangle$.

(*b*) Construct a quadrilateral PQRS having given $PQ=2''$, $QR=2{\cdot}7''$, $RS=1{\cdot}8''$, $QS=1{\cdot}3''$ and $RP=2{\cdot}8''$. On a line 5 cm. long construct a quadrilateral similar to PQRS.

VI. (*a*) Prove that the angle subtended by an arc of a circle at its centre is double that subtended by it at any point on the remaining part of the circumference.

(*b*) The bisector of an angle at the $\odot^{ce}$ of a $\odot$ bisects the arc on which the angle stands.

VII. (*a*) Prove that the internal bisector of an angle of a $\triangle$ divides the opposite sides internally in the ratio of the sides containing the angle.

(*b*) P is any point within a $\triangle$ABC. The bisectors of $\angle$PBC, CPA, APB meet BC, CA, AB in D, E, F

respectively ; prove that $\frac{BD}{DC}\times\frac{CE}{EA}\times\frac{AF}{FB}=1.$

Or,

D is the mid-point of the side BC of $\triangle$ABC. DE and DF are the bisectors of $\angle$ADB and $\angle$ADC respectively, meeting AB in E and AC in F. Show that EF $\parallel$ BC.

VIII. (*a*) Construct a $\triangle$ having given two sides and the median bisecting the third side.

Or.

Construct a $\triangle$ having given the perimeter, one angle and an altitude to one of its arms.

Or

Construct a $\triangle$ABC when m_1, m_2, m_3 are given.

(*b*) Construct a $\triangle$ABC when a, A, m_1 are given.

Or,

Construct a $\triangle$ having given the perimeter, and two base angles.

Or,

Construct a $\triangle$ having given a side, the angle opposite to it and the difference of the other two sides.

Model Test Paper No. 5.
MATHEMATICS PAPER B.

Time allowed : 3 hours. Maximum Marks : 100.

PART I

I. (*a*) Construct a regular octagon in a $\odot$ of a radius 2·3" long. Measure one of the angles B.

(*b*) Divide a line 2" long into two parts which will contain a rectangle of area equal to a quarter of

a square inch. Measure the parts and verify by calculation.

II. (*a*) "The sum of the angles of a triangle is equal to two right angles." Prove it.

(*b*) In a $\triangle$ the angle between the internal bisector of one base angle and the external bisector of the other is equal to one half the vertical angle.

III. (a) Prove that if the square on one side of a $\triangle$ be equal to the sum of the squares on the other two sides, the angle contained by these sides is a right angle.

(*b*) Show that the diagonal of a square is less than that of a rectangle of the same area.

IV. (*a*) The opposite angles of any quadrilateral inscribed in a $\odot$ are supplementary.

(*b*) If the bisectors of the angles of a quadrilateral be drawn forming another quadrilateral, then the latter is a cyclic one.

Or,

In a cyclic quadrilateral the bisectors of any angle and of the opposite exterior angle intersect on the circle circumscribing the quadrilateral.

PART II.

V. (*a*) Construct a rhombus whose diagonals are 9 cm. and 6 cm. respectively. Inscribe a circle in it and measure the radius.

(*b*) Draw graphically a line $\sqrt{7}$ inches long or find geometrically the value of $\frac{5}{2{\cdot}7 \times 1{\cdot}3}$.

VI. (*a*) The rectangles contained by the segments of two intersecting chords of a circle are equal.

(*b*) Two circles intersect and from a point in the common chord produced two secants are drawn. Prove that the four points so determined are cyclic.

VII. (*a*) 'If the three sides of one △ are proportional to the three sides of another, the triangles are equiangular.' Prove it.

(*b*) Prove that the △ formed by joining the mid-points of the sides of a triangle is similar to the whole triangle.

VIII. (*a*) Draw a circle to touch a given line AB at C and a given ⊙ PGT.

Or,

Draw a circle to pass through two given points and having its centre in a given straight line.

(*b*) Construct a ⊙ to touch a given ⊙ at a given point and to pass through a given point.

Or,

Draw a circle to pass through two given points and having its centre on the circumference of a given circle.

Model Test Paper No. 6.

MATHEMATICS PAPER B.

Time allowed : 3 hours. Maximum Marks : 100.

PART I.

I. (*a*) Construct a quadrilateral ABCD, AB=1·5″, BC=1·7″, CD=1·1″, DA=1·8″, AC=2·3″. Find the

distances of B and D from AC, hence find out the area of the quadrilateral.

(*b*) Draw three circles of radii 2 cm., 3 cm., and 2·5 cm. touching each other externally in pairs.

II. (*a*) Prove that if the hypotenuse and a side of one right-angled triangle be equal to the hypotenuse and side of another right-angled triangle, the two triangles are congruent.

(*b*) If two triangles have two sides of the one equal to two sides of the other, each to each, and also the angles which are opposite to one pair of equal sides, equal, prove that the $\angle$s opposite to the other pair of equal sides are either equal or supplementary.

III. (*a*) 'The area of a $\triangle$ is half that of a $\|^{m}$ on the same base and having the same altitude.' Prove it.

(*b*) 'Two $\triangle$s are in equal in area, if two sides of the [one are equal to two sides of the other, each to each, and the contained $\angle$s are supplementary'. Prove it.

IV. (*a*) Prove that one circle and only one can be drawn through any three points not in a straight line.

(*b*) 'If from a point within a circle, three equal straight lines can be drawn to the circumference the phint is the centre of the circle'. Prove it.

Or,

A straight line cannot cut a circle in more than two points.

PART II.

V. (*a*) Draw two parallel st. lines 2" apart and draw a transversal cutting each of them at 45°. Draw a circle touching the parallels and the transversal. Measure the smallest angle of the figure formed by joining the points of contact.

(*b*) Take a straight line 1·6" long and find graphically a line $(1 \cdot 6)^2$ inches long.

Or,

Construct a square whose area is 11 square inches.

VI. (*a*) "The angle in a semicircle is a right angle; the angle in a segment greater than a semicircle is an acute angle; and the angle in a segment less than a semicircle is an obtuse angle." Prove it.

(*b*) "Circles described on any two sides of a triangle as diameters intersect the third side in the same point." Prove it.

VII. (*a*) If a straight line is drawn parallel to one side of a triangle, prove that the other two sides are divided proportionally.

(*b*) P is any point within a $\triangle$ABC; a line DE $\parallel$ AB cuts PA, PB in D and E; FF is drawn $\parallel$ BC to cut PC in F; prove that DF $\parallel$ AC.

Or,

Prove that a parallel to the parallel sides of a trapezium divides the non-parallel sides proportionally.

VIII. (*a*) Draw an equilateral $\triangle$ in a square with its vertex at a point in the side.

Or,

Construct a square about a quadrilateral.

Or,

Divide a $\triangle$ into three equal parts by 3 lines $\parallel$ base.

(*b*) Construct an equilateral $\triangle$ equal to the area of a given $\triangle$.

Or,

Bisect a $\triangle$ by a straight line drawn through a point in one of its sides.

Or,

Divide a quad. into three equal parts.

Model Test Paper No. 7.

MATHEMATICS PAPER B.

Time allowed : 3 hours. Maximum Marks : 100.

PART I.

I. (*a*) About a $\odot$ of 3·6 cm. radius describe an equilateral $\triangle$ and measure its sides.

(*b*) Draw the plan of a quadrilateral piece of land PQRS, where PQ=80 yds , QR=150 yds. ; RS =120 yds., SP=114 yds , and the angle PQR is equal to 90°. Mark the position of a stone pillar which is 120 yds. from the corner S and 50 yds. distant from the side PQ. Measure out its distance from P.

II. (*a*) If two triangles have two angles and one side of the one equal to two angles and the corresponding side of the other, prove that the triangles are congruent.

(*b*) Two triangles ABC and BCD are drawn on opposite sides of BC such that A and D are equidistant from BC, show that BC bisects AD.

III. (*a*) If there are three or more parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are also equal.

(*b*) Prove that the straight lines joining the mid-points of opposite sides of a quadrilateral bisect each other.

IV. (*a*) Two straight lines AB and CD intersect at O, prove that $\angle AOC = \angle DOB$.

(*b*) If OE and OF be the bisectors of these angles, prove that they are in the same straight line.

PART II.

V. (*a*) Draw a $\triangle ABC$ with $\angle B = 60°$, $\angle C = 67\frac{1}{2}°$ and its height $= 4{\cdot}1$ cm. Construct a $\|^{m}$ of equivalent area having one of its diagonals $3{\cdot}5$ cm. long.

(*b*) In a $\odot$ of radius $2{\cdot}5$ cm. place a chord of $4{\cdot}3$ cm. in length. Measure and calculate its distance from the centre of the circle.

VI. (*a*) Prove that the locus of a point which is equidistant from two intersecting st. lines consists of the pair of st. lines which bisect the angles between the two given lines.

(*b*) What does it become when the given lines are parallel. Find the locus of a point which moves in such a way that of the perpendiculars drawn from

it to the sides AB, AC of a $\triangle$, the first is always twice as long as the second.

VII. (*a*) The tangent at any point of a circle and the radius through the point are perpendicular to each other.

(*b*) Chords of a $\odot$ touching a concentric $\odot$ are equal and are bisected at their point of contact.

VIII. Construct a circle of a given radius.

(1) To pass through a given point and touch a given st. line.

Or,

(2) To pass through a given point and touch a given circle.

Or,

(3) To touch two given st. lines.

Or,

(4) To touch two given circles.

Or,

(5) To touch a given st. line and a given circle.

Or,

(6) To pass through two given points.

Model Test Paper No. 8.

MATHEMATICS PAPER A.

Time allowed : 3 hours. Maximum Marks : 100.

PART I.

I. (*a*) Through a point O draw two straight lines including an angle of 120°, take a point S in one of them at a distance of 1·3″ from O and R in the other at a distance of 1.6″ from O. Find the distance of S from OR and of R from OS.

(*b*) Construct a triangle ABC in which AB =3 cm., BC=4 cm. and CA=5 cm. Construct another triangle of which A, B, C are the mid-points of the sides. Measure the sides of the other triangle.

II. (*a*) If two triangles have two sides of the one equal to two sides of the other and included angle of the one equal to included angle of the other, prove that the two triangles are congruent.

(*b*) Prove that the mid-point of the hypotenuse of a right-angled triangle is equidistant from the angular points.

III. (*a*) If two sides of a triangle are equal, prove that the angles opposite to these sides are equal.

(*b*) In a triangle if the bisector of an angle bisects the opposite side, prove that the triangle is isosceles.

Or,

Prove that the straight line bisecting the exterior angle at the vertex of an isosceles triangle is parallel to the base.

IV. (*a*) Enumerate and prove some of the most important properties of a parallelogram.

(*b*) What properties of a quadrilateral will convert it into a parallelogram ? Prove one of them.

Or,

Prove that the bisectors of the four angles of a parallelogram enclose a rectangle.

PART II.

V. (*a*) Find by geometrical construction a mean proportional between two lines of lengths 4·5 cm. and

·2 cm. respectively. Check the correctness of the drawing by measurement.

Or,

Taking 1″ as the unit of length construct the line whose length is $\sqrt[4]{30}$ inches.

(*b*) Draw a triangle with sides 5 cm., 6 cm., and 8 cm. Construct a triangle equal in area to it and having an angle of 60° and one side equal to 7 cm.

VI. (*a*) Prove that equal chords of a circlea re equidistant from the centre.

(*b*) Prove that any two equal chords of a circle are equally inclined to the diameter passing through the point of intersection.

VII. (*a*) If two circles touch, prove that the point of contact lies on the straight line through the centres.

(*b*) Three circles touch one another, two by two, externally at P, Q and R. Show that the tangents at P, Q and R are concurrent.

VIII. (*a*) Place a chord of a given length in a $\odot$ and passing through a given point when produced.

(*b*) Draw a st. line XY $\parallel$ BC the base of $\triangle$ ABC, so that $\triangle AXY = \frac{9}{7}$ fig. BCYX.

PUNJAB UNIVERSITY MATRICULATION EXAMINATION

PAPER 1940

1. (*a*) Draw the direct common tangents to two circles whose radii are 5 centimetres and 2 centimetres and centres are 3 inches apart.

(*b*) Draw a circle to touch a straight line AB=7 centimetres at a point C where AC=3 centimetres and to pass through a point D which is not on AB.

Or,

(*b*) Construct a rectangle whose diagonal is ·5 inches and one of the sides=one inch.

2. If the sides of a convex polygon are produced in order, prove that the sum of the angles so formed is equal to four right angles.

Prove that each $\angle$ of a convex regular polygon of a sides is $\frac{2n-4}{n}$ right angles.

The angles of a five-sided figure are $a+30$, $a+10$, $2a$, $a-12$ and $3a-40$ degrees. Find the value of a.

3. If a straight line touches a circle and from the point of contact a chord be drawn, prove that the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

State and prove the converse of this theorem.

4. If two triangles have one angle of the one equal to one angle of the other and the sides about these equal angles proportional, prove that the triangles are similar.

Prove that the lines joining the middle-points of the sides of a triangle make four triangles similar to the original triangle.

5. (*a*) **Prove that the triangles on the same base and of the same altitude are equal in area.**

(*b*) **Prove that the sum of the perpendiculars, drawn from a point within an equilateral triangle on the sides, is constant and equal to the altitude of the triangle.**

6. (*a*) Draw a square equal in area to a hexagon whose sides are 2·4 inches.

Or,

(*a*) Construct a triangle having given the base equal to 2 inches, sum of the other sides equal to 5·1 inches and one of the angles at the base equal to 60 degrees. Measure the sides.

(*b*) Find geometrically the mean proportional between the straight lines 6·4 and 3·6 centimetres long.

7. (*a*) **Prove that the bisector (inrernal of an angle of a triangle) divides the opposite sides internally in the ratio of the sides containing the angle.**

(*b*) **The base BC of a triangle ABC is bisected at at C. DE, DF bisect angles ADC, ACB meeting AC, AB E and F. Prove that EF is parallel to BC.**

8. (*a*) **Prove that if there are three or more parallel straight lines and the intercepts made by them on any straight line that cus them are equal, then the corresponding intercepts on any other sttaight line that cuts them are also equal.**

(b) In a trapezium ABCD, AB is parallel to CD. P is the middle point of AD and PQ is drawn parallel to DC, meeting BC in Q. Prove that BQ=CQ.

Hint. *Join AC cutting PQ in O. Then in $\triangle$ ACD, O is the mid-point of AC. $\therefore$ in $\triangle$ ABC, Q is the mid-point of BC. Hence BQ=CQ.*]

9 (a) **Prove that the opposite angles of a cyclic quadrilateral are supplementary.**

(b) ABCD is a cyclic quadrilateral in which AB and DC when produced meet in E and EA is equal to ED. Prove that AD is parallel to BC.

[**Hint.** $\because$ $EA=ED$. $\therefore$ $\angle A=\angle D$. Hence $\angle ECB$ $=\angle EBC$

$\therefore$ $EB=EC$. Hence $AB=CD$. Hence $BC \parallel AD$.]

Or,

10. (a) **Prove that in a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.**

(b) Upon the sides BC, CA, AB of a triangle ABC, perpendiculars PD, PE, PF are drawn from a point P within the triangle. Show that :

$$AF^2+BD^2+CE^2=BF^2+CD^2+AE^2.$$

1941

PART I

1. (*a*) Draw the transverse common tangents to two circles whose radii are 1·4 inches, 1·2 inches and centres are 3 inches apart.

(*b*) Draw a circle passing through the vertices of an equailateral triangle whose side is 2 inches,

Or,

(*b*) Find geometrically the square root of 15.

2. **If two triangles have two sides of the one equal to two sides of the other each to each and also the angles contained by those sides equal, show that the triangles are congruent.**

If the diagonals of a quadrilateral bisect each other at right angles, prove that the quadrilateral is either a rhombus or a square.

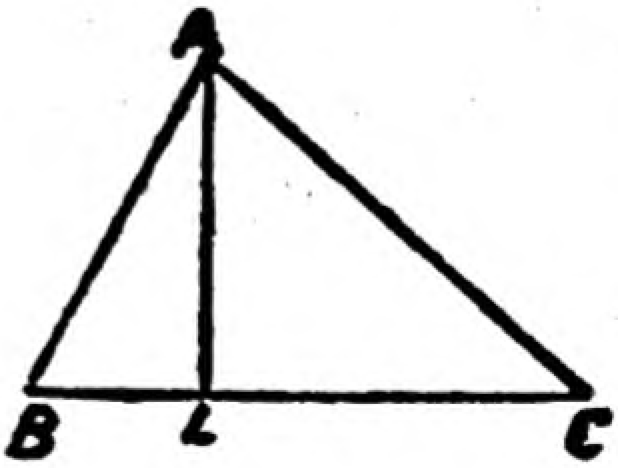
3. **Prove that the parallelograms on the same base and of the same altitude are equal in area.**

[*Prop. 20*]

Prove that the straight lines joining the middle points of the sides of a triangle form with the sides three parallelograms which are equal in area.

Note.—*Only one question is to be attempted from Questions 4 and 5.*

4. **Prove that the ratio of the areas of two similar triangles is equal to the ratio of the squares on the corresponding sides.** [*Prop. 37*]



Apply this theorem to prove that in a triangle ABC, right-angled at A, $AB^2+AC^2=BC^2$.

[**Hint :** *Draw $AL \perp BC$.*

$\because$ $\angle s\ ABL, ABC$ *are similar* $\therefore \frac{\triangle ABL}{\triangle ABC}=\frac{AB^2}{BC^2}$.

$\because$ $\triangle s\ ACL, ABC$ *are similar* $\therefore \frac{\triangle ACL}{\triangle ABC}=\frac{AC^2}{BC^2}$.

$\therefore$ $\frac{AB^2}{BC^2}+\frac{AC^2}{BC^2}=\frac{\triangle ABL}{\triangle ABC}+\frac{\triangle ACL}{\triangle ABC}$

Or $\frac{AB^2+AC^2}{BC^2}=\frac{\triangle ABL+\triangle ACL}{\triangle ABC}=\frac{\triangle ABC}{\triangle ABC}=1$

Hence $AB^2+AC^2=BC^2$.]

Or,

5. (*a*) **Prove that the angles in the same segment of a circle are equal. State the converse of this theorem.** [*Prop. 44*]

(*b*) **In a circle two chords AC, BD intersect at right angles in E. Join CD and draw AF perpendicular to CD meeting BD or BD produced in L.**

Show that BE=EL.

Hint.—*First prove* $\angle ABL=\angle C=\angle ALB$. *Then* $BE=EL$.]

PART II

6. (*a*) In and about a given circle of radius 3·8 centimetres inscribe and circumscribe an octagon.

(*b*) Draw a triangle ABC in which AB=6·5 centimetres, AC=5·3 centimetres and angle at C=90°. Construct the line which bisects AB at right angles and if it cuts AC at M, measure AN.

Or,

(*b*) Divide internally and externally a straight line 5·4 centimetres long in the ratio of 2 : 3.

7. **If two straight lines intersect, prove that the vertically opposite angles are equal.**

If the bisectors of the vertical angle of a triangle bisects the base, prove that the base angles are equal:

8. (*a*) **Prove that (1) the angle in a semicircle is a right angle : (2) the angle in a segment greater than a semicircle is less than a right angle ; (3) the angle in a segment less than a semicircle is greater than a right-angle.**

(*b*) **If a circle is described on one side of the equal sides of any isosceles triangle as diameter, prove that it passes through the middle point of the base.**

Note. *Only one question is to be attempted from Question 9 and 10.*

9. Enunciate and prove the geometrical theorems corresponding to the identities :—

(1) $(a-b)^2=a^2+b^2-2ab$.

(2) $(a+b)^2-(a-b)^2=4ab$.

Or,

10. (*a*) **If swo triangles are equiangular, prove that their corresponding sides are proportional.**

If AD be the bisector of the angles A of a triangle ABC, prove that $BA.AC=BD.CD+AD^2$.

1942

PART I

1. (*a*) Make a rectangle whose adjacent sides are 9 centimetres and 4 centimetres. Construct a square equal in area to this rectangle. Measure its side.

(*b*) Construct a triangle whose sides are 3 inches, 4 inches and 5 inches. Inscribe a circle in this triangle.

2. (*a*) **Prove that the angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.**

(*b*) **An arc AB of a circle subtends an angle of 80 degrees at the centre. P is a point on the remaining part of the circumference such that arc AP=arc BP. Calculate the angle PAB.**

[**Hint.**—$\angle APB=\frac{1}{2}\,80°=40°$. *Hence in the* $\triangle APB$, $\angle PAB=\frac{1}{2}(80°-40°)=20°$].

3. (*a*) **Prove that if a straight line touches a circle and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments.**

(*b*) **ABP is a segment of a circle capable of an angle of 60 degrees. The tangents to the circle at A and B meet in C. Prove that ABC is an equilateral triangle.**

[**Hint.**—$\because\ \angle APB=60°$.

$\therefore\ \angle APC=60°$ *and* $\angle ACB=60°$.]

Note.—*Only one question is to be attempted from Question 4 and 5.*

4. (*a*) **State and prove the converse of :—**

In a right-angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

(*b*) **ABD and BCD are two triangles on the opposite sides of the line BD. ∠BDC=90°, ∠BCD=30°, CD=10 inches, AB=3 inches and AD=4 inches. Prove that ∠BAD is a right angle.**

[**Hint.**—$BD=\frac{1}{2}CD=5''$
Hence in $\triangle ABD$,
$\because\ AB^2+AD^2=BD^2$.
$\therefore\ \angle BAD=90°$.]

5. (*a*) Prove that if two triangles have one angle of the one equal to one angle of the other, and the sides about these equal angles proportional, the triangles are similar.

(*b*) ABC and DEF are two triangles in which $\angle B=\angle E$. The bisectors of these equal angles divide the opposite sides in the same ratio. Prove that the triangles ABC and DEF are similar.

[**Hint.**—*Let* BL, EM *be the bisectors.*

Then $\frac{AB}{BC}=\frac{AL}{CL}$ *and* $\frac{ED}{EF}=\frac{DM}{MF}$.

But $\frac{AL}{CL}=\frac{DM}{MF}$.

$\therefore\ \frac{AB}{BC}=\frac{ED}{EF}$ *and* $\angle B=\angle E$.

$\therefore\ \triangle^s$ *ABC, DEF are similar.*]

PART II.

6. (*a*) Construct a quadrilateral ABCD in which $\angle A=60°$. AB=2·8 inches, BC=3 inches, CD=2·5 inches and DA=3·2 inches.

Reduce this figure to an equivalent triangle.

(*b*) Find geometrically the value of ;

$$\frac{3{\cdot}2\times1{\cdot}5}{1{\cdot}8}.$$

7. (*a*) Prove that internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

(*b*) AD is a median of the triangle ABC. DE and DF are the bisectors of the angles ADB and ADC respectively and meet the sides AB and AC in E and F. Prove that EF is parallel to BC.

8. (*a*) Prove that the locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.

(*b*) ABCD is a quadrilateral in which AB=BC. The perpendicular bisectors of AD and DC meet in E. B is joined to E. Prove that BE bisects the angle ABC.

[**Hint.** *Join AE, DE, CE.*
Then $AE=CE$.
Hence $\triangle ABE \equiv \triangle BEC$.
$\therefore \angle ABE = \angle CBE$.]

Note.—*Only one question is to be attempted from Questions 9 and 10.*

9. If three or more parallel straight lines make equal intercepts on any transversal they make equal intercepts on any other transversal. Prove it.

If through the middle point of the base of a triangle a straight line is drawn parallel to one of the sides, prove that its intercepts on the internal and external bisectors of the vertical angle is equal to the other sides.

10. Prove that—

(*i*) the medians of a triangle are concurrent;

(*ii*) the point of concurrence is a point of trisection of each median.

[Theorem 4 of concurrency of lines in a triangle.]

1943

PART I

1. (*a*) Draw two circles whose radii are 4·5 cm. and 1·5 cm. and whose centres are 5 cm. apart. Draw a direct common tangent to these two circles, and measure its length between the points of contact.

(*b*) Construct a triangle whose perimeter is 5·4 inches and the angles at the base are 60° and 90°.

2. (*a*) Prove that if two triangles are equi-angular, their corresponding sides are proportional.

(*b*) AB is a straight line AP and BQ are two parallel lines drawn on opposite sides of AB. P and Q are joined by a straight line which meets AB in C. If AP and BQ be 2·5 inches and 1·5 inches long, prove that $AC=\frac{5}{3}$ CB.

3. (*a*) Prove the following theorem and its con verse :—

The opposite angles of any quadrilateral inscribed in a circle are supplementary.

(*b*) Inscribe any six-sided figure ABCDEF in a given circle of two inches radius. Prove that the sum of its alternate angles, i.e., $\angle$B, $\angle$D and $\angle$F is equal to four right angles.

Note.—*Only one question is to be attempted from Questions 4 and 5.*]

4. (*a*) If a straight line joining two points subtends equal angles at two other points on the same side of it, prove that the four points lie on the same circle.

(*b*) If D, E and F are the middle points of the sides BC, CA and AB of a △ ABC, and P is the foot of the perpendicular from A to BC, show that D, P, E and F are concyclic.

Or,

5. (*a*) Prove that areas of similar triangles are in the ratio of the squares on their corresponding sides.

(*b*) ABC is a right-angled triangle. Equilateral triangles are constructed on all the three sides of this triangle. Prove that the area of the equilateral triangle described on the hypotenuse is equal to the areas of the equilateral triangles described on the other two sides.

PART II.

6. (*a*) In a circle of 1·3 inches radius, construct—

(*i*) a regular hexagon,

(*ii*) an equilateral △.

Measure the sides of the triangle and verify by calculation.

(*b*) Take a line 2·5 inches long. Divide it externally in the ratio 3 : 2.

7. (*a*) Define a convex polygon. If the sides of a convex polygon are produced in order, the sum of the exterior angles so formed is equal to four right angles.

(*b*) Apply the above proposition to prove that the sum of the angles of a triangle is equal to two right angles.

8. (*a*) Show that the locus of a point which is equidistant from two fixed intersecting straight lines, consists of a pair of straight lines which are right angles to each other.

What does this proposition become when the straight lines are parallel ?

(*b*) A chord of constant length moves inside a circle. Find the locus of its middle points.

[**Note**. *Only one question is to be attempted from Questions 9 and 10*].

9. (*a*) State and prove the geometrical theorem corresponding to the algebraical identity $a^2-b^2=(a+b)(a-b)$.

(*b*) Show that the longest side of a triangle is bisected by the shortest median.

Or,

10. (*a*) What is meant by the projection of a straight line upon another straight line ?

In an obtuse angled triangle the square on the side opposite to the obtuse angle is equal to.......

Complete the enunciation of the above theorem and prove it.

(*b*) ABC is an isosceles triangle. Its base BC is produced to D, and D is joined to A. Prove that $AD^2=AC^2+BD.CD$.

———

1944

PART I

1. (*a*) Draw two circles of radii 3 inches and 1 inch, their centres being 5 inches apart. Draw a transverse common tangent to these circles. Measure the length of the tangent and verify it by calculation.

(*b*) Construct a triangle ABC having $\angle B=30$ degrees, $BC=2$ inches, and $BA-CA=1$ inch.

Note :—Do not change the scale.

2. (*a*) In equal circles if two arcs subtend equal angles at the centres, they are equal. Prove.

(*b*) AB is a chord of a circle whose centre is O. OP is a radius of the circle and is drawn parallel to AB. A is joined to O and AO is produced to meet the circle again in D. Prove that arc PD=arc PB.

3. (*a*) If a straight line be drawn parallel to one side of a triangle, it divides the other two sides proportionately. Prove.

(*b*) Prove that the diagonals of a trapezium cut each other proportionately.

Note. *Only one question is to be attemped from Questions 4 and 5.*

4. (*a*) Equal chords of a circle are equidistant from the centre. Prove it.

(*b*) AB and AC are two chords of a circle and they are equally inclined to the diameter AD. Prove that AB=AC.

5. (*a*) Show that the external bisector of an

angle of a triangle divides the opposite sides externally in the ratio of the sides containing the angle.

What does this proposition become when the triangle is an isosceles triangle ?

(*b*) ABC is a triangle. O is a point inside. OD, OE and OF are the bisectors of the angles AOB, BOC and COA respectively. Show that AD.BE.CF=DB.EC.FA.

PART II.

6. (*a*) With sides 1·5″, 1·7″ and 1·9″ draw a triangle. Find a point within the triangle at which the three sides subtend equal angles.

(*b*) Find geometrically the value of $\frac{(1\cdot2)^2}{2}$ taking 1″ as unit of length.

7. (*a*) State the conditions under which two triangles can be congruent. Prove any one of them.

(*b*) Prove that a triangle is equilateral if its three altitudes are equal.

8. Show that a quadrilateral is a parallelogram if

(*a*) its opposite angles are equal ;

(*b*) its diagonals bisect each other ;

(*c*) each of its diagonals bisects the quadrilateral.

Note :—*Only one question is to be attempted from Question 9 and 10.*

9. (*a*) Show that areas of similar triangles are to one another as the squares on the corresponding sides.

(*b*) Bisect a triangle by a straight line drawn parallel to the base.

Prove the validity of your construction.

10. (*a*) P is a point outside a circle. PT is a tangent to it and PAB is any secant. Show that

$$PT^2 = PA.PB.$$

(*b*) ABC is an isosceles triangle in which AB=AC. A circle through B touching AC at the middle point intersects AB in P. Show that $AP = \frac{1}{4}AB$.

1945

PART I.

1. (*a*) Construct a quadrilateral ABCD given that AB=4″ BC=3″, BC=2·5″, $\angle ABC = 90°$ and $\angle BCD = 120°$. Measure AD.

(*b*) Draw a triangle equal in area to the above quadrilateral.

2. (*a*) Prove that if two sides of a triangle are equal, the angles opposite these sides are equal.

(*b*) ABC is a right angled triangle at A. AD is perpendicular from A to BC, P is any point on CB such that CP=CA. Prove that AP bisects $\angle BAD$.

[**Proof.** $\because$ CP=CA, $\angle CPA = \angle CAP$.

But in $\triangle ABP$, Ext $\angle CPA = \angle B + \angle BAP$.

and $\angle CAP = \angle CAD + \angle PAD$

$\therefore$ $\angle B + \angle BAP = \angle CAD + \angle PAD$

But $\angle B = \angle CAD$ (*being complements of* $\angle BAD$)

$\angle BAP = \angle PAD$.

Hence AP bisects $\angle BAD$.]

3. (*a*) If there are three or more parallel straight lines and the intercepts made by them on any straight line that cuts them are equal, then prove the corresponding intercepts on any other straight line that cuts them are also equal.

(*b*) Prove that a straight line drawn from the middle point of a side of a triangle parallel to the base, bisects the other side and is half of the base.

4. (*a*) Prove that two triangles on the same base and of the same altitude are equal in area.

(*b*) ABCD is a quadrilateral ; a line through D parallel to AC meets BC produced at P.

Prove that $\triangle ABP =$ quadrilateral ABCD.

[**Hint**. $\triangle ABP = \triangle ACD$ (*being on the same base and between the same* $\|^s$.]

$\therefore \triangle ABC + \triangle ACP = \triangle ABC + \triangle ACD.$

Hence $\triangle ABP =$ Quad. ABCD]

5. (*a*) In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides. Prove.

(*b*) ABC is a right-angled triangle at C, p is the length of the perpendicular from C to AB, prove that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

where $a = BC$, $b = CA$.

Proof :—$\triangle ABC = \frac{1}{2} AB.p = \frac{1}{2} cp$

Also $\triangle ABC = \frac{1}{2} AC.BC = \frac{1}{2} ab.$

$\therefore \frac{1}{2} cp = \frac{1}{2} ab$ or $cp = ab.$

$\therefore p = \frac{ab}{c}$. Hence $p^2 = \frac{a^2b^2}{c^2} = \frac{a^2b^2}{a^2+b^2}$

$(\because \; c^2 = a^2 + b^2)$

$$\because \quad \frac{1}{p^2}=\frac{a^2+b^2}{a^2b^2}=\frac{b^2}{p^2b^2}+\frac{a^2}{a^2b^2}$$
$$=\frac{1}{a^2}+\frac{1}{b^2}.$$

PART II

6. (*a*) Divide a straight line 2·3 cm. long in extreme and mean ratio.

(*b*) Draw a circle passing through two given points and touching a given straight line.

Or,

(*b*) Draw three circles of radii 1, ·8 and ·5 cm. touching each other externally two by two.

7. (*a*) Show that the medians of a triangle are concurrent and that they are divided in the ratio of 1 : 2 at the centroid.

(*b*) Show that the four times the sum of the medians of a triangle is greater than three times the sum of the sides.

8. (*a*) Prove that the internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle and likewise the external bisector externally.

(*b*) In the triangle ABC the internal and external bisectors of the angle A meet the side BC in L and M respectively.

Prove that

$$LM=\frac{2abc}{c^2-b^2}$$

where $BC=a$, $CA=b$, and $AB=c$.

9. (*a*) If a straight line touches a circle and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments. Prove.

(*b*) State the converse of the above theorem and apply it to draw a tangent to a circle at a given point on its circumference.

10. Prove that one and only one circle can be drawn through three points not in a straight line.

(*b*) Prove that a circle cannot cut a straight line in more than two points.

———

1946

1. (*a*) Construct a rt.-angled △ with hypotenuse equal to 2 inches and one side 1·3 inches, and then constrcut a parallelogram equal to it in area and having one angle equal to 75°.

(*b*) Divide a straight line 1·8 inches long externally in the ratio 5 : 2 and measure the two parts thus obtained.

(*c*) Find geometrically the value of $\frac{3}{1 \cdot 3}$.

[**Hint.**—This can be written as $\frac{3 \times 1}{1 \cdot 3}$.]

2. (*a*) Draw a transverse common tangent to two equal circles with radii 3 cm. each and with their centres 7·5 cm. apart. Measure its length and verify by calculation.

(*b*) Circumscribe an equilateral triangle about a circle of radius one inch ; measure the side.

3. (*a*) If the sides of a convex polygon are produced in order, prove that the sum of the exterior angles so formed is equal to four right angles.

(*b*) One angle of a regular polygon is $1\frac{8}{9}$ right angles ; find the sum of all the angles.

4. (*a*) Prove that the locus of a point, which remains equidistant from two fixed points, is the perpendicular bisector of the straight line joining the two points.

(*b*) ABC and ABD are two isosceles triangles on a common base AB. Prove that CD bisects AB at right angles.

5. Prove that the sum of the squares on any two sides of a triangle is equal to twice the square on the median bisecting the third side together with twice the square on half the third side.

Hence show that the median bisecting the hypotenuse of a right-angled triangle is half the hypotenuse.

6. (*a*) State and prove the converse of the following theorem :—

'Angles in the same segment of a circle are equal.'

(*b*) Prove that any four vertices of a regular pentagon are concyclic.

7. (*a*) Show that the opposite angles of any quadrilateral inscribed in a circle are supplementary.

(*b*) ABC is an isosceles triangle with AB equal to AC. A circle is drawn passing through B and C, cutting AB in P. Prove that AP=AQ.

8. (*a*). What are similar figures? Prove that equiangular triangles are always similar.

(*b*) ABC is a right-angled triangle and AD is perpendicular to the hypotenuse BC. Prove that $AB^2 = BD.BC$.

9. (*a*) Prove that the ratio of the areas of similar triangles, is equal to the ratio of the squares on the corresponding sides.

(*b*) ABC is a triangle and PQ is a straight line meeting AB in P and AC in Q. If AP=1 cm., PB=3 cm., AQ=1·5 cm., and QC=4·5 cm., prove that the triangle APQ is one-sixteenth of the triangle ABC in area.

10. (*a*) Why is a straight angle so called? How many right angles does it contain by definition?

Use these facts to prove that if a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles.

(*b*) How many lines can be drawn joining two given points? Which of them is the shortest?

Use these facts to prove that any two sides of a triangle are together greater than the third.

(*c*) What is the difference between equal and congruent triangles? Illustrate your answer by a suitable figure.

1947

1. (*a*) Construct a quadrilateral ABCD with AB $=BC=1\cdot6''$, $CD=1\cdot9''$, $DA=2\cdot1''$, and $\angle ABC=105°$. Reduce this to an equivalent triangle.

Either (*b*) Construct a triangle ABC with $AB=2''$, $\angle ABC=60°$, and $\angle ACB=45°$, Describe a circle to it touching CB and CA produced.

Or

(*b*) Construct an isosceles triangle ABC with base $BC=1\cdot5''$, and corresponding altitude$=1\cdot2''$. Find a point equidistant from A and B and at a distance of $2''$ from C.

[**Hint.**—Draw PQ the right bisector of AB.

With centre C and radius equal to $2''$, draw an arc cutting PQ in P. Then P is the required point.]

2. Attempt *any three* parts :—

(*a*) Divide a st. line AB, 5 cm. long internally at P, so that $AB.BP=AP^2$.

[**Hint**. *Divide AB internally in the mean and extreme ratio.*

(*b*) Construct a st. line equal to $\sqrt{2\cdot3}$ inches in length.

[**Hint.** *Find the mean proportional between $2\cdot3''$ and $1''$.*

(*c*) Through a given point on a circle draw its tangent without using the centre.

(*d*) Draw three circles of radii 2·5 cm. 3 cm., and 3·5 cm. respectively touching one another externally.

3. (*a*) If a straight line cuts two parallel straight lines the alternate angles are equal.

(*b*) A quadrilateral has a pair of opposite sides equal and parallel. Show that it is a $\|^{gm}$.

4. (*a*) If two angles of a triangle are unequal, the greater angle has the greater side opposite to it.

(*b*) The base BC of an isosceles triangle ABC is produced to D, and a straight line DPQ is drawn cutting AC internally at P and meeting AB in Q. Prove that AP is greater than AQ.

[**Hint**. In $\triangle BDQ$, ext. $\angle AQP >$ int. opp. $\angle B > \angle ACB$.

In $\triangle PCD$, ext. $\angle ACB >$ int. opp. $\angle CPD > \angle APQ$.

Hence $\angle AQP > \angle APQ$. $\qquad \therefore AP > AQ$.

5. (*a*) **State and prove the converse of this theorem :—**

In a right angled $\triangle$, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

(*b*) ABCD is a rectangle with AB=9″ and BC=24″. DC is produced to K so that CK=7″. If L be the middle point of AD, calculate LB, LK, and BK, and show that $\angle BLK$ is a right angle.

[**Hint**. (1) In $\triangle ALB$, $LB^2 = AB^2 + AL^2 = 9^2 + 12^2$
$= 81 + 144 = 2{\cdot}5$

$\therefore LB = \sqrt{225} = 15''$.

(2) In $\triangle DLK$, $LK^2 = DK^2 + DL^2 = 16^2 + 12^2$
$= 256 + 144 = 400$

$\therefore LK = \sqrt{400} = 20''$.

(3) In $\triangle BCK$, $BK^2 = BC^2 + CK^2 = 24^2 + 7^2$
$= 576 + 49 = 625$

$\therefore$ $BK=\sqrt{625}=25$ inches.

Now $\because$ $25^2=20^2+15^2$ or $RBK^2=LK^2+LB^2$.

$\therefore$ $\angle BLK=90°$].

6. (*a*) Show that triangles on the same base and between the same parallels are equal in area.

(*b*) Show how to construct an isosceles triangle equal in area to a given triangle with one of its sides as base. Give proof for your construction.

[**Steps.** Let ABC be the given $\triangle$ with BC as its base.

(1) Through A draw $AX \parallel BC$

(2) Draw ED the rt. bisector of BC meeting AX in D.

(3) Join BD, CD. Then Isos. $\triangle DBC=\triangle ABC$.

Proof. $\because$ D lies on the rt. bisector of BC, $\therefore$ $BD=CD$.

Hence $\triangle DBC$ is an isosceles $\triangle$.

$\because$ $\triangle$s ABC, DBC, lie in the same base and between the same $\parallel$s $\therefore$ $\triangle ABC=\triangle DBC$.]

7. (*a*) Show that there is one circle and only one which passes through three given points not in a straight line.

(*b*) Given a pentagon ABCDE. If the quadrilaterals ABCD and BCDE are cyclic, prove that the pentagon is also cyclic.

8. (*a*) The angle subtended by an arc of a circle at the centre is double that subtended by it at any point on the remaining part of the circumference.

(*b*) The radius of a circle is $\sqrt{2}$ inches. The circle is divided into segments by a chord 2 inches in length. Prove that the angle of the larger segment is 45 degree.

Proof. $\because$ chord$=2''$ and radius$=\sqrt{2}$ inches.

$\therefore$ the $\triangle$ formed by the chord and the two radii is rt. angled.

Hence the chord subtends a rt. angle at the centre.

$\therefore$ it will subtend an angle of $45°$ at the circumference.]

9. (*a*) If two triangles have one angle of the one equal to one angle of the other and the sides about these angles proportional, the triangles are similar.

(*b*) Chord AB of a circle is produced to O. If T be a point on the circle such that $OA.OB=OT^2$, show that OT is a tangent to the circle.

10. (*a*) Show that the internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

(*b*) AD is a median of a triangle ABC ; DP and DQ are the internal bisectors of angles ADB and ADC meeting AB and AC in P and Q respectively. Show that PQ is parallel to BC.

———

1948

Note. 1. *Attempt first three questions and five more from the remaining seven :—*

Note. 2. *For the first two questions :—(i) the use of set squares and protractor is not allowed ; (ii) proofs are not required but the construction steps must be clearly stated ; (iii) the scale is not to be changed.*

1. Attempt part (a) and one more from the remaining two.

(a) Construct a right-angled triangle with hypotenuse equal to 2·5 inches and an acute angle equal to 30°. Then draw a circle passing through its vertices.

(b) Find geometrically the approximate value of $\frac{4\cdot3}{3\cdot4}$ and check your result by calculation.

(Take any scale you please, but state your result clearly.

[Hint. This is the same thing as $\frac{4\cdot3\times1}{3\cdot4}$]

(c) Divide a straight line 1·8″ in length, externally in the ratio of 5 : 2. Point out clearly the two segments and measure them.

2. ***Attempt any two parts :—***

(a) Draw a transverse common tangent to two circles with radii 2 cm. and 3 cm. respectively and centres 7 cm. apart.

(b) In a given circle inscribe a triangle equiangular to a given triangle.

(*c*) Construct a triangle ABC, given AB=5 cm., AC=3 cm. and median AD=3 cm.

3. (*a*) If two angles of a triangle are equal, the sides opposite to these angles are also equal.

(*b*) The external bisector of the vertical angle of a triangle is parallel to the base; show that the triangle is isosceles.

Note. *Attempt any five questions from the following seven :—*

4. (*a*) If there are three or more parallel straight lines and the intercepts made by them on any st. line that cuts them are equal, then the intercepts made by them on any other straight line that cuts them will also be equal.

(*b*) C is the middle point of a straight line AB and AD and BE are perpendiculars from A and B on any other straight line, meeting it in D and E respectively. Prove that CD=CE.

5. (*a*) Prove that the medians of a triangle are concurrent.

(*b*) From the figure of the last theorem deduce that the point of concurrence of the medians of a triangle trisects each medians. Use this result to prove that if two medians of a triangle are equal, the triangle is isosceles.

6. (*a*) The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

(b) The sum of the squares on the diagonals of a rhombus is equal to four times the square on a side.

[**Remember** (1) *The diagonals of a rhombus bisect each other at right-angles.*

(2) *Square on a line*$=4$ *times the square on half the line.*]

7. (a) In equal circles (or, in the same circle) if two arcs are equal, the chords of the arcs are also equal.

(b) If two sides of a cyclic quadrilateral are parallel, the other two must be equal.

8. (a) The tangent at any point of a circle and the radius through the point are perpendicular to each other.

(b) The straight line joining the point of contact of two parallel tangents of a circle must pass through the centre.

9. (a) If two chords of a circle intersect outside the circle, the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

(b) OAB, OCD and OEF are three straight lines through a point O. If the points A, B, C, D and C, D, E, F are separately concyclic, show that the points A, B, E, F are also concyclic.

[**Proof.** $\because$ A, B, C, D are concyclic. $\therefore$ OA.OB$=$ OC.OD.

$\because$ C, D, E, F are concyclic. $\therefore$ OE.OF$=$ OC.OD.

Hence $OA \cdot OB = OE \cdot OF$. $\therefore$ A, B, E, F are concyclic.]

10. (*a*) If two triangles are equiangular, their corresponding sides are proportional.

(*b*) ABC is a triangle inscribed in a circle with centre O. AO is joined and produced to meet the circumference in L, and AD is drawn perpendicular to BC. Show that the $\triangle$s ABL and ABC are similar.

Const. Join BL. The $\angle C = \angle L$ and $\angle ADC = \angle ABL$.]

East Punjab University Matriculation Examination

August 1948

Group 1—*Problems.*

Note—(*i*) *The use of set squares and protractor is not allowed.*

(*ii*) *The scale is not to be changed.*

(*iii*) *Proofs are not required, but construction steps must be clearly stated.*

Construct a parallelogram with diagonals equal to 2·3″ and 3·2″ respectively and one side equal to 1·5″. Then construct a square equal in area to it.

Or,

Construct a pentagon ABCDE ; given

$AB = BC = CD = 2{\cdot}5$ cm. ;

$DE = EA = 6$ cm. ;

$\angle B = 120°$ and $\angle C = 105°$.

Reduce it to an equivalent triangle.

2. Find geometrically the value of $\sqrt{5}$.

Or,

Construct a triangle with perimeter 3″ and base angles of 45° and 60° respectively.

3. Circumscribe an equilateral triangle about a circle of radius one inch.

Or,

Through a given point on the circumference of a circle draw its tangent without using the centre.

Group 2.

Note—Attempt *any three* questions from this group.

4. (*a*) State and prove the converse of the following theorem :—

> 'If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles.

(*b*) ABCD is a rhombus and O is a point outside it. If OA=OC, show that OBD is a straight line.

5. (*a*) If two triangles have two sides of the one equal two sides of the other, each to each and the angles included in those sides are equal the triangles are congruent. Prove it.

(*b*) On the sides BC, CA, AB of a triangle ABC. equilateral triangles BCA′, CAB′ and ABC′ are constructed all outwards. Show that AA′=BB′=CC′.

6 (*a*) Illustrate and explain the geometrical theorem corresponding to the algebraical identity :—

$$(a+b)^2=a^2+b^2+2ab.$$

(*b*) Prove geometrically the algebraical identity :

$$(a+b)^2-(a-b)^2=4ab.$$

7. (*a*) The locus of a point, which remains equidistant from two fixed points, is the perpendicular bisector of the straight line joining these points. Prove it.

(*b*) Hence or otherwise show that the straight line joining the vertices of two isosceles triangles with a common base bisects the base at right angles.

8. (*a*) In any triangle the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median bisecting the third side. Establish it.

(*b*) Prove that the sum of the squares on the sides of a parallelogram is equal to the sum of the squares on its diagonals.

Group 3

Note. Attempt *any two* questions from this group.

9. (*a*) Show that the opposite angles of a cyclic quadrilateral are supplementary.

(*b*) Given a quadrilateral ABCD ; P is a point on AB between A and B, and Q is a point on CD between C and D. quadrilaterals APQD and PBCQ are both cyclic, show that AD is parallel to BC.

10. (*a*). If two circles touch one another internally or externally, their centres and the point of contact are in the same straight line. Prove it.

(*b*) Two circles touch one another externally at A, and PAQ is a straight line meeting one circle in P and the other in Q. Show that the tangents at P and Q are parallel.

11. (*a*) Show that the internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

(*b*) AC is a diameter of a circle of radius $2{\cdot}5''$; B and D are two points on the circle on opposite sides of AC, such that AB=BC and AD=3″. If BD cuts AC at M, show that $AM=2\frac{1}{7}''$.

[**Hint.** Join DC. An accurate figure is not needed.]

12. (*a*) Prove that the areas of two similar triangles are proportional to the squares on the corresponding sides.

(*b*) OAB and OCD are two straight lines through O, such that OA=2cm., AB=4 cm., OC=3 cm., and CD=6 cm. Prove that the triangle OAC is one-eighth of the quadrilateral ABCD in area.

1949

(Group 1—*Problems*)

Note (*i*) *The use of set squares and protractor is not allowed.*

(*ii*) *The scale is not to be changed.*

(*iii*) *Proofs are not required but construction steps must be clearly stated.*

1. Divide a straight line 3·5″ long into three parts proportional to the numbers 2, 3, 4. Hence construct a triangle with perimeter 3·5″ and sides proportional to the numbers 2, 3, 4.

Or,

Construct a triangle with base$=2{\cdot}5''$, altitude$=1{\cdot}5''$, and median bisecting the base$=2''$.

II. Find geometrically the value of $\frac{(1{\cdot}9)^2}{1{\cdot}3}$, taking one inch as the unit of length.

Or,

In a given straight line AB, find a point P such that $AP^2=AB.BP$.

III. Draw two parallel straight lines one inch apart and another straight line intersecting them at an angle of 75°. Construct a circle touching these three straight lines.

Or,

Draw an angle of 105°, and construct a circle of radius $1{\cdot}5''$ touching its arms.

Group 2

Note—*Attempt* **any three** *full questions from this group. Full question means parts* (*a*) *and* (*b*) *of the* **same** *question and not of different questions.*]

IV (*a*) Two straight lines are cut by a transversal. Show that these two lines will be parallel if—

(*i*) a pair of a alternate angles are equal, or

(*ii*) a pair of corresponding angles are equal.

[*Both the parts are to be done.*]

(*b*) Two parallel straight lines are cut by a transversal. Show that the bisectors of the four interior angles enclose a rectangle.

V. (*a*) What is the difference between congruent and equal triangles ?

If two right-angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, prove that the triangles are congruent.

(*b*) If the middle point of the base of a triangle is equidistant from its sides, prove that the triangle is isosceles.

VI. (*a*) If two sides of a triangle are unequal, prove that the greater side has greater angle opposite to it.

(*b*) AB is the shortest and CD the longest side of a quadrilateral $ABCD$. Prove that angle A is greater than angle C.

7. (*a*) State and prove the converse of this theorem :

'In a right-angled triangle, square on the hypotenuse is equal to the sum of the squares on the other two sides.'

(*b*) $ABCD$ is a rectangle in which $AB=5$ units and $BC=2$ units. P is a point in CD such that $CP=1$ unit. Prove that AB subtends a right angle at P.

[*An accurate figure is not needed*]

8. (*a*) Prove that the locus of a point which remains equidistant from intersecting straight lines consists of the pair of straight lines bisecting the angles between those lines.

[*Give full proof in two parts*]

(*b*) Given a triangle ABC with unequal sides. Show how to find a point equidistant from AB and AC, and also equidistant from B and C. Give proof for your construction.

Group 3

Note—*Attempt* **any two** *full questions from this group.* [*Full question means parts* (*a*) *and* (*b*) *of the* **same** *question and not of different questions.*]

9. (*a*) Prove that equal chords of a circle are equidistant from the centre.

(*b*) If two equal chords of a circle meet outside the circle, show that their parts lying outside the circle are equal.

10 (*a*) If the straight line joining two points subtends equal angles at two other points on the same side of it, show that the four points lie on a circle.

(*b*) In an acute angled triangle ABC, the altitudes AD and BE intersect in H. Prove that triangles HED and HAB are equiangular.

11. (*a*) If two chords of a circle intersect *outside* the circle, prove that the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

(*b*) If two circles intersect, prove that their common chord, when produced, bisects their common tangents.

12. (*a*) What are similar figures ?

Prove that equiangular triangles are similar.

(*b*) Show that any straight line, drawn parallel to the base of a triangle and meeting the two sides, is bisected by the median which bisects the base.

E. P. U. MATRICULATION EXAMINATION

1950

Group 1 (*Problems*)

Note. (*i*) *The use of set squares and protractor is not allowed.*

(*ii*) *The scale is not to be changed.*

(*iii*) *Proofs are not required, but construction steps must be clearly stated.*

1. Construct a quadrilateral ABCD with AB= 1·5″, BC=1″, CD=1·8″, ∠B=105°, and ∠C=135°. Reduce it to an equivalent triangle.

Or,

Construct a regular hexagon of side 1·5″ and reduce it to an equivalent square

2. Construct a triangle ABC with BC=2″, ∠A= 60°, and ∠B=75°. Draw a circle passing through its vertices and measure its radius.

Or,

Draw three circles of radii 2 cm., 3 cm., and 4 cm. respectively, each touching the other two externally.

3. Construct two equal circles with radius one inch each and centres one inch apart. Draw a direct common tangent to them and write down its length.

Or,

Find geometrically the mean proportional between two straight lines whose lengths are 1·8″ and 3·2″. Verify your result by calculation.

Group 2 (*Theorems and Riders*)

Note. *Attempt any* three *full questions from this group.*

['*Full question' means parts* (*a*) *and* (*b*) *of the* **same** *question and not of different questions.*]

4. (*a*) Prove that the sum of the angles of a triangle is equal to two right angles.

(*b*) ABC is a triangle in which $\angle A = 64°$; the *external* bisectors of angles B and C meet in O. Find the magnitude of $\angle BOC$. [*Give reason for each step of the process : no ready-made formula will be accepted.*]

5. (*a*) (*i*) Define a parallelogram.

(*ii*) Prove that the opposite sides of a parallelogram are equal and its diagonals bisect each other.

(*b*) If the opposite angles of a quadrilateral are equal, show that it is a parallelogram.

6. (*a*) Show that in any triangle, the square on the side opposite to an acute angle is less than the sum of the squares on the sides containing the acute angle by twice the rectangle contained by either of these sides and the projection upon it of the other.

(*b*) The vertical angle A of an isosceles triangle ABC is acute and BD is perpendicular from B on AC. Prove that $BC^2 = 2AC.CD$.

7. (*a*) Show that the medians of a triangle are concurrent.

(*b*) If two medians of a triangle are equal prove that the triangle is isosceles.

8. (*a*) If a straight line touch a circle and from the point of contact a chord be drawn, show that the angles between the chord and the tangent are equal to the angles in the alternate segments.

(*b*) A tangent to a circle is drawn parallel to a chord. Prove that the point of contact bisects an arc cut off by the chord.

Group 3 (*Theorems and Riders*)

Note. *Attempt any* **two** *full questions from this group*

[*'Full question' means parts* (*a*) *and* (*b*) *of the* **same** *question and not of different questions.*]

9. (*a*) Prove that the locus of a point which remains equidistant from two fixed points is the perpendicular bisector of the line joining these points.

(*b*) In the sides AB and AC of an isosceles triangle ABC points R and Q are taken respectively, such that AR=AQ straight lines BQ and CR intersect in O and P is the middle point of BC. Prove that AOP is a straight line.

10. (*a*) Prove that the internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle.

(*b*) Angle C of a triangle ABC is a right angle ; the internal bisector of angle B meets AC in D. If AB=2·5″ and BC=1·5″, calculate the length of AD and DC.

11. (*a*) Prove that the opposite angles of a cyclic quadrilateral are supplementary.

(*b*) If the opposite sides of a cyclic quadrilateral are equal, show that it is a rectangle.

12. (*a*) Show that the tangent at any point of a circle and the radius through the point are perpendicular to one another.

(*b*) Prove that the straight line joining the points of contact of two parallel tangents to a circle passes through the centre.

1951

Group I—(*Problems.*)

Note.— (*i*) *The use of set squares and protractor is not allowed.*

(*ii*) *The scale is not to be changed.*

(*iii*) *Proofs are not required but construction steps must be clearly stated.*

1. Construct a triangle with two sides 2·6″ and 1·3″, so that its area may be maximum.

Or,

Draw two circles of radii 1·1 cm. and 2·3 cm. their centres being 4·3 cm. apart. Draw the two external common tangents, and measure them.

2. Construct a regular hexagon about a circle of radius 1·2″. Draw a circle about this hexagon, and measure its radius.

Or,

Construct a rhombus whose diagonals are 8 cm. and 6 cm. Measure its side. Reduce the rhombus to a rectangle of equal area.

3. Construct geometrically a straight line of length $\sqrt{17}$ cm. With this as hypotenuse, construct a right-angled triangle with one angle 60°.

Or,

Divide a straight line of length 3″ internally in the extreme and mean ratio.

Group II.—(*Theorems and Riders*)

Note.—*Try* **any three** *'complete questions' from this group.* [*Complete question means parts* (*a*) *and* (*b*) *of the* **same** *question.*]

4. (*a*) If a straight line cuts two other parallel straight lines then (*i*) the pairs of alternate angles are equal, (*ii*) the pairs of corresponding angles are equal, (*iii*) the sum of a pair of interior angles on the same side of the cutting line is equal to two right angles.

(*b*) ABCD is a parallelogram. Show that the bisectors of any two adjacent angles of the parallelogram are at right angles.

5. (*a*) If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.

(*b*) ABC is a triangle in which AC > AB. The bisectors of the base angles B and C meet in O.

Prove that OC > OB.

6. (*a*) Prove that the right bisectors of the sides of a triangle meet in one point.

(*b*) ABCD is a quadrilateral in which AB = BC, and the right bisectors of AD and CD meet in O. Prove that BO bisects the angle ABC.

7. (*a*) Prove that in a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

(*b*) Prove that the sum of the squares on the four sides of a rhombus is equal to the sum of the squares on the two diagonals.

8. (*a*) Prove that the angle which an arc of a circle subtends at the centre is double the angle which it subtends at any point on the remaining part of the circumference.

(*b*) O is the circumcentre of a triangle ABC; D is the middle point of BC. Prove that angle BOD= angle BAC.

Group III—(*Theorems and Riders*)

Note—*Try* **any two** *'complete questions' from this group.*

9. (*a*) The locus of a point, which is equidistant from two intersecting straight lines, consists of a pair of straight lines which bisect the angles between the two given lines.

(*b*) ABC is a triangle. On the side BC find a a point D, equidistant from AB and AC. Give reasons for your construction.

10. (*a*) Prove that angles in the same segment of a circle are equal.

(*b*) AB and CD are two chords of a circle intersecting at O. show that the triangles AOD and BOC are similar.

11. (*a*) It two triangles are equiangular, show that the corresponding sides are proportional.

(*b*) AD is perpendicular to hypotenuse BC of a right-angled triangle ABC. Prove that $AD^2 = BD.DC$.

12. (*a*) If two chords of a circle intersect either inside or outside the circle, the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

(*b*) Prove that the common chord of two intersecting circles, when produced, bisects the common tangents to the two circles.

Printed at The Swatantra Bharat Press,
423, Kucha Bulaqi Begum, Esplanade Road, DELHI.